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Search for the EOB Hamiltonian describing two black holes with arbitrary eccentricities

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Outline

I. Introduction

II. An EOB Hamiltonian for a system of two BHs in general

- Scattering angle

III. Discussion

I. Introduction

✓ BBH Sources for GWs

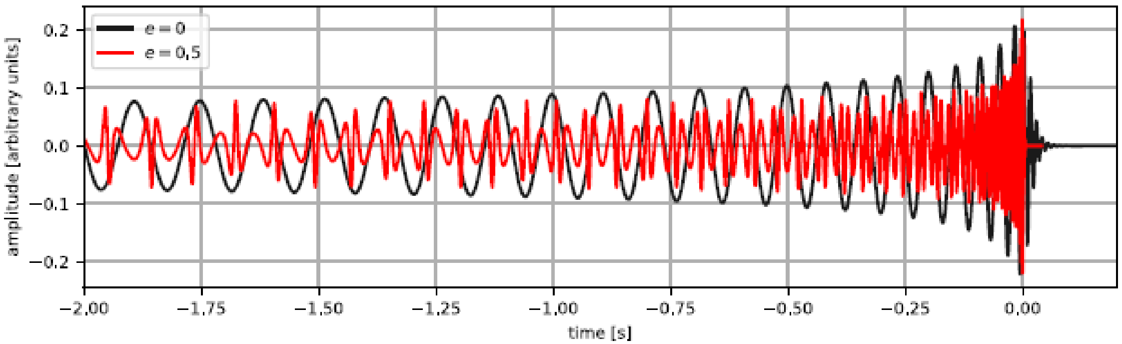
- O1 & O2: 11 events = 10 BBH + 1 BNS
- O3: 48 candidates = 31 BBH + 5 BNS + 6 BHNS + 6 others

- BBH systems are characterized by $M = m_1 + m_2, q = \frac{m_1}{m_2}, \vec{s}_1, \vec{s}_2, e, \ldots$
- For modelled search in LIGO, mostly, $m: 1\sim100\ M_\odot, q \leq 10, \vec{s}\sim\vec{J}, e\sim0, l = 2$

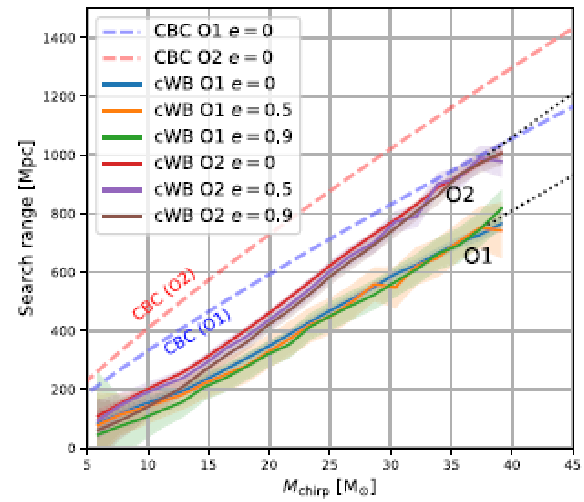
- Mostly, $\boldsymbol{e\sim0}$, e.g., "quasi-circular orbits", is assumed.

- Un-Modelled search using data from O1 & O2: Abbott+ 2019

"Search for Eccentric Binary Black Hole Mergers with Advanced LIGO and Advanced Virgo during Their First and Second Observing Runs"

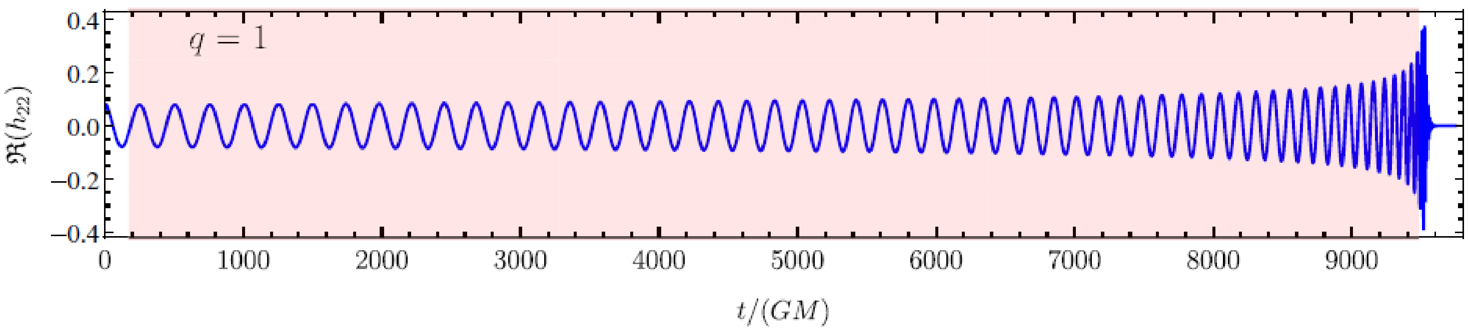


- With the cWB (coherent WaveBurst) algorithm for $(5-50)\ M_{\odot}$ and $e \in [0, 0.99]$



➔ Eccentric binary formation channels with rates $\gtrsim 100\ Gpc^{-3}\ yr^{-1}$ for $e > 0.1$ are ruled out, assuming a black hole mass spectrum with a power-law index $\lesssim 2$.

- Waveform with circular orbit assumption:



PN approximation

NR simulation

Perturbation theory

Takes a long time, e.g., ~2 days, and computationally expensive!

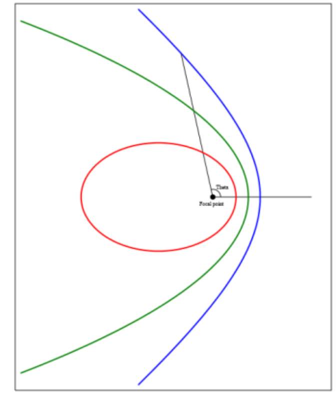
➔ Effective One Body model: EOB Hamiltonian

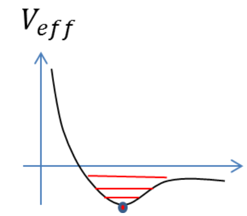
➔ IMR waveforms for $e = 0$ with spin: SEOBNR pipeline

- In general,

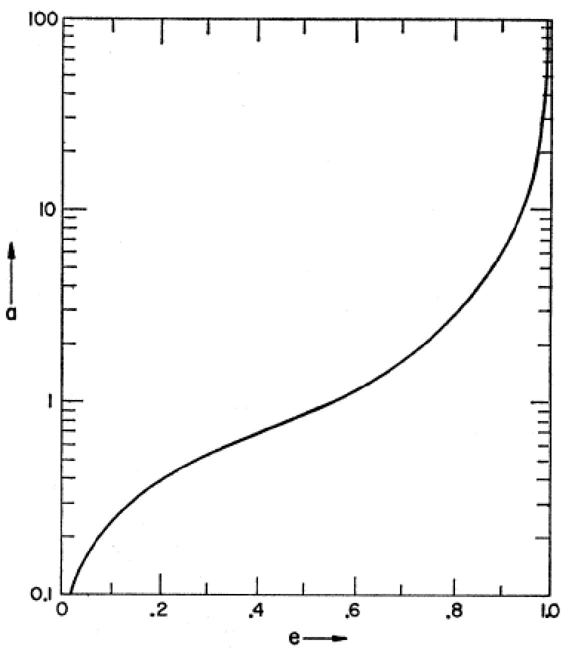
(E, L)

