2- and 3-pion Coulomb interactions from perturbative QED

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Zimanyi winter workshop 2019

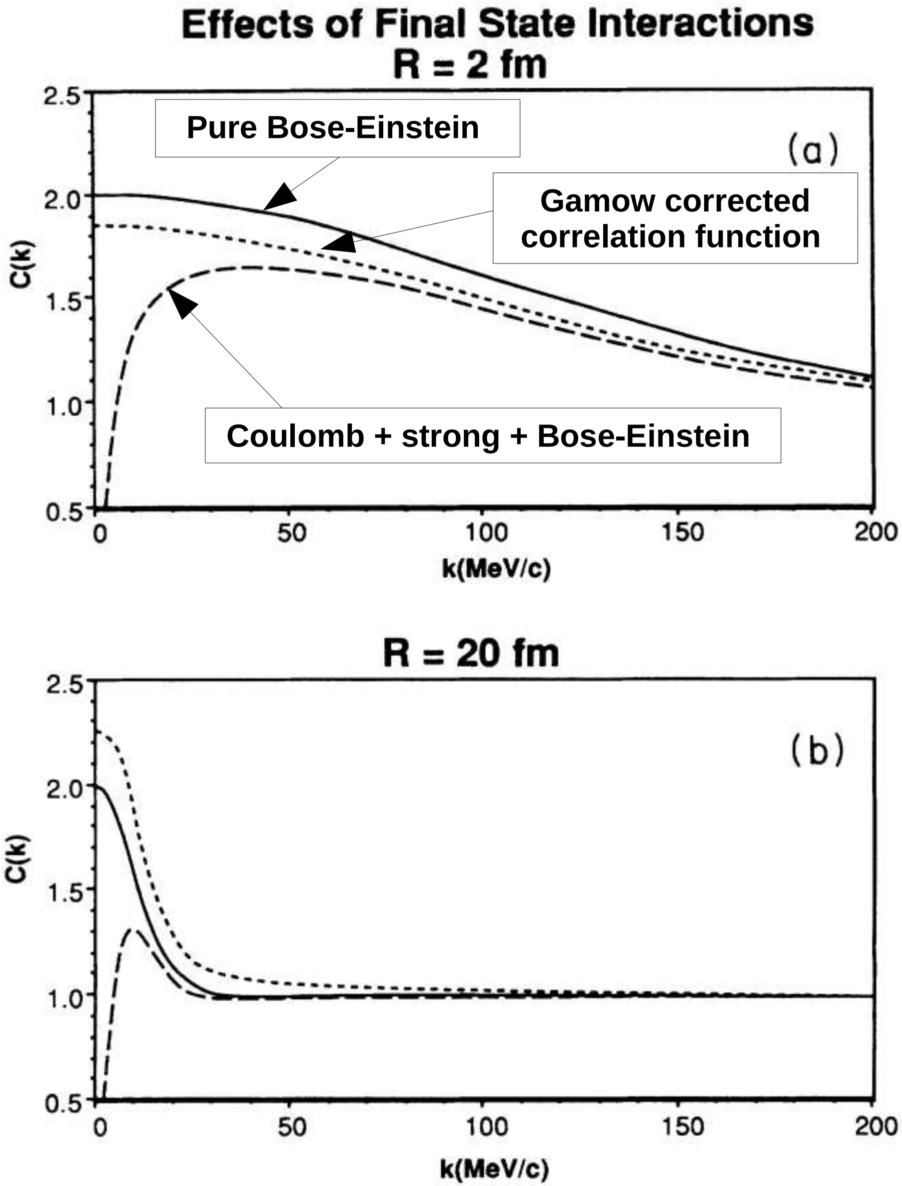
Outline

Central question

How large are the 3-body Coulomb interactions between the particles produced in high-energy hadronic collisions?

1) Why are Coulomb interactions important?
2) Feynman diagram approach to calculating final-state interactions.
3) Scalar QED Feynman rules.
4) 2-pion Coulomb calculations at NLO.
5) 3-pion Coulomb calculations at NLO.

Motivation I:
Why are 2-pion Coulomb interactions important?



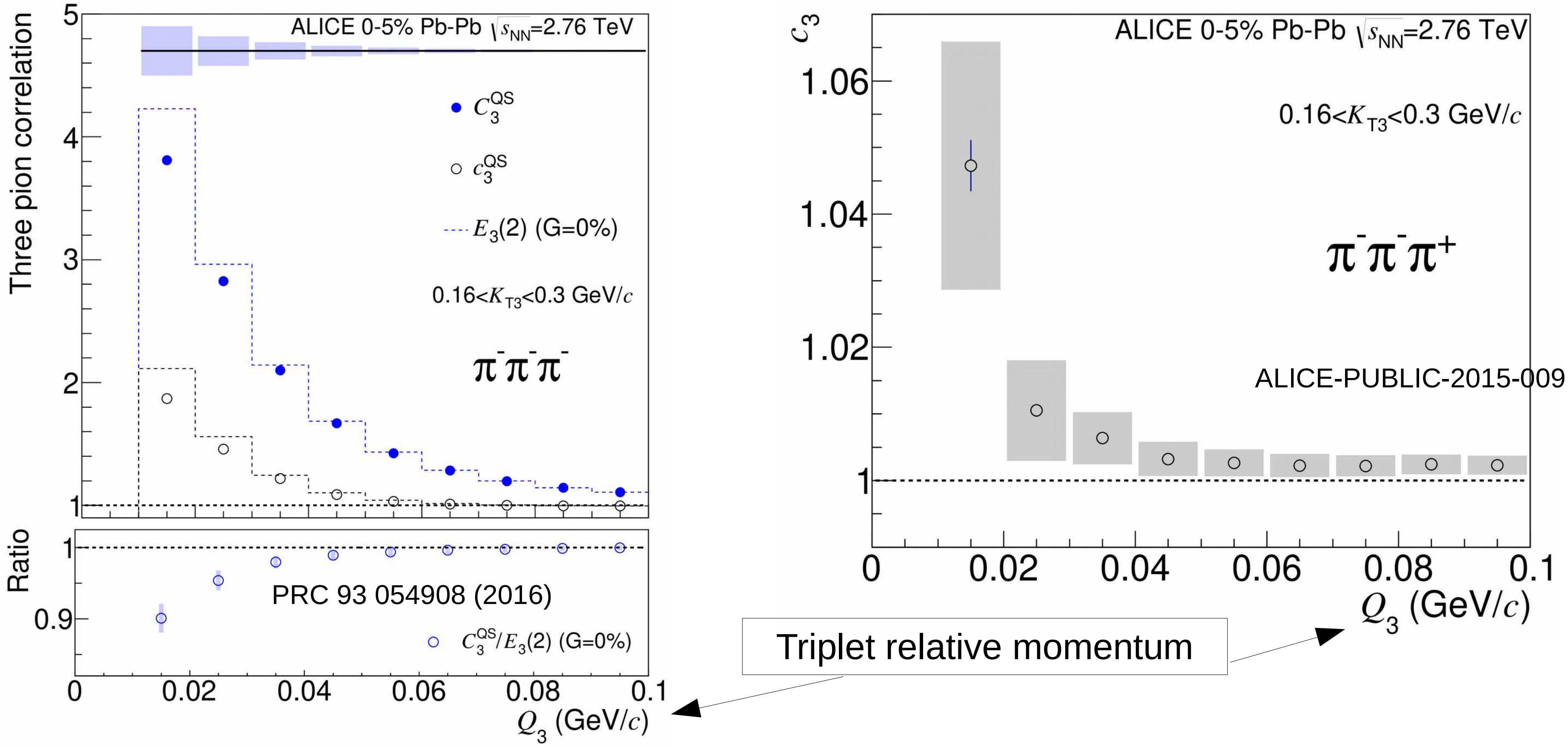
The size of the medium produced in high-energy collisions is important to the understanding of the dynamics of multibody QCD matter, e.g. a QGP.

