Phase diagram of QCD-like matter from upgraded PNJL model

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Two phases predicted for QCD matter:

- **Hadronic phase:**
  Quarks and gluons are bound into hadrons: **confinement**
  This is hadronic matter, we can observe it experimentally

- **QGP phase:**
  Quarks and gluons are free in the medium
  We don’t directly observe this phase experimentally
**QGP and phase diagram**

- **Equation of state**
- **QGP and phase diagram**

- **Chemical Potential**
  - *T (GeV)*
  - *μ (GeV/fm³)*

- **Critical End Point?**
  - *T_c ≈ 0.15*

- **LHC**, **RHIC**, **SPS, GSI**, **CFL**, **Color Superconductor?**

- **Hadronic Gas**

- **Neutron Star**

*Figure – Phase Diagram of nuclear matter*
QCD lagrangian: life is tough

\[ \mathcal{L}_{QCD} = i \delta_{ij} \bar{\psi}_i^j \gamma^{\mu} \partial_{\mu} \psi_k^j + g_s \bar{\psi}_i^j \gamma^{\mu} \lambda_{ij}^a A_{\mu}^a \psi_k^j - m_k \bar{\psi}_k^i \psi_k^j - \frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} \]

Perturbative approach pQCD

- Need of a small coupling constant for convergence of the perturbative series, works at high energy/high T, \( \mu \).
- Not working at phase transition, the coupling constant is large.

Lattice approach lQCD

- Space-time discretized on a lattice. Matter on the node, gluons are the lines connecting the nodes.
- Static study, no dynamics on lattice, only thermodynamics.
- Does not work at finite chemical potential, only at finite temperature.
A. Schmitt from ect* summer school lectures

- **data**
  - heavy-ion collisions: \( T \gg \mu \)
    (see however FAIR, NICA)
  - compact stars: \( T \ll \mu \)
Introduction

(P)NJL models

Equation of state

QGP and phase diagram

- GSI, FAIR
- NICA
- BES program (RHIC)
- SPS (CERN)

Lower temperature and higher density: search for critical end point, phase transitions and neutron star physics.

Needs prediction to know where to search. Those predictions can only be made using effective models
(P)NJL Model
Effective model

Works only in a special domain of energy but allows finite chemical potential studies.

Contact interaction

Static approximation: no gluons propagating the interaction

\[ \frac{1}{p^2 - \epsilon_g^2} = -\frac{1}{\epsilon_g^2} \]

if \( p \ll \epsilon_g^2 \)
Nambu-Jona-Lasinio (NJL) Lagrangian

\[ \mathcal{L}_{NJL} = \delta_{ij} \bar{\psi}_k^i (i \gamma^\mu \partial_\mu - m) \psi_k^j + G (\bar{\psi}_k^i \lambda_{ij} \psi_k^j)^2 + \text{'t Hooft term} \]

Symmetries

- Chiral symmetry $SU_L(3) \otimes SU_R(3)$
- Color symmetry $SU_c(3)$ (but global)
- Flavour symmetry $SU_f(3)$

Free parameters

- $m_q^0 = 0.0055 \text{GeV}$
- $m_s^0 = 0.134 \text{GeV}$
- $\Lambda = 0.569 \text{GeV}$
- $G = \frac{2.3}{\Lambda^2} \text{GeV}^{-2}$
- $K = \frac{11}{\Lambda^5} \text{GeV}^{-5}$

Problem

- Center symmetry is missing
- Confinement is not described
Polyakov loop

Confinement = effective potential $U(\phi, \bar{\phi}, T)$, $\phi$ the Polyakov loop.

PNJL = Frozen gluons + Thermal gluons.