$\boldsymbol{e}$: **0~1**

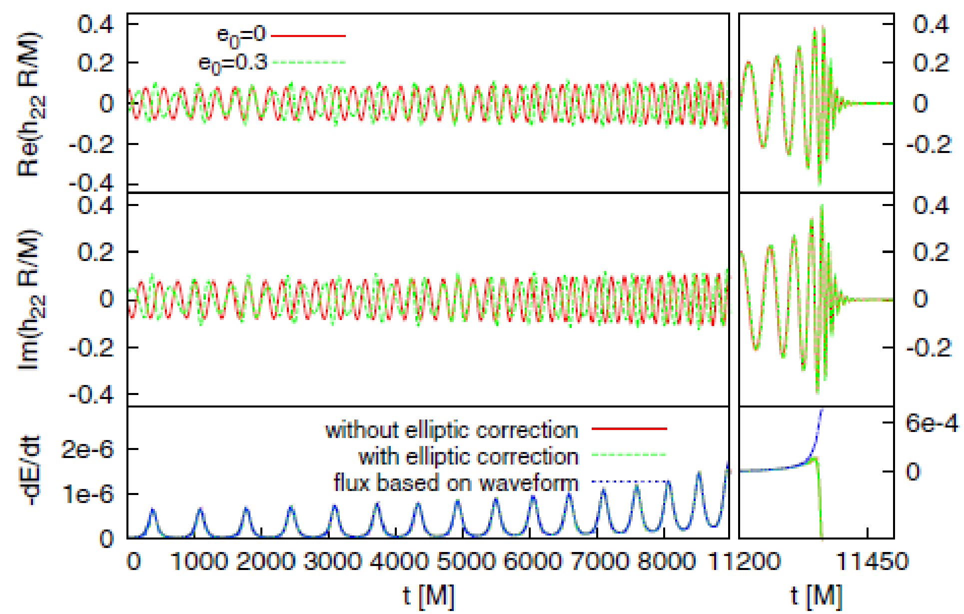




- "Circularization":

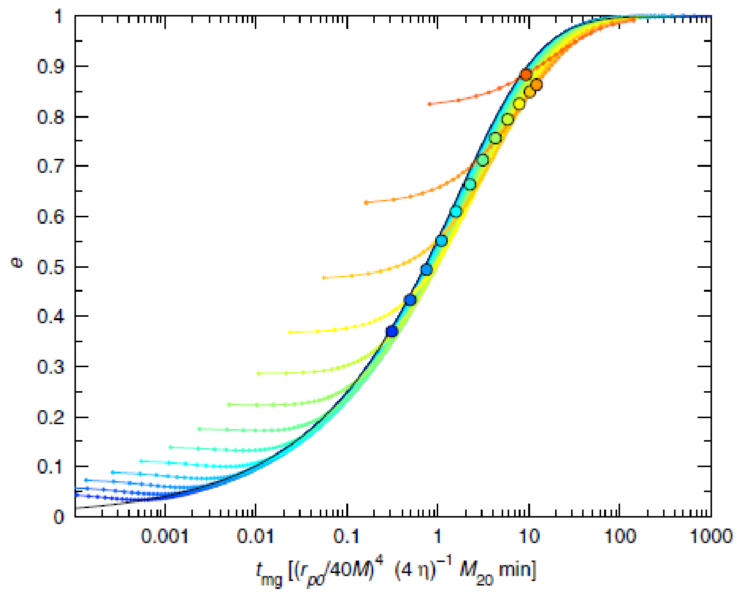


Peters (1964): Evolution of eccentricity ('2.5'PN)

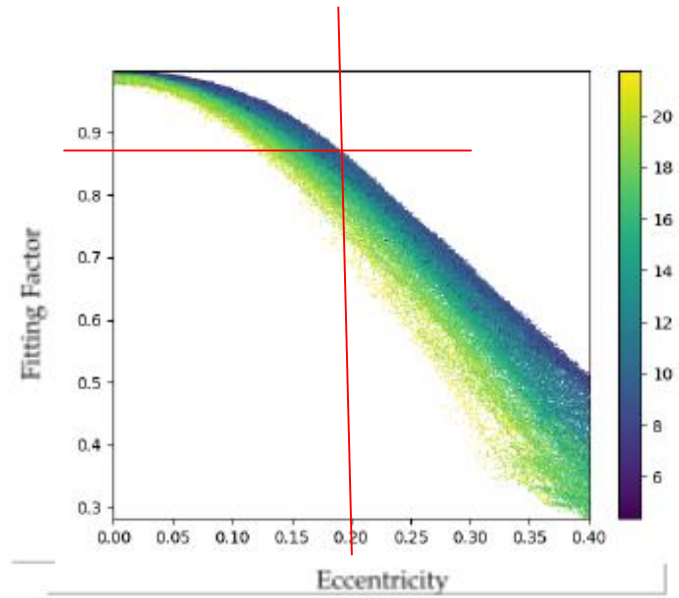


Cao & Han (2017)

- Higher-order (3.5) PN calculation: Kocsis & Levin ('12)



- Fitting Factor: Tiwari *et al*



- BBH waveforms with eccentricity for inspiral phase only:

  ➔ TaylorF2ecc (e: 0.0001~0.2) by C. Kim, J. Kim, H. Lee +, ...

- Including the plunge phase??: "EOB Hamiltonian with eccentricity"

✓ Develop an EOB Hamiltonian for General Eccentricity

- IMR waveforms with eccentricity:
  - Cao & Han 2017; Hinderer & Babak 2017; Hinder et al. 2018; Huerta et al. 2018; Ireland et al. 2019

- Ex) Cao & Han 2017: works up to $e \sim 0.2$ with overlap factor $\gtrsim 0.98$, compared to NR simulations

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}}, \qquad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} + \vec{\mathcal{F}}.$$

$$H = M\sqrt{1 + 2\eta\left(\frac{H_{\rm eff}}{M\eta} - 1\right)},$$

- Conservative part
- Same as SEOBNR's

$$\vec{\mathcal{F}} = \frac{1}{M\eta\omega_\Phi|\vec{r}\times\vec{p}|}\frac{dE}{dt}\vec{p}, \qquad -\frac{dE}{dt} = \frac{1}{16\pi}\sum_{\ell}\sum_{m=-\ell}^{\ell}|\dot{h}_{\ell m}|^2.$$

l=2, 3, …, 8

$$h_{22}^{\rm insp-plun} = h_{22}^{(C)} + h_{22}^{(PNE)}, \qquad h_{22}^{(PNE)} = h_{22} - h_{22}|_{\dot{r}=0},$$