Source sizes are most often estimated using the Bose-Einstein correlations between identical pions.

The proper treatment of Coulomb + strong final-state interactions is crucial in order to extract the underlying Bose-Einstein correlation.

Pratt, Csorgo, & **Zimanyi**
PRC 42 2646 (1990)

Motivation II:
Why are 3-pion Coulomb interactions important?



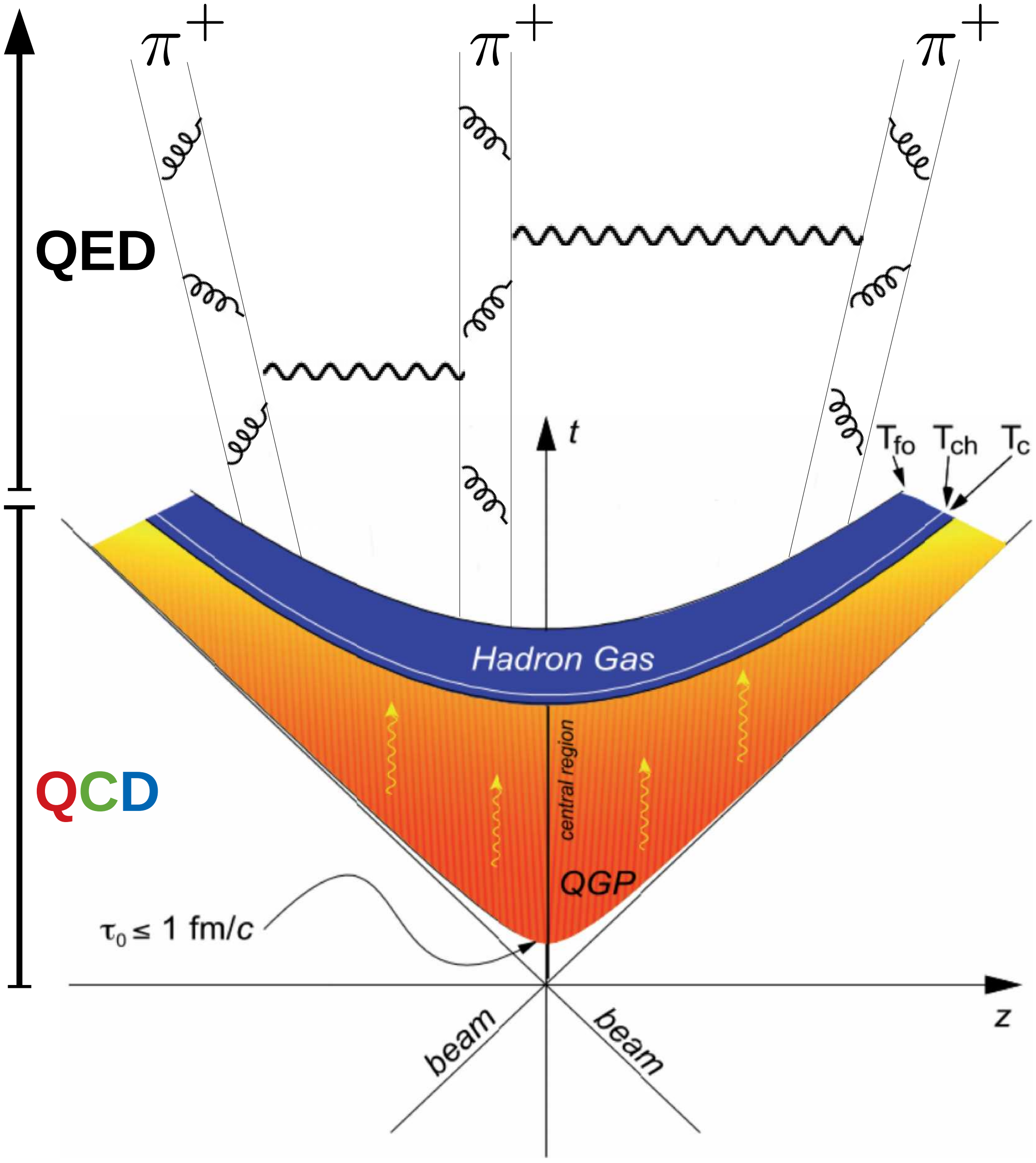
3-pion correlations have been Coulomb corrected according to an “asymptotic ansatz”: $K_3 = K_2^{12}\, K_2^{13}\, K_2^{23}$

Left plot: The measured 3-pion correlations differ significantly wrt the expectations from 2-pion measurements (dashed lines).

There is an unexplained suppression on the left, and an unexplained residue on the right.

Both may be due to genuine 3-body Coulomb interactions which were not taken into account.

Feynman diagram approach to calculating final-state interactions



Before freeze-out,
QCD processes dominate.

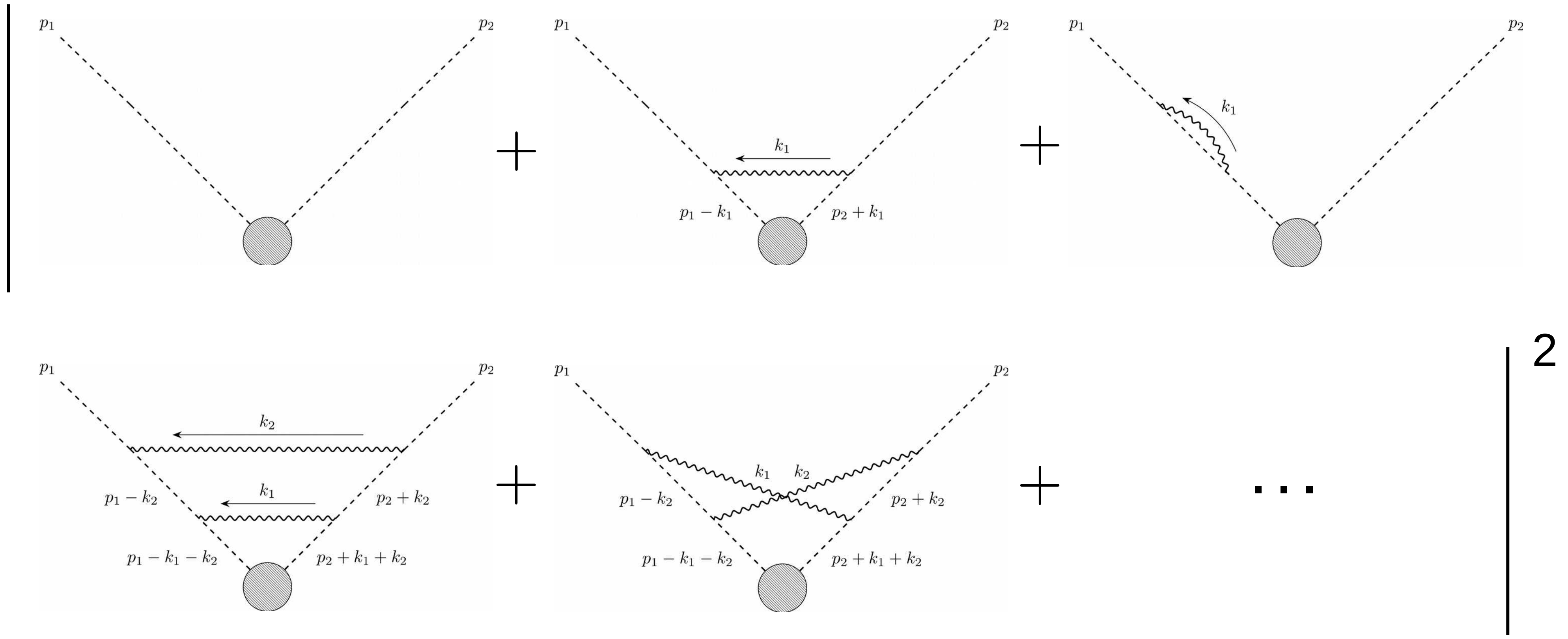
After freeze-out,
QED dominates the interaction between charged pions.