Polyakov extended NJL Lagrangian

$$\mathcal{L}_{PNJL} = \overline{\psi}_k (i\gamma_0 - m) \psi_k + G (\overline{\psi}_k \lambda_i \psi_k)^2 + '{t Hooft - U(\phi, \bar{\phi}, T)$$
$U(\phi, \bar{\phi}, T)$:

- mean field in which quarks propagate, gives a pressure to the medium
- It corresponds to the thermodynamics of the $\frac{1}{4} F_{\mu\nu}^a F_{a\mu\nu}$ term in the QCD lagrangian.
- The parameters are determined by fitting with the $P_{YM}$ of lQCD.

$$\frac{U(\phi, \bar{\phi}, T)}{T^4} = -\frac{b_2(T)}{2} \bar{\phi}\phi - \frac{b_3}{6}(\bar{\phi}^3 + \phi^3) + \frac{b_4}{4}(\bar{\phi}\phi)^2$$

with the parameters: $b_2(T) = a_0 + a_1(\frac{T_0}{T}) + a_2(\frac{T_0}{T})^2 + a_3(\frac{T_0}{T})^3$

<table>
<thead>
<tr>
<th>$a_0$</th>
<th>$a_1$</th>
<th>$a_2$</th>
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From quarks to hadrons: mesons
Quark-antiquark bound states

In NJL, degrees of freedom are quarks. Mesons need to be built from quark-antiquarks bound states.

\[
iU(k^2) = \Gamma \frac{-ig_m^2}{k^2 - m^2} \Gamma
\]

Mesons masses

The mass is given by the poles: \( m = k \)
Bethe-Salpeter equation

\[ iU(k^2) = \Gamma \frac{2ig_m}{1-2g_m\Pi(k^2)} \Gamma \]

\[ \text{Mesons masses} \]

By analogy, the mass is given by the poles:

\[ 1 - 2G\Pi(k_0 = m, \vec{k} = 0) = 0 \]
Limitations of the model

**Good things**

- ✓ Lagrangian which quite shares the symmetries of the QCD lagrangian
- ✓ Works at finite density and in the phase transition region
- ✓ Degrees of freedom = quarks but hadronic matter made from bound states

**Bad things**

- Dynamical gluons do not participate in the interaction : low energy approximation.
- 4-point interactions are non renormalizable : need of a cut-off.
Equation of State
As always in statistical physics, we need the partition function:

\[
Z[\bar{q}, q] = \int D\bar{q} Dq \left\{ \int_0^\beta d\tau \int_V d^3x L_{NJL} \right\}
\]

Using the bosonisation procedure, we obtain the mean field partition function:

\[
Z[\bar{q}, q] = \exp \left\{ - \int_0^\beta d\tau \int_V \frac{\sigma_{MF}^2}{4G} + Tr \ln S_{MF}^{-1} \right\}
\]

\[
\Omega_{NJL}(T, \mu) = -\frac{T}{V} \ln Z[\bar{q}, q]
\]
NJL grand potential

\[ \Omega_{NJL} = -2 \int_0^\Lambda \frac{d^3 p}{(2\pi)^3} E_p \]

\[ +2T \int_0^\infty (\ln[1 + \exp(-\beta(E_p - \mu))] + \ln[1 + \exp(-\beta(E_p + \mu))] \]

\[ + 2G \sum_k < \bar{\psi}_k \psi_k >^2 - 4K \Pi_i < \bar{\psi}_k \psi_j > ) \]

PNJL grand potential

\[ \Omega_{PNJL} = -2 \int_0^\Lambda \frac{d^3 p}{(2\pi)^3} E_p \]

\[ +2T \int_0^\infty (\ln[1 + L^+ \exp(-\beta(E_p - \mu))] + \ln[1 + L \exp(-\beta(E_p + \mu))] \]

\[ + 2G \sum_k < \bar{\psi}_k \psi_k >^2 - 4K \Pi_i < \bar{\psi}_k \psi_j > + U_{PNJL} \]
Figure from 3rd student Laurence Pied
\[ (P)NJL \text{ models} \]

Equation of state

\[ \frac{1}{N_c} \text{ expansion} \]

\[ '{t \ Hooft \ scaling :} \quad g \bar{\psi} A_\mu \psi \rightarrow g N_c \bar{\psi} \frac{A_\mu}{N_c} \psi \quad \text{with} \quad g N_c = \text{cst} \]

\[ g^{2l} N_c^k \equiv (g N_c)^{2l} N_c^{k-2l} \]

$k$ is the number of fermion lines and $l$ is the number of interaction lines.