$h_{22}^{(C)}$: for a circular orbit; h_{22}: circular + eccentricity

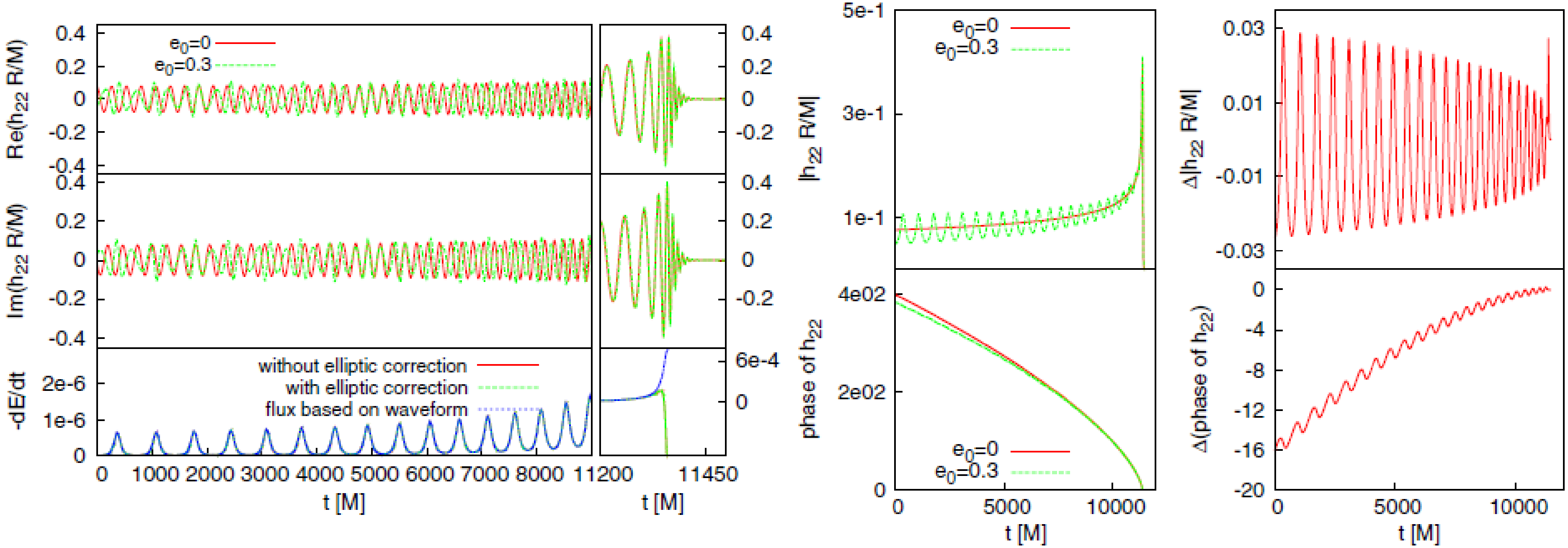
$$h_{\ell m}^{\rm merger-RD} = \sum_{n=0}^{N-1} A_{\ell mn} e^{-i\sigma_{\ell mn}(t - t_{\rm match}^{\ell m})}$$

$$M_{\rm final} = M[1 + 4(m^0 - 1)\eta + 16m^1\eta^2(\chi_1 + \chi_2)],$$

$$\chi_{\rm final} \equiv \frac{a_{\rm final}}{M_{\rm final}} = \chi^0 + \eta\chi^0(t_4\chi^0 + t_5\eta + t_0) + \eta(2\sqrt{3} + t_2\eta + t_3\eta^2),$$

SEOBNRE (Spinning EOBNR Eccentric)

SEOBNRv1 vs SEOBNRE

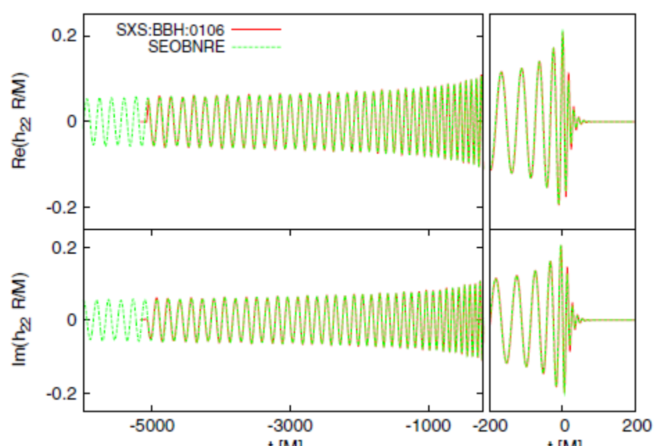


$$\mathcal{O}(h,s) \equiv \frac{\langle h|s\rangle}{\sqrt{\langle h|h\rangle\langle s|s\rangle}}, \qquad \langle h|s\rangle \equiv 4\mathrm{Re}\int_{f_{\min}}^{f_{\max}} \frac{\tilde{h}(f)\tilde{s}^*(f)}{S_n(f)}df$$