The production amplitude of a pair or triplet at freeze-out is referred to as M_0.

For simplicity,
we treat M_0 as momentum independent (point-source Gamow approximation).

We are interested in the **QED** interactions after freeze-out.

Diagrammatic illustration of 2-pion Coulomb scattering probability



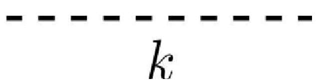
The complete 2-pion Coulomb scattering amplitude is represented by the sum over all possible intermediate processes.

In perturbation theory, we calculate them order by order to the desired accuracy.

Scalar QED Feynman rules

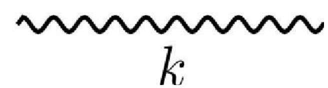
Lagrangian: $$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + (D_\mu\phi)^*(D^\nu\phi) - m^2\phi^*\phi$$

Propagators:



$$= \frac{i}{k^2 - m^2 + i\epsilon}$$

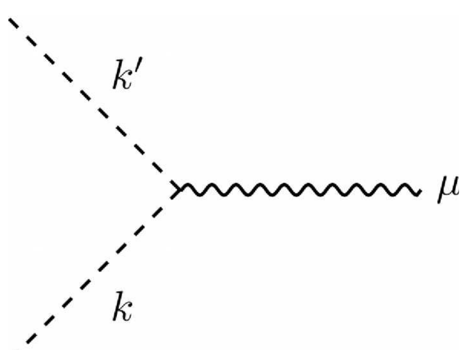
Pion mass



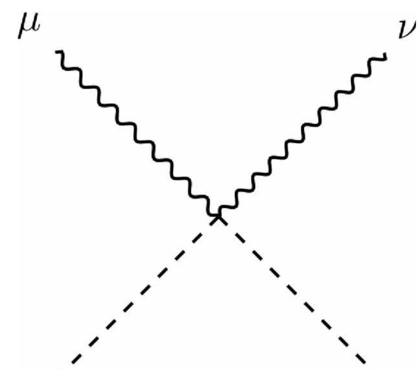
$$= \frac{-ig^{\mu\nu}}{k^2 - \lambda^2 + i\epsilon}$$

Photon regularization mass

Vertices:



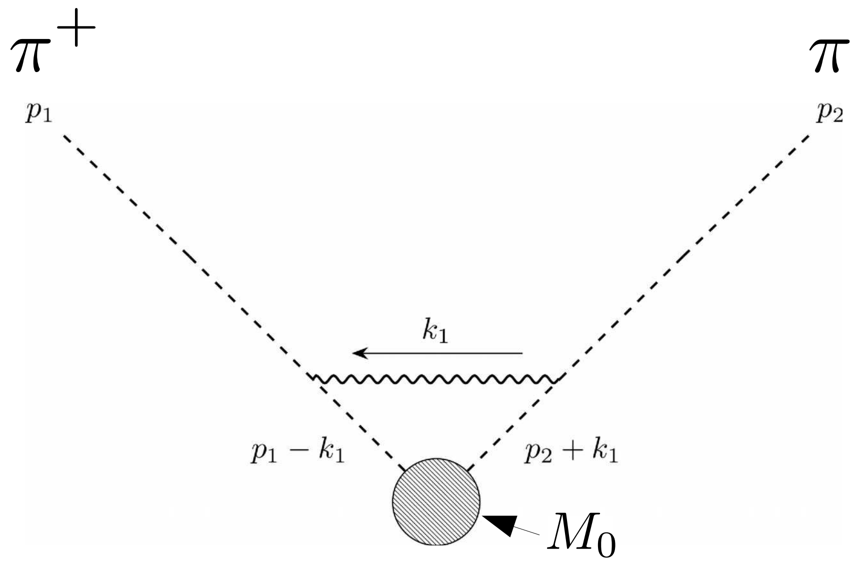
$$= -ie(k + k')^\mu$$



$$= 2ie^2 g^{\mu\nu}$$

Peskin & Schroeder
Intro to QFT

2-pion Coulomb leading order amplitude: I_1



$$I_1 = M_0 \int_{-\infty}^{\infty} \frac{d^4k_1}{(2\pi)^4} \frac{i}{(p_1-k_1)^2 - m^2 + i\epsilon} [-ie(2p_1 - k_1)^\mu] \frac{-ig^{\mu\nu}}{k_1^2 - \lambda^2 + i\epsilon} [-ie(2p_2 + k_1)_\nu] \frac{i}{(p_2+k_1)^2 - m^2 + i\epsilon}$$

Use non-relativistic & pair-rest-frame simplification

$$p_{1,2} = (m, \pm p, 0, 0) \qquad p \ll m$$

$$= M_0 \frac{-ie^2}{(2\pi)^4} 4m^2 \int_{-\infty}^{\infty} d^4k_1 \frac{1}{(k_1^2 - 2p_1k_1 + i\epsilon)} \frac{1}{(k_1^2 - \lambda^2 + i\epsilon)} \frac{1}{(k_1^2 + 2p_2k_1 + i\epsilon)}$$

2-pion Coulomb leading order amplitude: I_1

To identify the important terms in the integrand, it is convenient to make a scale transformation:

$$\mathbf{k}_1 \to p\,\mathbf{k}_1$$

$$k_1^0 \to \frac{p^2}{m}\,k_1^0$$

Baier and Fadin
Sov. Phys. JETP 30 127 (1970)

After this transformation, it is clear that 3 types of terms in the denominator can be ignored.

$$M_0 \frac{ie^2}{(2\pi)^4} \frac{4m}{p} \int dk_1^0 d\mathbf{k}_1 \frac{1}{(\mathbf{k}_1^2 - 2k_1^0 + 2\mathbf{k}_1\mathbf{n} - i\epsilon)} \frac{1}{(\mathbf{k}_1^2 + \lambda^2 - i\epsilon)} \frac{1}{(\mathbf{k}_1^2 + 2k_1^0 + \mathbf{k}_1\mathbf{n} - i\epsilon)}$$

<u>Methods to evaluate each integral:</u>

dk^0 : Use residue theorem. Two simple poles @ $k_1^0 = \pm[\mathbf{k}_1\mathbf{n} + \frac{\mathbf{k}_1^2}{2} - \frac{i\epsilon}{2}]$

$d\varphi$: Trivial 2π

$d(\cos\theta)$: $1 / \cos\theta$ integrand results in a logarithm.

dk : Use residue theorem. Two simple poles and a branch cut.