We go beyond mean field approximation (orange, red) in the $N_C$ expansion.

\[ iS_\Sigma(p) = iS(p)(O(1)O(N_c)) + O((g N_c)^2)O(1) + \]
\[ \text{O}(g N_c^2)O\left(\frac{1}{N_c}\right) + O((g N_c)^4)O\left(\frac{1}{N_c}\right) + \ldots \]

\[ \sum \]

\[ = \begin{array}{c} \text{green} \end{array} + \begin{array}{c} \text{orange} \end{array} + \begin{array}{c} \text{red dashed} \end{array} + \begin{array}{c} \text{purple} \end{array} + \ldots \]
Mesonic grand potential

\[ \Omega_M = -\frac{g_M}{8\pi^3} \int dp^2 \int \frac{ds}{\sqrt{s+p^2}} \left[ \frac{1}{\exp(\beta(\sqrt{s+p^2}-\mu)-1)} + \frac{1}{\exp(\beta(\sqrt{s+p^2}+\mu)-1)} \right] \delta_M \]

Phase shift: the physics

The phase shift depends on the mesons masses

\[ \delta_M = -\arg[1 - 2K_M \Pi_M] \]
Traditional PNJL - Before

One of the parameter is $T_0 = 270$ MeV, the critical temperature for confinement. This is the pure Yang-Mills critical temperature.

Quarks are here too! - Better

Shift in the critical temperature if we gluons can split into $q$-$\bar{q}$ pairs. We use the reduced temperature to quantify it.

$$T_{\text{eff}} = \frac{T - T_c}{T_c} \rightarrow T_{\text{YM}}^{\text{eff}} \simeq 0.57 T_{rs}^{\text{eff}}$$

This rescales the critical temperature to $T_0 = 190$ MeV

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Different quark-gluon interaction

We include a temperature dependence in the rescaling:

\[ \tau = 0.57 \frac{T - T_0(T)}{T_0(T)} \]

where: \( T_0 = a + bT + cT^2 + dT^3 + e\frac{1}{T} \)

and: \( b_2(T) = a_0 + \frac{a_1}{1+\tau} + \frac{a_2}{(1+\tau)^2} + \frac{a_3}{(1+\tau)^3} \)

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<td>0.082</td>
<td>0.36</td>
<td>0.72</td>
<td>-1.6</td>
<td>-0.0002</td>
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DF, T Steinert, J. Aichelin arxiv 1908.08122
We reproduce lattice results at 0 $\mu$

We have an effective model based on a lagrangian that shares QCD symmetry and match lattice results. This is an effective theory, no sign problem, we can expand to finite chemical potential.
**Mesonic contributions to the pressure**

As expected, Mesons have significant contribution at low temperature.

**Critical temperature**

Minimum of speed of sound: localisation of the cross over region.

*DF, T Steinert, J. Aichelin arxiv 1908.08122*
Lattice at finite $\mu$

Lattice can perform Taylor expansion around zero chemical potential.

$$\frac{T_c(\mu_B)}{T_c(0)} = 1 - \kappa \left( \frac{\mu_B}{T_c(\mu_B)} \right)^2 + \ldots$$

"On the critical line of 2+1 flavor QCD" Cea, Cosmai, Papa

The $\kappa$ coefficient is the second order derivative of our function:

$$\kappa = -T_c(0) \left. \frac{\partial T_c(\mu_B)}{\partial \mu_B^2} \right|_{\mu_B=0}$$

"On the critical line of 2+1 flavor QCD" Cea, Cosmai, Papa

Our critical temperature

At $\mu_B = 0$, we get the critical temperature:

$$T_c = 146 \text{MeV}$$

DF, T Steinert, J. Aichelin arxiv 1908.08122

Large $\mu$ comparison

- Match pQCD predictions at large $\mu$
To determine the critical chemical potential, we first calculate the two solutions for bare and dressed quarks mass. Region with three solutions, meaning that we have a first order transition.