$e_0 = 0.03 \quad \mathcal{O} = 0.99300.$

$e_0 = 0.3 \quad \mathcal{O} = 0.46942.$

NR (SXS) vs SEOBNRE

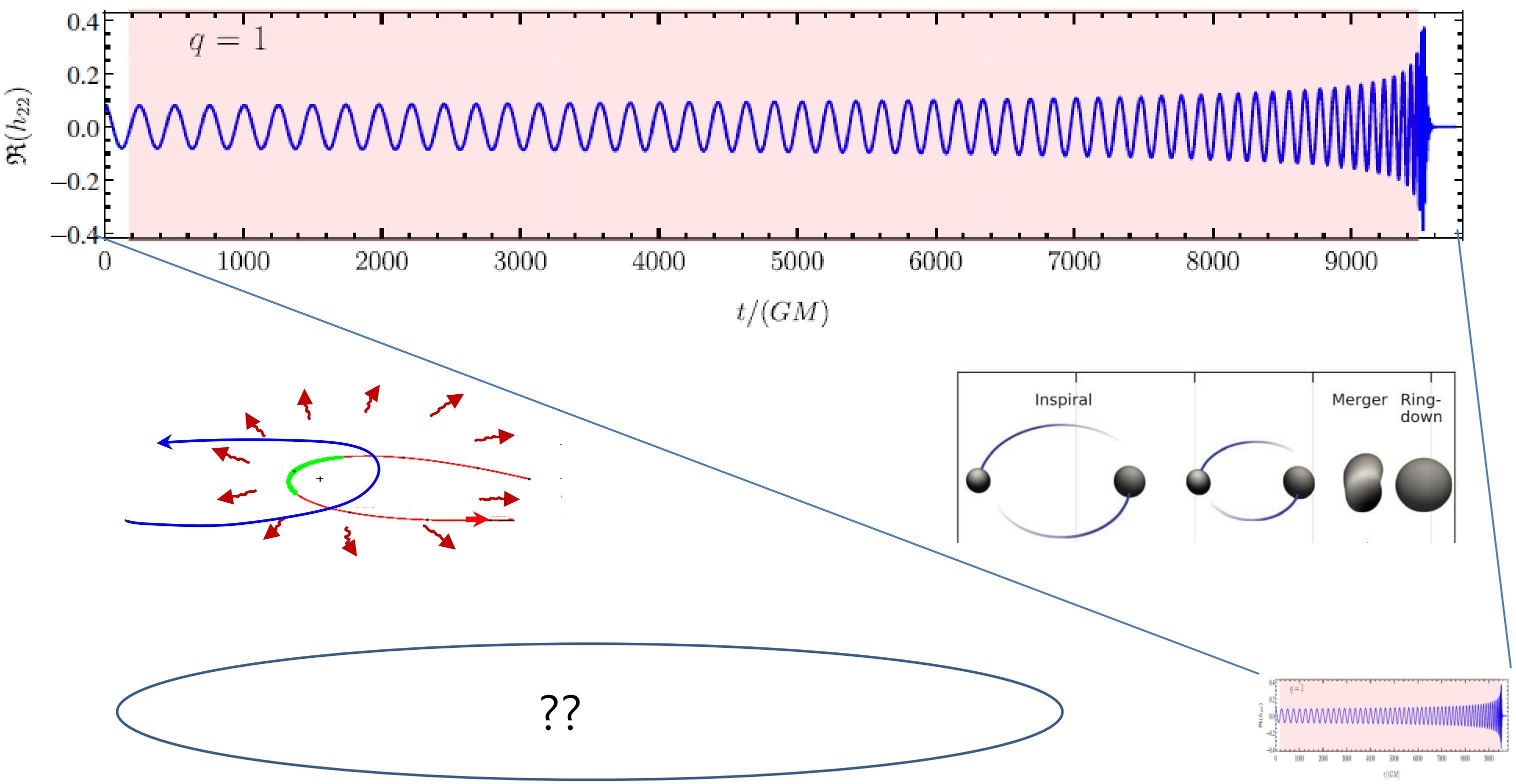


$e_0 = 0.1 \quad \mathcal{O} = 0.99739.$

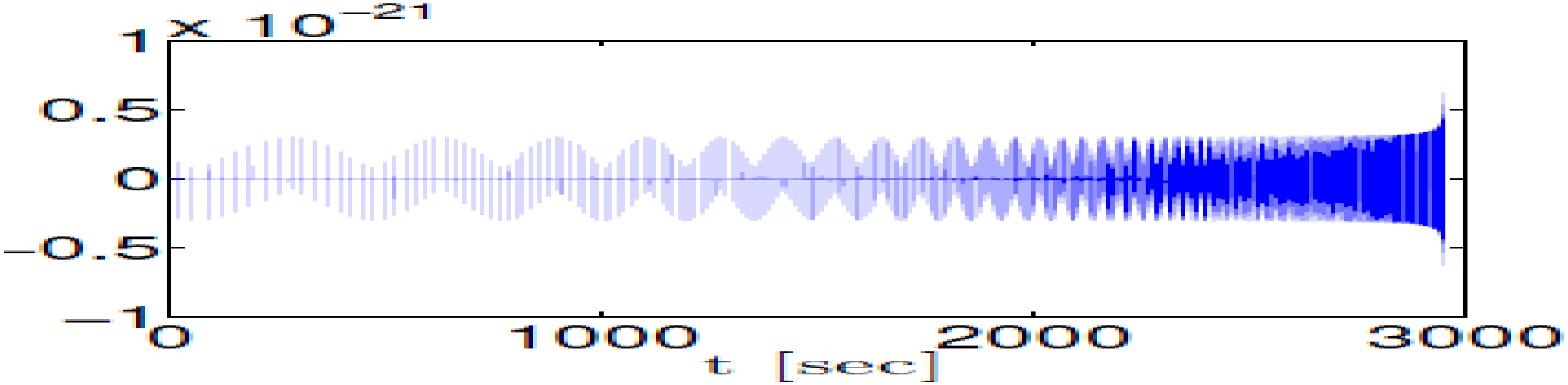
$e_0 = 0.3 \quad \mathcal{O} = 0.98171.$

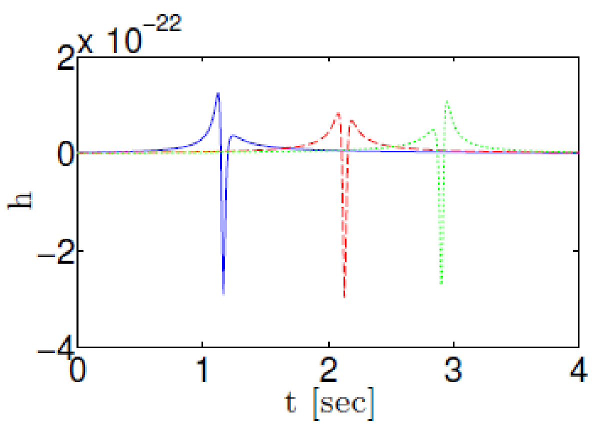
SEOBNRE is good up to $e = 0.2$

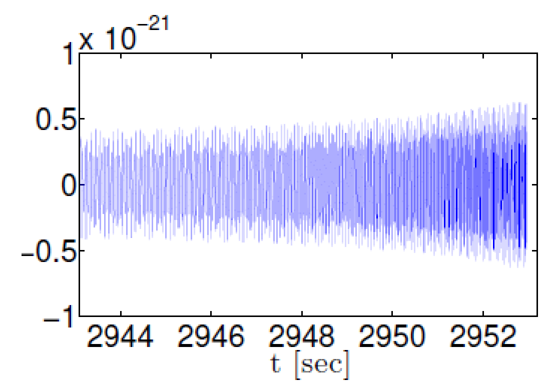
- EOB Hamiltonian which covers $\mathbf{0.2 < e < 1}$?
- **What about even for $e > 1.0$?**