2-pion Coulomb at LO and all orders

Leading order amplitude

relative velocity

$$I_1 = -M_0 \frac{\alpha}{v}\left[\frac{\pi}{2} + i \ln \lambda/2\right] \qquad \alpha = \frac{e^2}{4\pi} = \frac{1}{137} \qquad v = \frac{2p}{m} \approx \frac{q_{\rm inv}}{m}$$

All orders

$$I_n = \left(\frac{i\alpha}{-v}\right)^n \int_0^\infty d\beta e^{-\beta} \frac{1}{n!}\left[\int_{i\lambda\beta/2}^\infty dt \frac{e^{-t}}{t}\right]^n$$

Baier and Fadin
Sov. Phys. JETP 30 127 (1970)

$$\begin{aligned} M &= M_0 \sum_{n=0}^{\infty} I_n \\ &= M_0 e^{i\frac{\alpha}{v}\left(\gamma + \ln\frac{\lambda}{2p}\right)} e^{-\alpha\pi/2v}\Gamma\left(1 + \frac{i\alpha}{v}\right) \end{aligned}$$

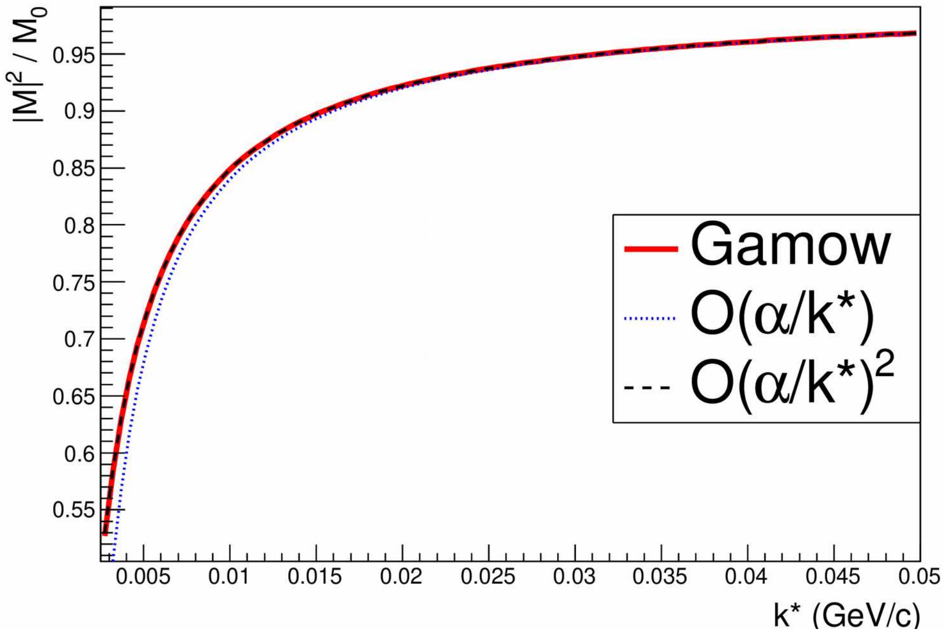
Euler-Mascheroni constant

Mod square of M gives the well known Gamow factor!

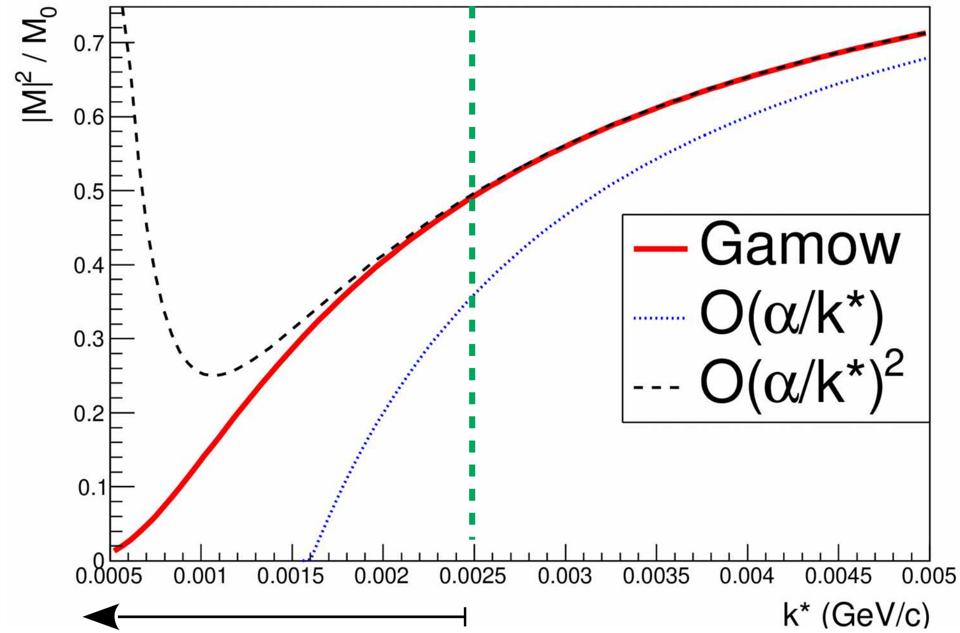
2-pion LO and NLO compared to Gamow

$\pi^+ \pi^+$

Relevant k* interval
for heavy-ion data



Very small k* interval



Unmeasurable region

Existing detectors can distinguish pairs of particles with $k^* = \frac{q_{\text{inv}}}{2} \gtrsim 0.0025\ \text{GeV/c}$

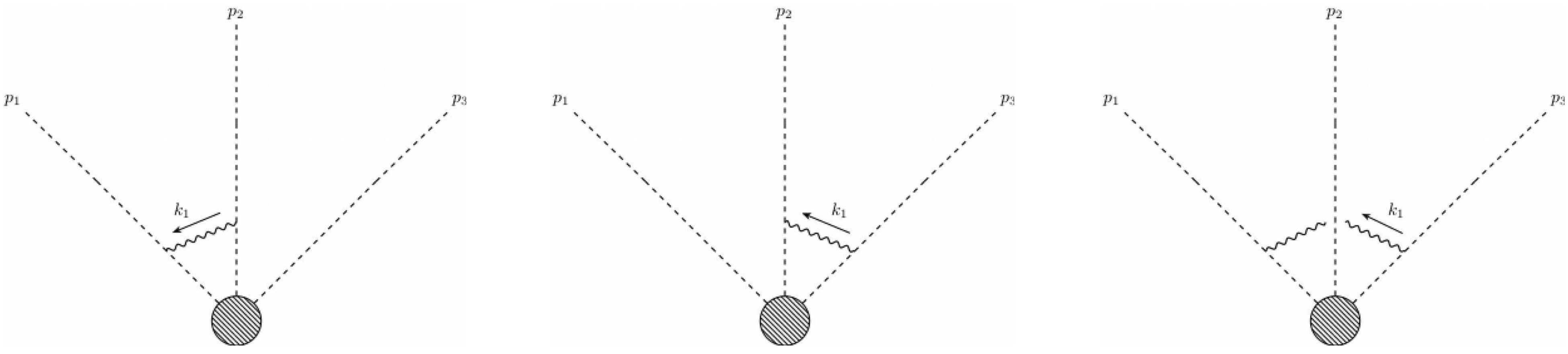
For *k** > 2.5 MeV/c, NLO accurately represents the full Gamow solution:

$$\frac{|\text{NL0} - \text{Gamow}|}{1 - \text{Gamow}} \lesssim 0.01$$

3-pion LO diagrams

<u>2 vertices</u>

$$\mathcal{O}(\alpha)$$



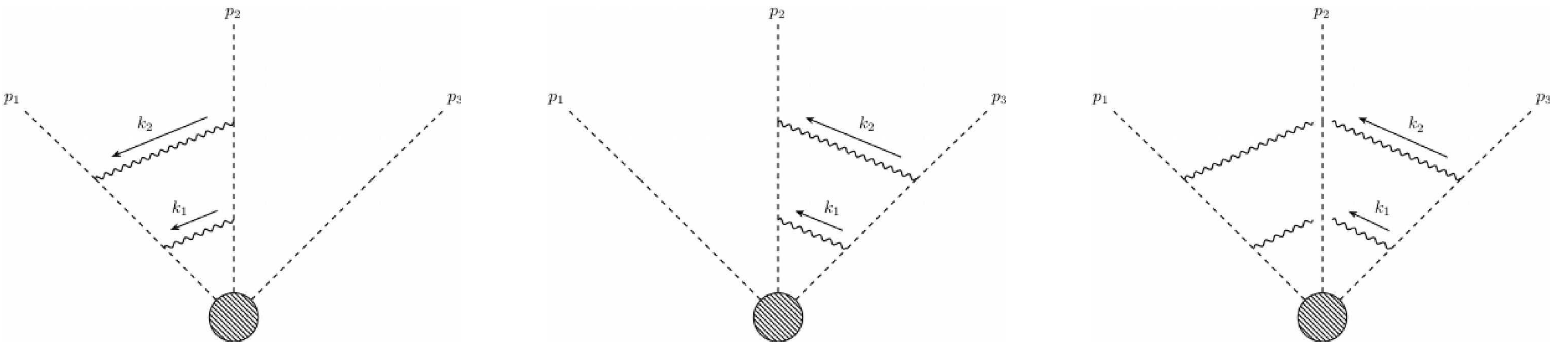
p_1 and p_3 photon exchange

To leading order, only 2-pion interactions contribute.