Figure from 4th student Fabien Mathieu
To determine precisely the value of $\mu_{\text{crit}}$, we use the same process but for the grand potential.

![Graph showing the relationship between pressure (P) and quark chemical potential ($\mu_q$) with two lines representing dressed mass and bare mass solutions. The value obtained is 0.425 GeV for $T=0$.]

**Critical chemical potential**

The value obtained is 0.425 GeV for $T=0$. 
Critical End Point

\[ g_u(\mu, T, mq, ms, \phi, \bar{\phi}) = 0 \quad g_s(\mu, T, mq, ms, \phi, \bar{\phi}) = 0 \]
\[ \frac{\partial \Omega_{\text{PNJL}}(\mu, T, mq, ms, \phi, \bar{\phi})}{\partial \phi} = 0 \quad \frac{\partial \Omega_{\text{PNJL}}(\mu, T, mq, ms, \phi, \bar{\phi})}{\partial \bar{\phi}} = 0 \]
\[ \frac{\partial \mu}{\partial mq} = 0 \quad \frac{\partial^2 \mu}{\partial mq^2} = 0 \]

The solution obtained has the coordinates:
\( (T_{\text{CEP}} = 0.11 \text{ GeV}, \mu^\text{CEP}_q = 0.32 \text{ GeV}) \).


DF, T Steinert, J.Aichelin arxiv 1908.08122
At finite $\mu$

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DF, T Steinert, J. Aichelin arxiv 1908.08122
Conclusion:

- PNJL : effective model to study the phase diagram at finite $\mu$.

\[
\text{PNJL} + T_0(T) + \text{Pressure beyond mean field (mesons)}
\]

- Lattice equation of state at $\mu = 0$.
- Lattice equation of state at $\mu \simeq 0$.
- PQCD results for pressure at large $\mu$
- Cross over transition for $T$ (speed of sound, $T_{\text{mott}}$)
- First order transition localized at $\mu = 0.425$ GeV at $T = 0$
- Critical End Point coordinates:
  \[
  (T_{\text{CEP}} = 0.11 \text{ GeV}, \mu_q^{\text{CEP}} = 0.32 \text{ GeV})
  \]
- Phase diagram of QCD matter
Thank you for your attention!!
Appendix

Sign problem

- Partition function: \( Z = \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp(-S) \)
- With the action:
  \[ S = \int d^4x \bar{\psi}(\gamma_v (\partial_v + iA_v) + \mu \gamma_4 + m)\psi = \int d^4x \bar{\psi} M\psi \]
- \( \mu \) appears as an \( A_4 \) imaginary quadrivector and:
  \[ M = \gamma_v \partial_v + i\gamma_v A_v + \mu \gamma_4 + m \]
- We then have:
  \[ M^\dagger(\mu) = M(-\mu^*) \]
- The action is now complex. It can be seen using the hermiticity of the \( \gamma_5 \) matrix. \( M \) hermiticity valid at \( \mu = 0 \) and but not for finite \( \mu \).
$U_A(1)$ anomaly

- Classical action invariant $\rightarrow$ symmetry.

- Quantum action not invariant $\rightarrow$ symmetry broken.

- Symmetry broken by quantum fluctuation : Anomalies!
S matrix

\[ S(p, E) = \exp(2i\delta(\vec{p}, E)) = \frac{F_J(\vec{k}, E^*)}{F_J(k, E)} \]

The zeroes of the Jost function are the poles of the S-matrix.

- S-matrix has a pole at \( k = +i\kappa \): Bound states have exponentially decaying solutions.
- Poles in the lower half plane can be written as \( k = -i\kappa + \gamma \)
  - \( \gamma = 0 \), resonances
  - \( \gamma = 0 \), antibound or virtual states.