- Levin, McWilliams & Contreras ('11): 3.5PN





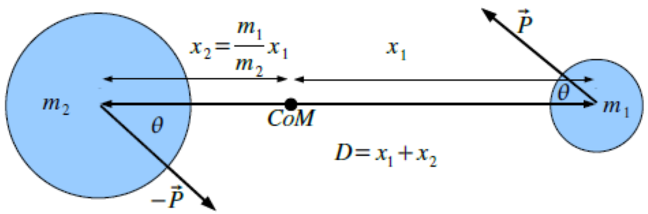


Repeated Burst (**RB**) phase

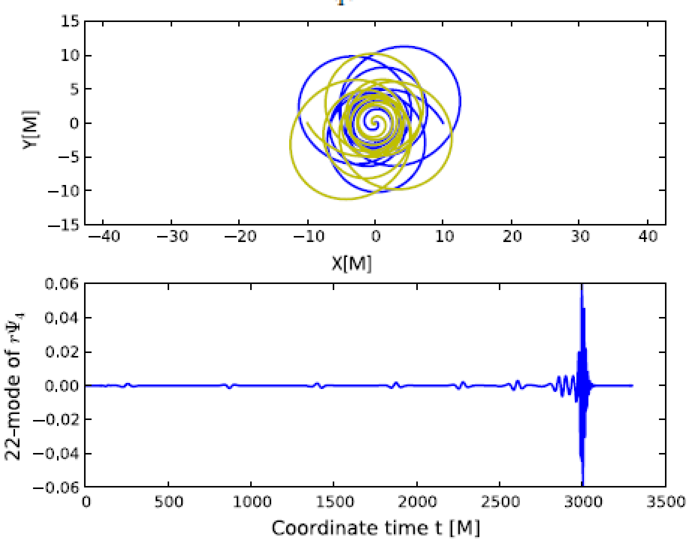
Inspiral phase

~Thousands

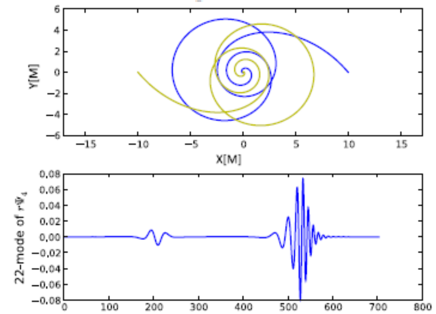
- **Gold & Brugmann ('13):**



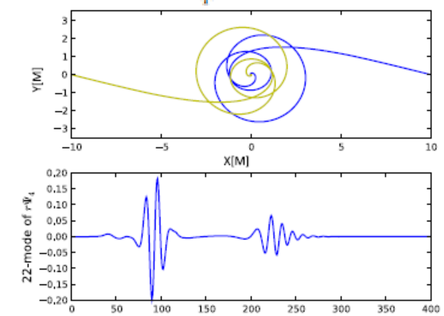
$P = P_{qc}, \ \Theta = 60°$



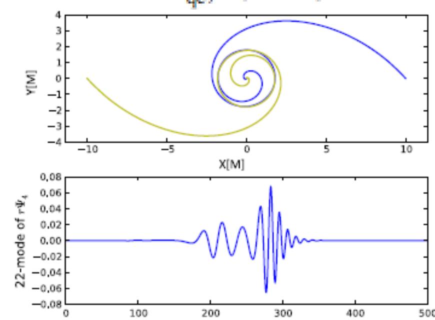
$P = P_{qc}, \ \Theta = 50°$:



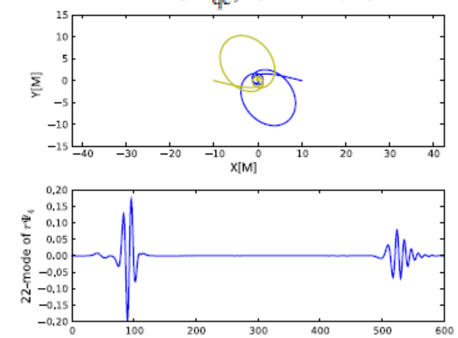
$P = 5P_{qc}, \ \Theta = 14.15°$



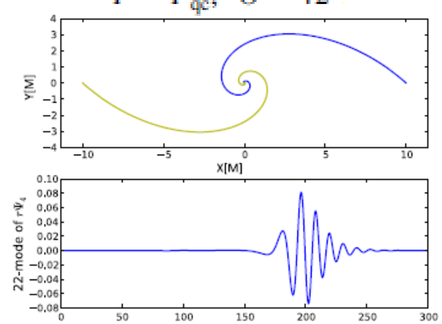
$P = P_{qc}, \ \Theta = 48°$



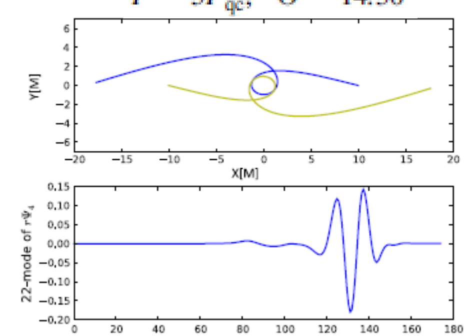
$P = 5P_{qc}, \ \Theta = 14.20°$:



$P = P_{qc}, \ \Theta = 42°$

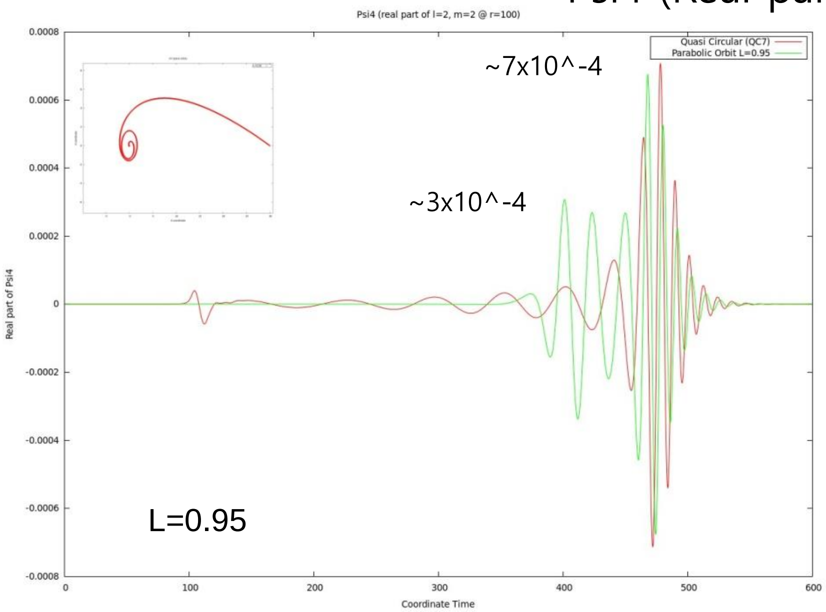


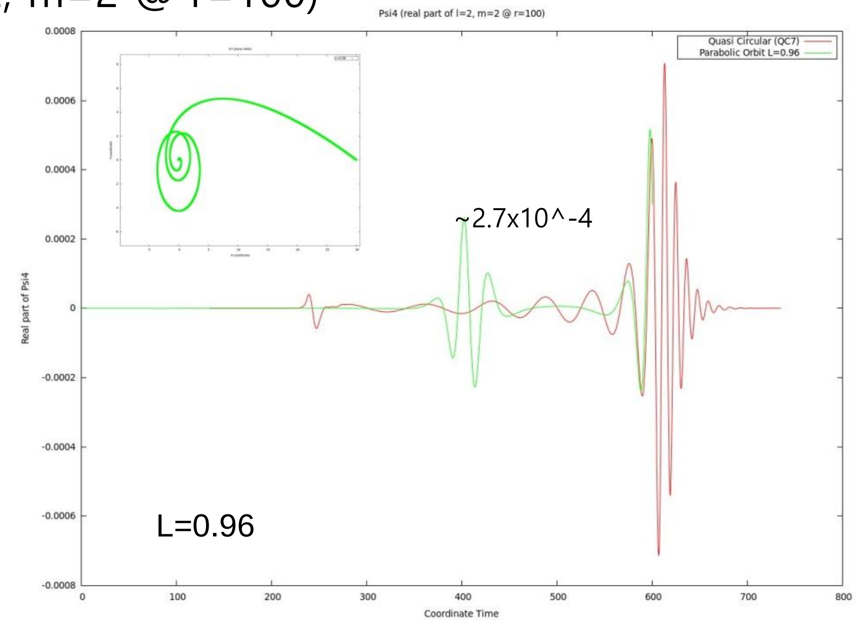
$P = 5P_{qc}, \ \Theta = 14.30°$

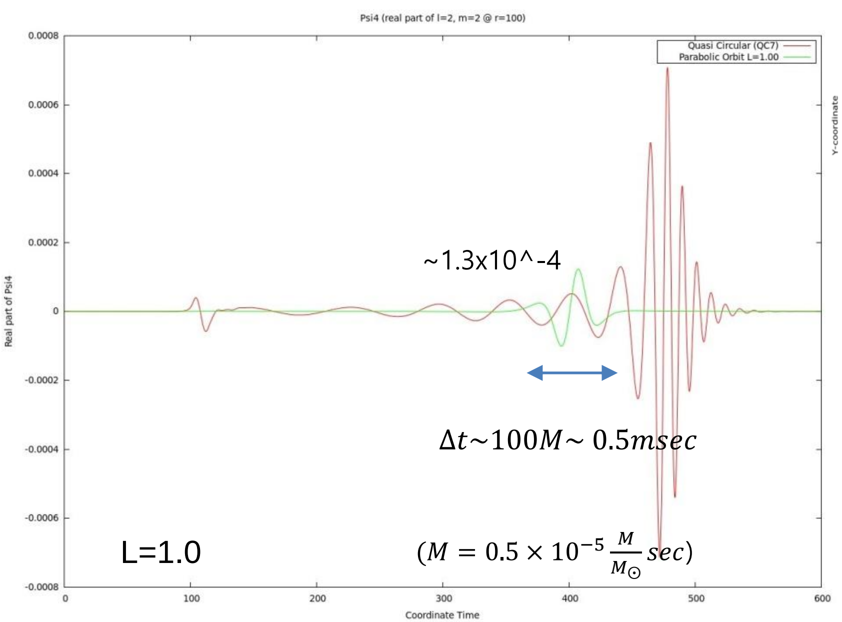


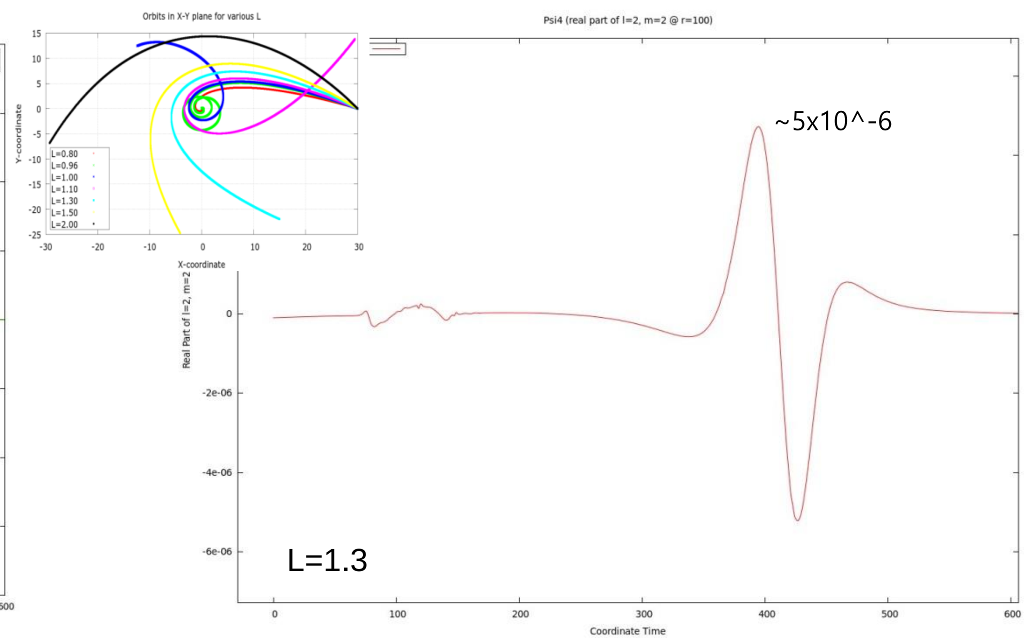
- How weak?: w/ J. Hansen, P. Diener, F. Loefler & H. Kim ('13)

Psi4 (Real part of l=2, m=2 @ r=100)





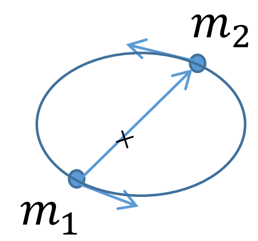




We would like to construct an EOB Hamiltonian with an arbitrary eccentricity which is accurate enough for GW detection analyses.

II. An EOB Hamiltonian for a system of two BHs in general

$$H^{\rm EOB} = M\sqrt{1+2\nu\left(\frac{H^{\rm eff}}{\mu}-1\right)}.$$



For $\nu = \frac{m_1 m_2}{(m_1+m_2)^2} \to 0$, i.e., $m_1 \gg m_2$, we know the answer

$$H_{\rm S}^2 = \left(1-\frac{2GM}{r}\right)\left[\mu^2 + \frac{L^2}{r^2} + \left(1-\frac{2GM}{r}\right)p_r^2\right]$$

✓ **Antonelli, Buonanno+ 2019:**

Then, the post-Schwazschild EOB Hamiltonian at 3PM would be

$$(\hat{H}^{\rm eff,PS})^2 = \hat{H}_{\rm S}^2 + (1-2u)[u^2 q_{\rm 2PM} + u^3 q_{\rm 3PM} + \mathcal{O}(G^4)] \qquad \hat{H}^{\rm eff} = \frac{H^{\rm eff}}{\mu}, \quad \cdots$$

$$q_{\rm 2PM} = \frac{3}{2}(5\hat{H}_{\rm S}^2-1)\left(1-\frac{1}{\sqrt{1+2\nu(\hat{H}_{\rm S}-1)}}\right), \quad q_{\rm 3PM} = -\frac{2\hat{H}_{\rm S}^2-1}{\hat{H}_{\rm S}^2-1}q_{\rm 2PM} + \frac{4}{3}\nu\hat{H}_{\rm S}\frac{14\hat{H}_{\rm S}^2+25}{1+2\nu(\hat{H}_{\rm S}-1)}$$

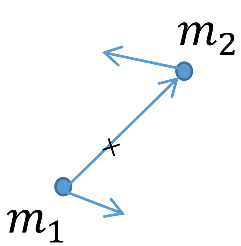
$$+\frac{8\nu}{\sqrt{\hat{H}_{\rm S}^2-1}}\frac{4\hat{H}_{\rm S}^4-12\hat{H}_{\rm S}^2-3}{1+2\nu(\hat{H}_{\rm S}-1)}\sinh^{-1}\sqrt{\frac{\hat{H}_{\rm S}-1}{2}}.$$

Angular momentum: $l \equiv |\boldsymbol{L}|/(GM\mu)$.