3-pion NLO diagrams: single-pair exchanges

4 vertices

$\mathcal{O}(\alpha^2)$



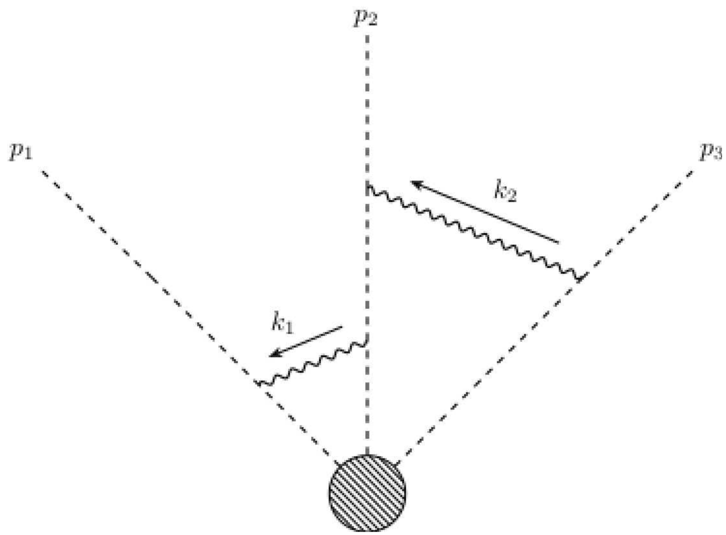
p_1 and p_3 photon exchanges

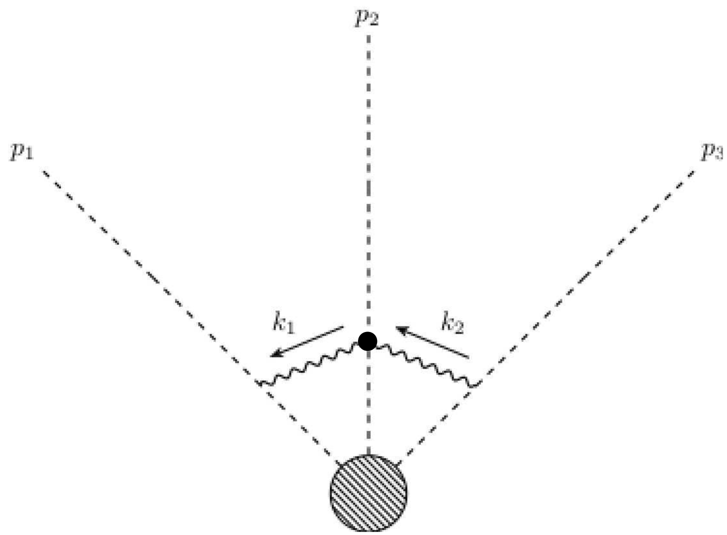
At NLO, purely 2-pion interactions still contribute.

3-pion NLO diagrams: Genuine 3-body contributions

<u>4 vertices</u>

$$\mathcal{O}(\alpha^2)$$





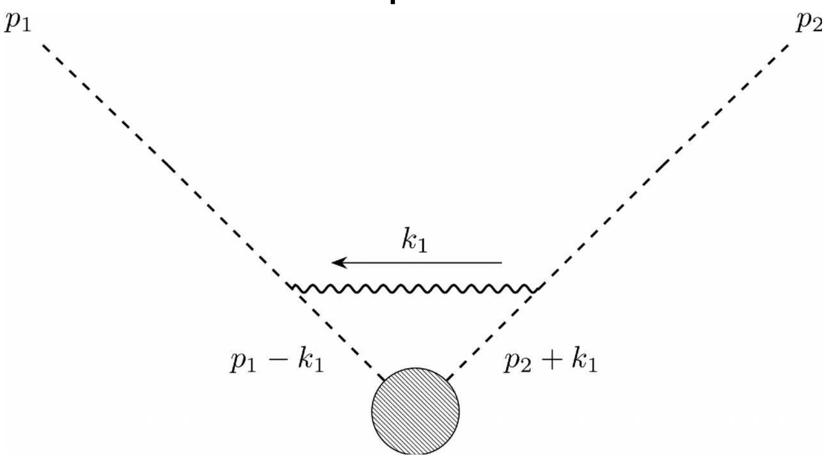
These are the lowest order **genuine** 3-body contributions.

The characteristic magnitude of a diagram

To get a sense for the magnitude of a diagram,
first apply the scale transformation as before. Then, consider the non-relativistic limit.

$$\mathbf{k} \to p\,\mathbf{k} \qquad k^0 \to \frac{p^2}{m}\,k^0$$

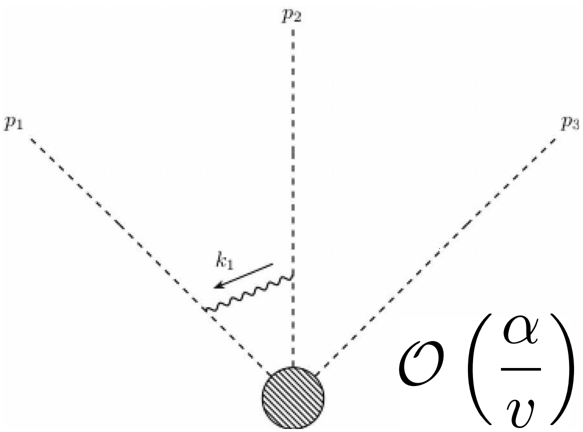
Example case



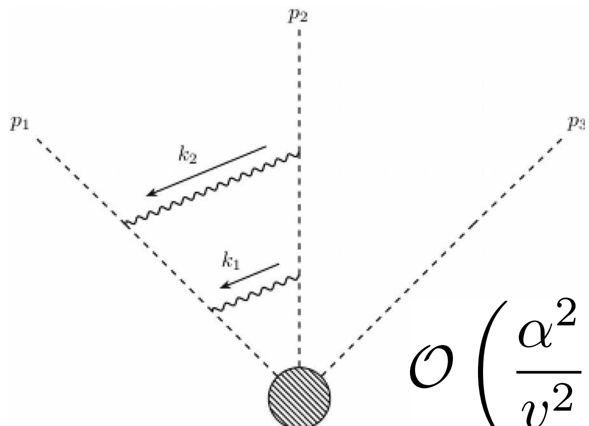
- The two vertices each give a factor of e*m*
- $d^4k \to \frac{p^5}{m} d^4k$
- $\frac{1}{k^2 - 2pk + i\epsilon} \to \frac{1}{p^2} \cdot \frac{1}{\mathbf{k}^2 - 2k^0 + 2\mathbf{kn} + i\epsilon}$

$$\int d^4k (\text{propagator})^3 (\text{vertex})^2 \propto \frac{p^5}{m}\frac{1}{p^6}(e\,m)^2 = 4\pi\alpha\frac{m}{p} \propto \boxed{\frac{\alpha}{v} \lesssim 0.1}$$

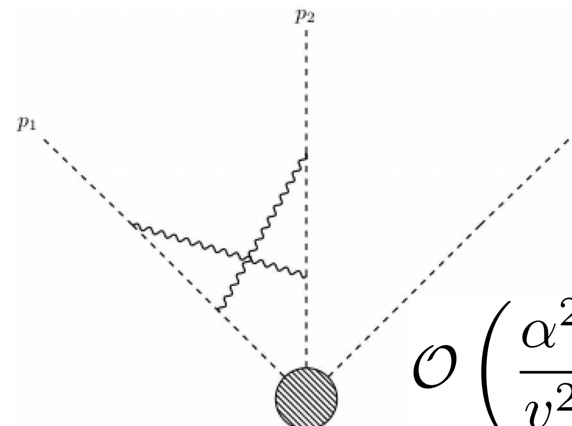
The characteristic magnitude of a diagram