✓ 3PM Hamiltonian:

- Recently, Bern et al (PRL, '19) have obtained the Hamiltonian at the third post-Minkowskian (3PM) order describing the scattering amplitude for two massive spinless particles in the context of effective field theory.

$$H(\boldsymbol{p},\boldsymbol{r}) = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + V(\boldsymbol{p},\boldsymbol{r}), \qquad V(\boldsymbol{p},\boldsymbol{r}) = \sum_{i=1}^{\infty} c_i(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)^i,$$



$$c_1 = \frac{\nu^2 m^2}{\gamma^2 \xi}(1-2\sigma^2), \qquad c_2 = \frac{\nu^2 m^3}{\gamma^2 \xi}\left[\frac{3}{4}(1-5\sigma^2) - \frac{4\nu\sigma(1-2\sigma^2)}{\gamma\xi} - \frac{\nu^2(1-\xi)(1-2\sigma^2)^2}{2\gamma^3\xi^2}\right],$$

$$c_3 = \frac{\nu^2 m^4}{\gamma^2 \xi}\left[\frac{1}{12}(3-6\nu+206\nu\sigma-54\sigma^2+108\nu\sigma^2+4\nu\sigma^3) - \frac{4\nu(3+12\sigma^2-4\sigma^4)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\right.$$
$$- \frac{3\nu\gamma(1-2\sigma^2)(1-5\sigma^2)}{2(1+\gamma)(1+\sigma)} - \frac{3\nu\sigma(7-20\sigma^2)}{2\gamma\xi} - \frac{\nu^2(3+8\gamma-3\xi-15\sigma^2-80\gamma\sigma^2+15\xi\sigma^2)(1-2\sigma^2)}{4\gamma^3\xi^2}$$
$$\left. + \frac{2\nu^3(3-4\xi)\sigma(1-2\sigma^2)^2}{\gamma^4\xi^3} + \frac{\nu^4(1-2\xi)(1-2\sigma^2)^3}{2\gamma^6\xi^4}\right],$$

$$E_{1,2} = \sqrt{\boldsymbol{p}^2 + m_{1,2}^2}, \quad m = m_1 + m_2, \quad \nu = m_1 m_2/m^2 \quad E = E_1 + E_2, \quad \xi = E_1 E_2/E^2,$$
$$\gamma = E/m, \quad \sigma = p_1 \cdot p_2 / m_1 m_2.$$

- PN/PM corrected:

$$H^{\mathrm{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H^{\mathrm{eff}}}{\mu} - 1\right)}, \qquad H_{\mathrm{S}}^2 = \left(1 - \frac{2GM}{r}\right)\left[\mu^2 + \frac{L^2}{r^2} + \left(1 - \frac{2GM}{r}\right)p_r^2\right]$$

$$\hat{H}^{\mathrm{eff}} = \frac{H^{\mathrm{eff}}}{\mu}, \qquad \hat{H}_{\mathrm{S}} = \frac{H_{\mathrm{S}}}{\mu}, \qquad u = \frac{GM}{r}, \quad \hat{p}_r = \frac{p_r}{\mu}, \qquad l \equiv \hat{p}_\phi = \frac{L}{GM\mu},$$

$$[\hat{H}^{\mathrm{eff,PS}}(u, \hat{p}_r, l)]^2 = \hat{H}_{\mathrm{S}}^2 + (1 - 2u)\hat{Q}^{\mathrm{PS}}(u, \hat{H}_{\mathrm{S}}, \nu)$$

$$\hat{H}_{\mathrm{S}}^2 = (1 - 2u)[1 + l^2u^2 + (1 - 2u)\hat{p}_r^2]$$

$$\hat{Q}^{\mathrm{PS}} = u^2 q_{\mathrm{2PM}}(\hat{H}_{\mathrm{S}}, \nu) + u^3 q_{\mathrm{3PM}}(\hat{H}_{\mathrm{S}}, \nu) + \Delta_{\mathrm{3PN}}(u, \hat{H}_{\mathrm{S}}, \nu) + \Delta_{\mathrm{4PN}}(u, \hat{H}_{\mathrm{S}}, \nu) + \mathcal{O}(\mathrm{5PN})$$

$$q_{\mathrm{2PM}} = \frac{3}{2}(5\hat{H}_{\mathrm{S}}^2 - 1)\left(1 - \frac{1}{\sqrt{1 + 2\nu(\hat{H}_{\mathrm{S}} - 1)}}\right)$$

$$q_{\mathrm{3PM}} = -\frac{2\hat{H}_{\mathrm{S}}^2 - 1}{\hat{H}_{\mathrm{S}}^2 - 1} q_{\mathrm{2PM}} + \frac{4}{3}\nu\hat{H}_{\mathrm{S}}\frac{14\hat{H}_{\mathrm{S}}^2 + 25}{1 + 2\nu(\hat{H}_{\mathrm{S}} - 1)} + \frac{8\nu}{\sqrt{\hat{H}_{\mathrm{S}}^2 - 1}}\frac{4\hat{H}_{\mathrm{S}}^4 - 12\hat{H}_{\mathrm{S}}^2 - 3}{1 + 2\nu(\hat{H}_{\mathrm{S}} - 1)}\sinh^{-1}\sqrt{\frac{\hat{H}_{\mathrm{S}} - 1}{2}}$$

$$\Delta_{\mathrm{3PN}} = \left(\frac{175}{3}\nu - \frac{41\pi^2}{32}\nu - \frac{7}{2}\nu^2\right)u^4$$

$$\Delta_{\mathrm{4PN}} = \sum_{n=2}^{5}\alpha_{4n}u^n(\hat{H}_{\mathrm{S}}^2 - 1)^{5-n} + (\alpha_{44,\ln}u^4(\hat{H}_{\mathrm{S}}^2 - 1) + \alpha_{45,\ln}u^5)\ln u$$

$$\alpha_{42} = \left(-\frac{1027}{12} - \frac{147432}{5}\ln 2 + \frac{1399437}{160}\ln 3 + \frac{1953125}{288}\ln 5\right)\nu,$$

$$\alpha_{43} = \left(-\frac{78917}{300} - \frac{14099512}{225}\ln 2 + \frac{14336271}{800}\ln 3 + \frac{4296875}{288}\ln 5\right)\nu,$$

$$\alpha_{44} = \left(-\frac{43807}{225} + \frac{296\gamma_E}{15} - \frac{33601\pi^2}{6144} - \frac{9771016}{225}\ln 2 + \frac{1182681}{100}\ln 3 + \frac{390625}{36}\ln 5\right)\nu + \left(-\frac{405}{4} + \frac{123}{54}\pi^2\right)\nu^2 + \frac{13}{2}\nu^3,$$

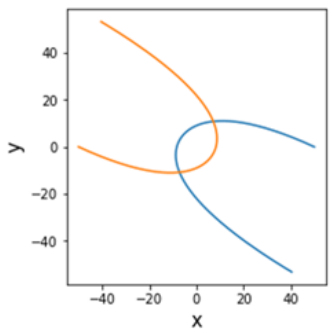
$$\alpha_{45} = \left(-\frac{34499}{1800} + \frac{136}{3}\gamma_E - \frac{29917}{6144}\pi^2 - \frac{254936}{25}\ln 2 + \frac{1061181}{400}\ln 3 + \frac{390625}{144}\ln 5\right)\nu + \left(-\frac{2387}{24} + \frac{205}{64}\pi^2\right)\nu^2 + \frac{9}{4}\nu^3,$$

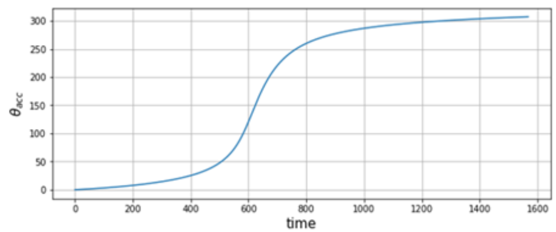
and

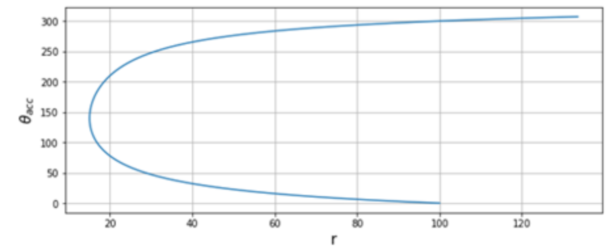
$$\alpha_{44,\ln} = \frac{148}{15}\nu, \qquad \alpha_{45,\ln} = \frac{68}{3}\nu.$$

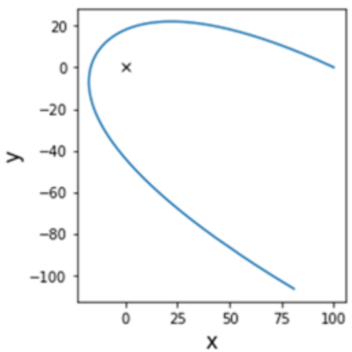
$\gamma_E = 0.57721...$ is the Euler-Mascheroni constant.

- **Scattering angles:** NR vs Newtonian vs EOB vs 3PM









$$H(r, p_r, L) = E$$

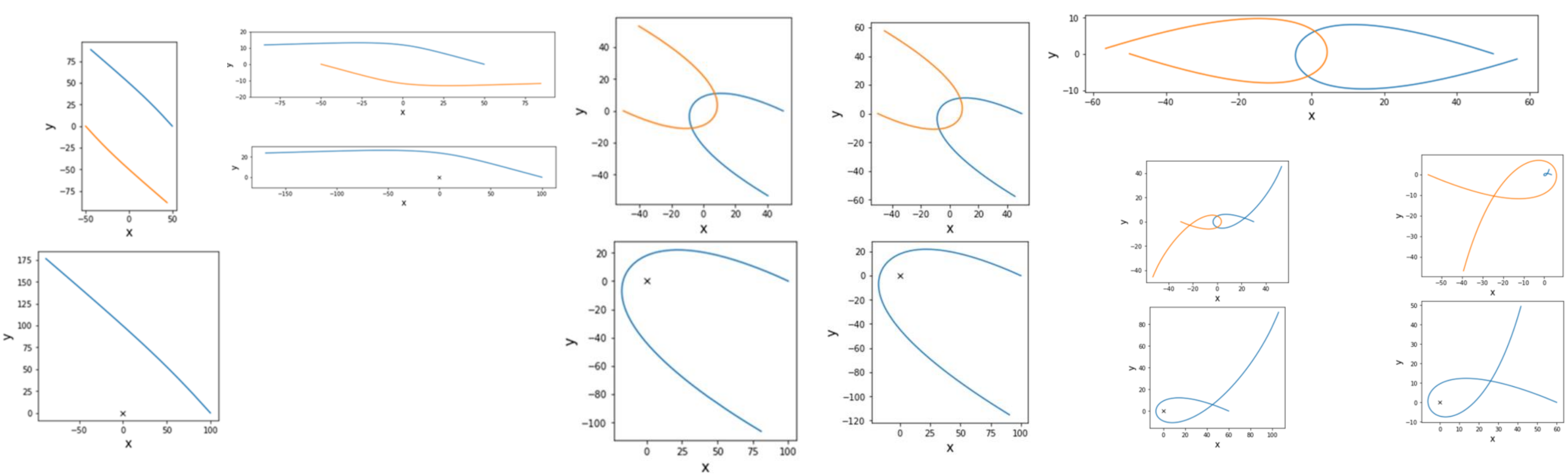
$$p_r(r, E, L)$$

$$\left(\frac{\partial H}{\partial L}\right)\left(r, p_r, \frac{\partial p_r}{\partial L}, L\right) = 0$$

$$\Delta\phi = \pi + \chi(E, L) = -2\int_{r_{\min}}^{\infty} dr \frac{\partial}{\partial L} p_r(r, E, L)$$

- Results:

Case: $\vec{P} = (P_x, P_y)$	NR	Newtonian	3PM	EOB	$\Delta\chi/\chi$ (%)
(0.090, 0.099)	90.13	87.97	90.34	89.78	-0.388
$(0.214, 0.058)$	165.11°	155.00°	165.73°	165.30	0.115
$(0.0326, 0.015)$	300.11°	258.51	299.98	300.52	0.137
(0.0331, 0.015)	299.08	257.63	299.12	299.67	0.197
(0.0376, 0.012)	356.05	273.65	370.20	359.91	1.084
(0.043, 0.0185)	376.41	262.64	359.44	384.72	2.208
(0.00957,0.004) (q=16)	407.92	265.04	429.96	405.05	-0.704



III. Discussion

- Motivations for constructing the EOB Hamiltonian with an arbitrary eccentricity are introduced.

- Comparison with NR results are shown for the scattering angle.

- Try PN corrected EOB Hamiltonian with eccentricity:

  G. Cho, A. Gopakumar, M. Haney & H. Lee ('18); 3PN for hyperbolic orbits

THANKS!

$$e = \sqrt{1 + 2EL^2/(G^2\mu {m_1}^2 {m_2}^2)}$$

$$L = b \times \mu v_\infty = b\sqrt{2\mu E}$$