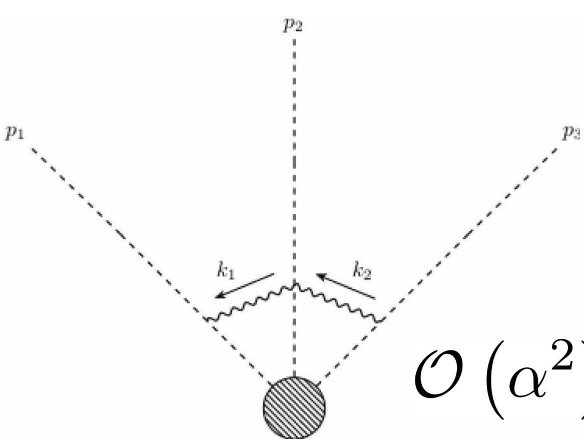
$\mathcal{O}\left(\frac{\alpha}{v}\right)$



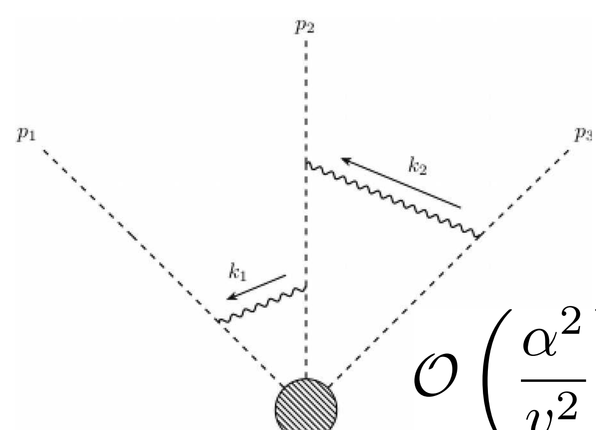
$\mathcal{O}\left(\frac{\alpha^2}{v^2}\right)$



$\mathcal{O}\left(\frac{\alpha^2}{v^2}\right)$

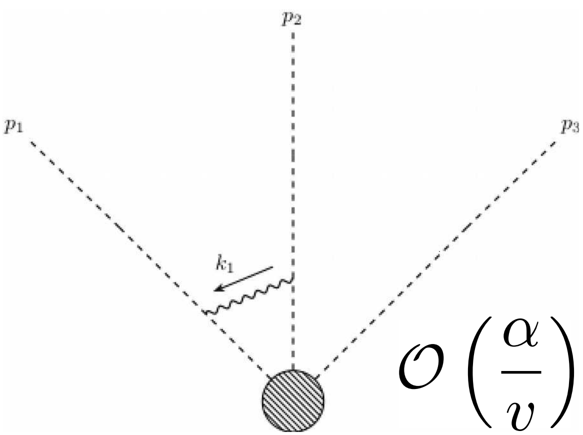


$\mathcal{O}\left(\alpha^2\right)$

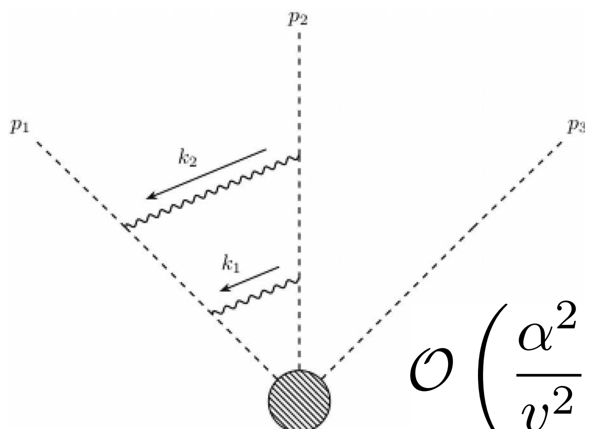


$\mathcal{O}\left(\frac{\alpha^2}{v^2}\right)$

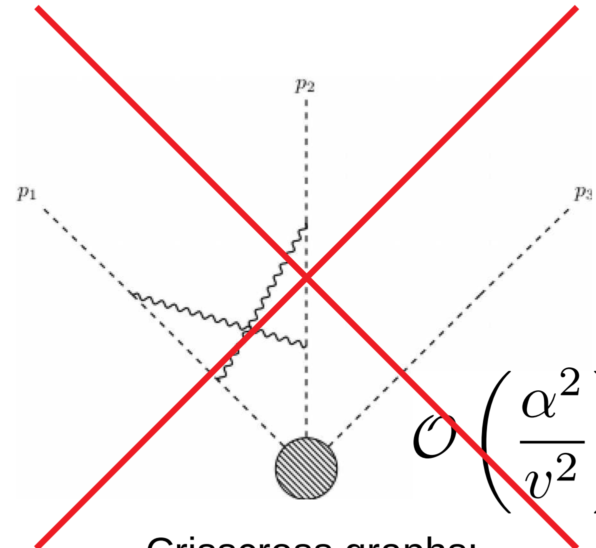
The characteristic magnitude of a diagram



$$\mathcal{O}\left(\frac{\alpha}{v}\right)$$

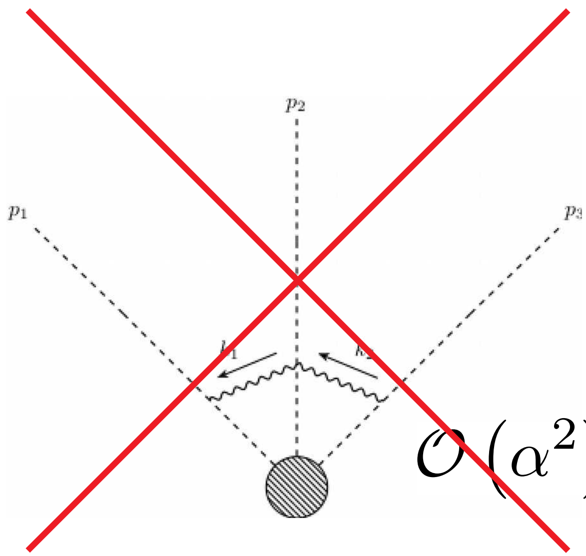


$$\mathcal{O}\left(\frac{\alpha^2}{v^2}\right)$$



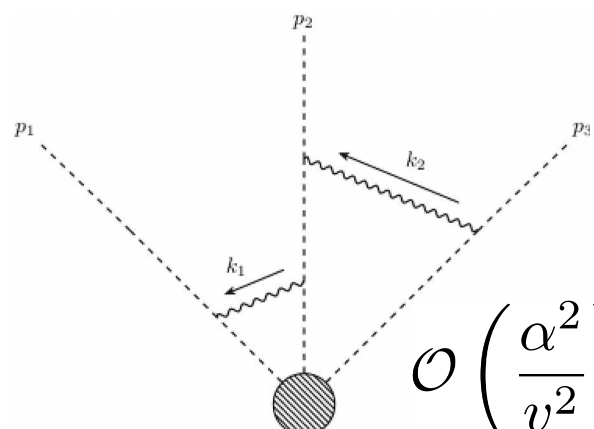
$$\mathcal{O}\left(\frac{\alpha^2}{v^2}\right)$$

Crisscross graphs:
Unimportant in non-relativistic limit



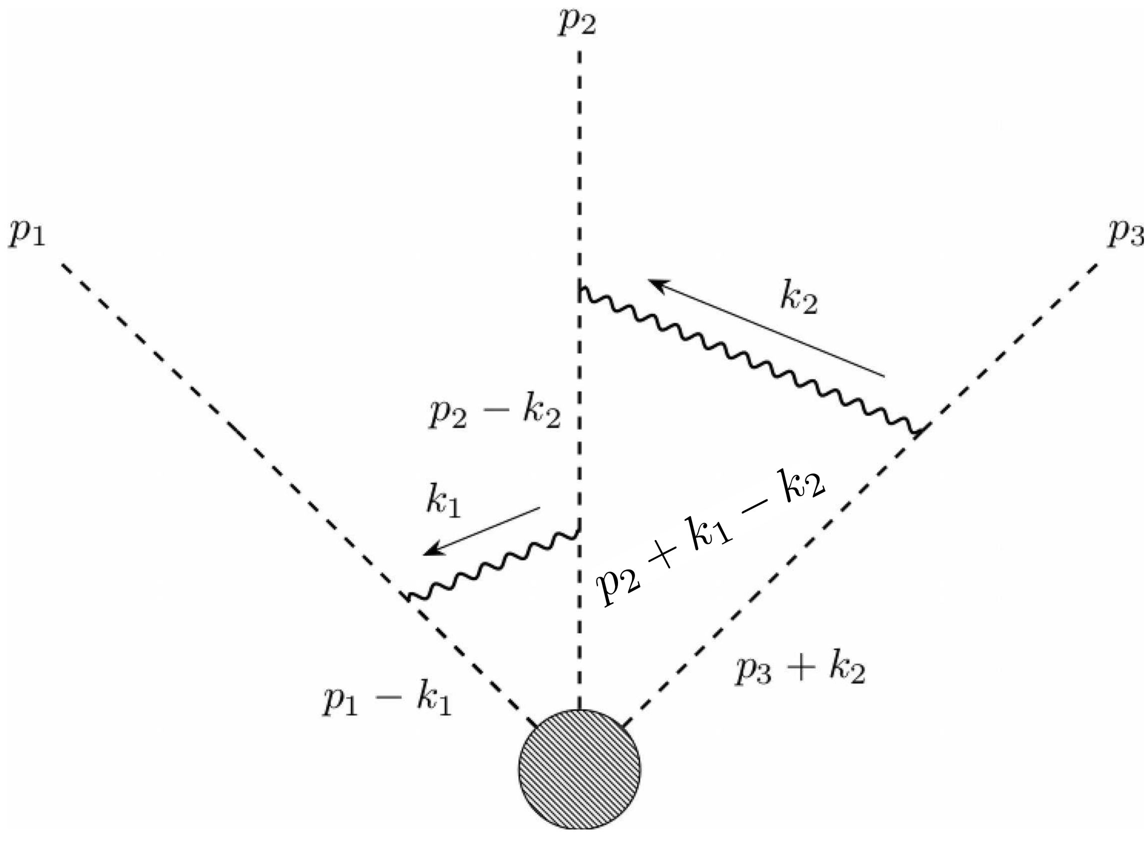
$$\mathcal{O}\left(\alpha^2\right)$$

No external momentum dependence.
Unimportant



$$\mathcal{O}\left(\frac{\alpha^2}{v^2}\right)$$

Calculation of the dominant genuine 3-pion diagram



$$\frac{-e^4}{(2\pi)^8}(2m)^4 \int d^4k_1 d^4k_2 \quad \cdot \quad \frac{1}{(p_1-k_1)^2 - m^2 + i\epsilon} \cdot \frac{1}{k_1^2 - \lambda^2 + i\epsilon} \cdot \frac{1}{(p_2+k_1-k_2)^2 - m^2 + i\epsilon}$$
$$\cdot \quad \frac{1}{(p_2-k_2)^2 - m^2 + i\epsilon} \cdot \frac{1}{k_2^2 - \lambda^2 + i\epsilon} \cdot \frac{1}{(p_3+k_2)^2 - m^2 + i\epsilon}$$

Calculation of the dominant genuine 3-pion diagram

$$\int d^4k_1 d^4k_2 \quad \cdot \quad \underbrace{\frac{1}{(p_1-k_1)^2-m^2+i\epsilon}}_{1/A} \cdot \underbrace{\frac{1}{k_1^2-\lambda^2+i\epsilon}}_{1/B} \cdot \underbrace{\frac{1}{(p_2+k_1-k_2)^2-m^2+i\epsilon}}_{1/C}$$

$$\cdot \quad \underbrace{\frac{1}{(p_2-k_2)^2-m^2+i\epsilon}}_{1/D} \cdot \underbrace{\frac{1}{k_2^2-\lambda^2+i\epsilon}}_{1/E} \cdot \underbrace{\frac{1}{(p_3+k_2)^2-m^2+i\epsilon}}_{1/F}$$

Use Feynman parameters to combine the first three denominators

$$\frac{1}{A\cdot B\cdot C} = \int_0^1 dx\,dy\,dz\;\delta(x+y+z-1)\frac{2}{(l^2-\Delta_1+i\epsilon)^3}$$

$$l = k_1 + (zp_2 - zk_2 - xp_1)$$

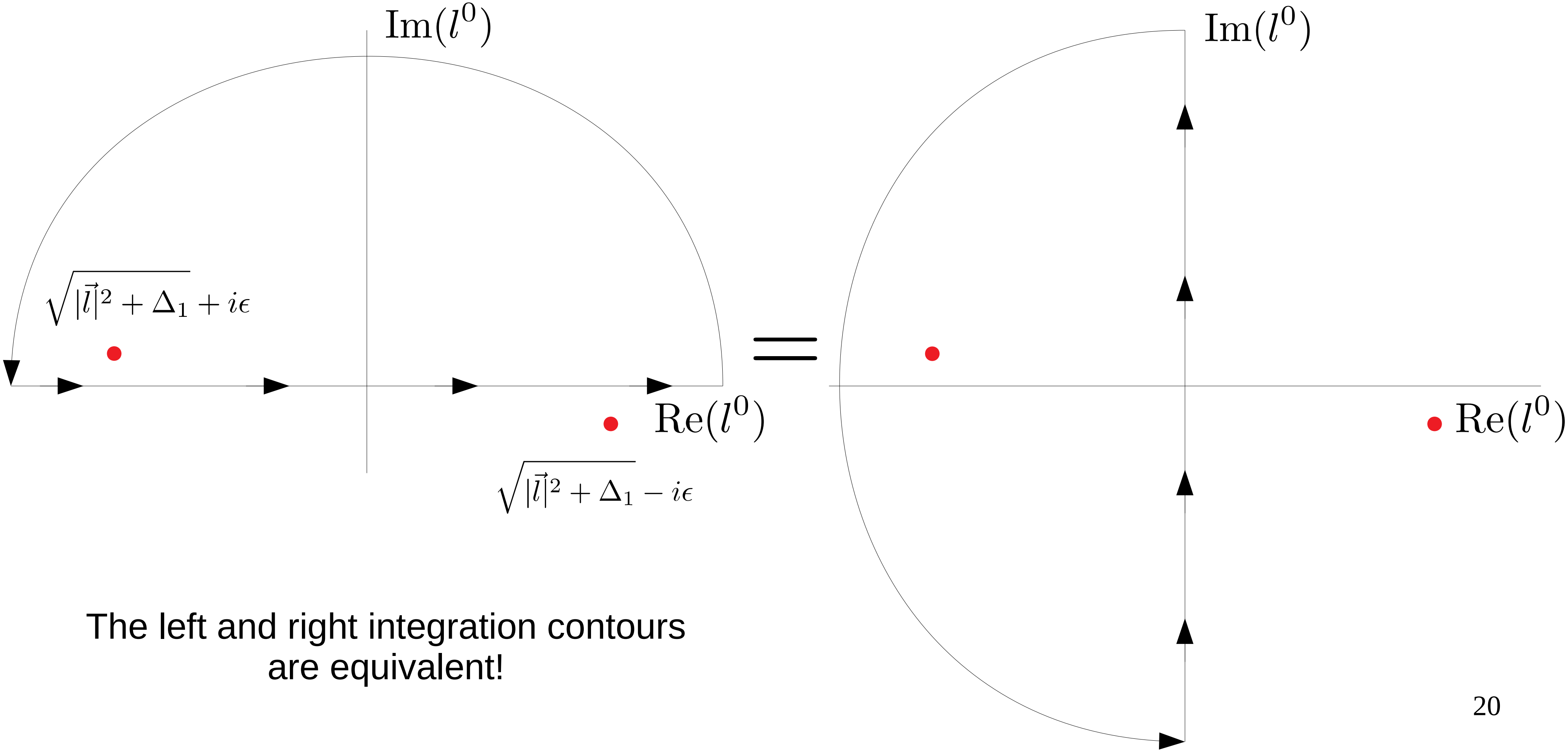
$$\Delta_1 = k_2^2(z^2-z) + 2k_2(xzp_1 - z^2p_2 + zp_2) + ((zp_2-xp_1)^2 + y\lambda^2)$$

Wick rotation in the l^0 complex plane

$$\int_{-\infty}^{\infty} dl^0 \frac{1}{(l_0^2 - \vec{l}^2 - \Delta_1 + i\epsilon)^3}$$

Consider the l^0 integral first.

- There are only two singularities with convenient locations
- The integrand vanishes at l^0 = infinity



The left and right integration contours are equivalent!

Wick rotation in the l^0 complex plane

Wick rotation

$$\int_{-\infty}^{\infty} d^4 l \frac{1}{(l_0^2 - \vec{l}^2 - \Delta_1 + i\epsilon)^3} = \int_{-i\infty}^{i\infty} d^4 l \frac{1}{(l_0^2 - \vec{l}^2 - \Delta_1 + i\epsilon)^3}$$

Scale transformation

$$l^0 \to i l_E^0 \qquad \vec{l} \to \vec{l}_E$$

Evaluate using hyperspherical coordinates

$$= \frac{i}{(-1)^3} \int_{-\infty}^{\infty} d^4 l_E \frac{1}{(l_E^2 + \Delta_1 - i\epsilon)^3} = -i \int d\Omega_4 \int_0^{\infty} \frac{l_E^2}{(l_E^2 + \Delta_1 - i\epsilon)^3}$$

$$= -i\pi^2 \frac{1}{2(\Delta_1 - i\epsilon)}$$

Evaluating the integral over k_2

$$\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int_{-\infty}^{\infty} d^4k_2\,\frac{1}{\Delta_1 - i\epsilon}\cdot\frac{1}{D\,E\,F}$$

<u>Again, use Feynman parameters to combine the 4 denominators</u>

$$= \int_0^1 dx\,dy\,dz\,dx'\,dy'\,dz'\,dw'\,\delta(x+y+z-1)\,\delta(x'+y'+z'+w'-1)$$

$$\times \frac{1}{a^4}\int_{-\infty}^{\infty} d^4l_2\,\frac{1}{(l_2^2-\Delta_2+i\epsilon(1-2x'))^4}$$

$$= i\pi^2\frac{1}{6}\int_0^1 dx\,dy\,dz\,dy'\,dz'\,dw'\,\delta(x+y+z-1)\,\delta(x'+y'+z'+w'-1)\left[\int_0^{1/2}\frac{dx'}{a^4\,\Delta_2^2}-\int_{1/2}^{1}\frac{dx'}{a^4\,\Delta_2^2}\right]$$

a and Δ_2 are complicated polynomials of the Feynman parameters and external momenta ($\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$)

Evaluation of the Feynman parameter integrals

7 integrals

$$\int_0^1 \underline{dx}\,\underline{dy}\,\underline{dz}\,\underline{dy'}\,\underline{dz'}\,\underline{dw'}\;\delta(x+y+z-1)\,\delta(x'+y'+z'+w'-1)\left[\underline{\int_0^{1/2}\frac{dx'}{a^4\,\Delta_2^2}-\int_{1/2}^{1}\frac{dx'}{a^4\,\Delta_2^2}}\right]$$

- 2 of the integrals are trivial due to the delta functions.
- 1, at least, can be evaluated analytically.
- That leaves **4** remaining integrals.

NVIDIA P2000

GPU numerical integration

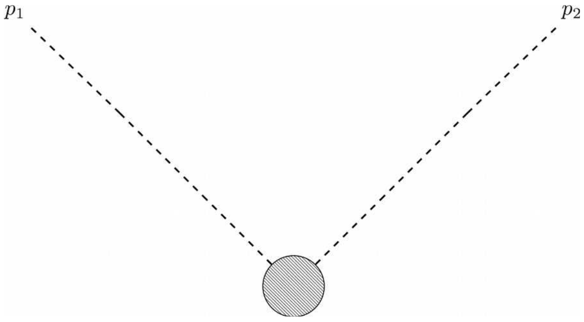
- Discretize each Feynman parameter integral into ~100 bins.
- $100^4 = 10^8$ computations.
- 1 - 10 ms per computation.
- 1000 cores with the NVIDIA P2000.
- Therefore, about 2 - 20 min to evaluate all integrals.
- This has to be done for several configurations of external momenta ($\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$).

Remarks & Summary

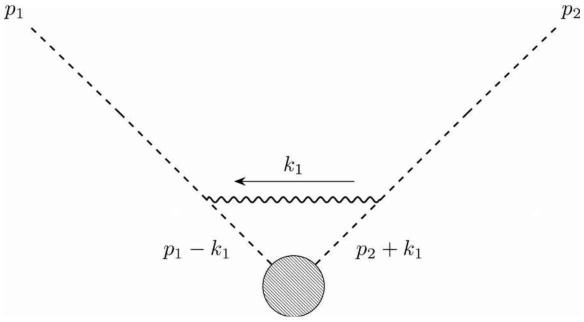
- An approach to evaluating 3-pion Coulomb interactions at Next-to-Leading Order in QED has been presented.
- The characteristic expansion parameter in the perturbation series is $\frac{\alpha}{v} \lesssim 0.1$
- Thus, the series rapidly converges.
- For 2-pion Coulomb correlation functions, NLO reproduces Gamow to better than 1%.
- The Gamow assumption (point-source) overestimates Coulomb interactions in hadronic collisions but will provide a useful upper limit to 3-body wrt to 2-body Coulomb interactions.
- With the non-relativistic simplification, there are no UV divergences.
- There are IR divergences, which are tamed with the fictitious photon mass, λ. IR terms cancel at NLO for 2-pion case. Should cancel in the 3-pion case too when all NLO Feynman diagrams are included.

Backup

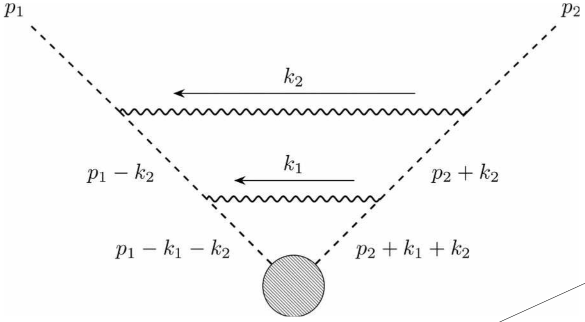
Cancellation of infrared divergences from different elements of the cross-section



$$I_0 = 1$$



$$I_1 = \frac{\alpha}{v}\left[-\frac{\pi}{2} + i\ln\lambda/2\right]$$



$$I_2 = \frac{\alpha^2}{2v^2}\left[\frac{1}{12}\pi^2 - \ln^2\lambda/2 - i\pi\ln\lambda/2\right]$$

NLO IR divergent term

from $|I_1|^2$ cancels that from 2*I_2