A new family of exact solutions of relativistic viscous hydrodynamics

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Introduction and motivation
Viscous Relativistic Hydro
Multipole solutions with
temperature dependent speed of sound
Family of solutions with temperature profile

Outlook
Summary

Partially supported by
NKTIH FK 123842, FK123959
K 133046 and EFOP 3.6.1-16-2016-00001
Context

Renowned exact solutions, reviewed in arXiv:1805.01427

Landau-Khalatnikov solution: \( \frac{dn}{dy} \sim \text{Gaussian} \)

Hwa solution (1974) – Bjorken: same solution + \( \varepsilon_0 \) (1983)

Chiu, Sudarshan and Wang: plateaux, Wong: Landau revisited

Revival of interest: Zimányi, Bondorf, Garpman (1978)

Buda-Lund model + exact solutions (1994-96)

Biró, Karpenko, Sinyukov, Pratt (2007)

Bialas, Janik, Peschanski, Borsch+Zhdanov (2007)

CsT, Csanád, Nagy (2007-2008)

CsT, Csernai, Grassi, Hama, Kodama (2004)

Gubser (2010-11)

Hatta, Noronha, Xiao (2014-16)

CsT, Kasza, Csanad, Jiang (2017-18)

New simple solutions \rightarrow Evaluation of measurables

Rapidity distribution \rightarrow Advanced initial energy density (Kasza)

Viscous solutions \rightarrow Theorems
Goal

Need for solutions that are:
- explicit
- simple
- accelerating
- relativistic
- viscous

realistic / compatible with the data:
- lattice QCD EoS
- ellipsoidal symmetry
- multipole solution (spectra, $v_2$, $v_3$, $v_4$ ..., HBT)
- finite $dn/dy$

Some of our recent new exact hydro solutions:
- finite in $y$: CsT, Kasza, Csanád, Jiang (CKCJ): arXiv.org:1805.01427
- viscous: Csanád, Nagy, Jiang, CsT: arxiv.org:1909.02498

This has been generalized to $T$-dependent speed of sound to multipole solutions and to spatially dependent $T$ profiles
Self-similar solutions for perfect fluid

Publication (for example):

3D spherically symmetric HUBBLE flow:
No acceleration:
\[ u^\mu \partial_\mu u_\nu = 0. \]
\[ u^\mu = \frac{x^\mu}{\tau} \]

Define a scaling variable for self-similarly expanding ellipsoids:
\[ s = \frac{r_x^2}{X_0^2 t^2} + \frac{r_y^2}{Y_0^2 t^2} + \frac{r_z^2}{Z_0^2 t^2} \]

EoS: (massive)
ideal gas
\[ \epsilon = mn + \kappa p, \]
\[ p = nT. \]
\[ \epsilon_Q = m_Q n_Q + \lambda_\epsilon n_Q T + B, \]
\[ p_Q = \lambda_p n_Q T - B, \]
\[ n(t, r) = n_0 \left( \frac{\tau_0}{\tau} \right)^3 V(s) \]
\[ T(t, r) = T_0 \left( \frac{\tau_0}{\tau} \right)^{3/\kappa} \frac{1}{V(s)} \]
\[ p(t, r) = p_0 \left( \frac{\tau_0}{\tau} \right)^{3+3/\kappa} \]

Scaling function \( V(s) \) can be chosen freely.
Shear and bulk viscous corrections in NR limit: known analytically.
New, exact solutions with viscosity

\[ T^{\mu \nu} = e u^\mu u^\nu - p \Delta^{\mu \nu} + \Pi^{\mu \nu} = T_0^{\mu \nu} + \Pi^{\mu \nu}, \]

\[ \Pi^{\mu \nu} = \pi^{\mu \nu} + \Delta^{\mu \nu} \Pi, \]

\[ e = \kappa p, \quad p = nT, \]

\[ \partial_\mu (N^\mu) = 0. \]

\[ D e = -(e + p + \Pi) \theta + \sigma_{\mu \nu} \pi^{\mu \nu}, \]

\[ (e + p + \Pi) D u^\alpha = \nabla^\alpha (p + \Pi) - \Delta^\alpha_{\nu} u_\mu D_{\pi^{\mu \nu}} - \Delta^\alpha_{\nu} \nabla_\mu \pi^{\mu \nu}, \]

\[ S = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2}, \]

\[ u^{\mu} = \frac{x^{\mu}}{\tau} = \gamma \left( 1, \frac{\dot{X}}{X} r_x, \frac{\dot{Y}}{Y} r_y, \frac{\dot{Z}}{Z} r_z \right), \]

\[ \theta = \partial_\mu u^\mu = \frac{d}{\tau}, \quad D = u^\mu \partial_\mu = \frac{\partial}{\partial \tau}. \]

\[ \tau = \sqrt{t^2 - r^2} \]

\[ \eta_s = \frac{1}{2} \ln \left( \frac{t + r_z}{t - r_z} \right) \]

\[ \sigma_{\mu \nu} \pi^{\mu \nu} = 0, \]

\[ \zeta/s \quad \eta/s \]

Csanád, Nagy, Jiang, CsT:

arxiv.org:1909.02498

Zimanyi’19@Budapest, 2019/12/05
Exact solution with viscosity

Csanád, Nagy, Jiang, CsT:
arxiv.org:1909.02498

Case A: No conserved $n$, and constant $\zeta$:
$\zeta = \zeta_0 \text{ (const)}, \quad \varepsilon = \kappa p, \quad p = p_0 (T/T_0)^{\kappa+1}$.

Case B: With conserved $n$, and constant $\zeta$:
$\zeta = \zeta_0 \text{ (const)}, \quad \varepsilon = \kappa p, \quad p = nT$.

Case C: No conserved $n$, and $\zeta \propto s$:
$\zeta = \zeta_0 (T/T_0)^{\kappa}, \quad \varepsilon = \kappa p, \quad p = p_0 (T/T_0)^{\kappa+1}$.

Case D: With conserved $n$, and $\zeta/n = \text{const}$:
$\zeta = \zeta_0 (n/n_0), \quad \varepsilon = \kappa p, \quad p = nT$.

Case E: With conserved $n$, and “$\zeta \propto s$”:
$\zeta = \zeta_0 (T/T_0)^{\kappa}, \quad \varepsilon = \kappa p, \quad p = nT$.

**Case A:**

$$p(\tau) = \left[ p_0 - \frac{d^2}{(\kappa+1)d - \kappa \tau_0} \right] \left( \frac{\tau_0}{\tau} \right)^{\frac{\kappa+1}{\kappa}} + \frac{d^2}{(\kappa+1)d - \kappa \tau} \frac{\zeta_0}{\tau_0}.$$

**Case C:**

$$p(\tau) = p_0 \left\{ \left( 1 + \frac{d^2}{(\kappa+1)(\kappa-d) p_0 \tau_0} \right) \left( \frac{\tau_0}{\tau} \right)^{\frac{\kappa}{\kappa+1}} - \frac{d^2}{(\kappa+1)(\kappa-d) p_0 \tau_0} \frac{\zeta_0}{\tau_0} \right\}^{\frac{\kappa+1}{\kappa}},$$

**Case D:**

$$p(\tau) = \left[ p_0 + \frac{d^2}{\kappa-d \tau_0} \zeta_0 \left( \frac{\tau_0}{\tau} \right)^{\frac{\kappa+1}{\kappa}} - \frac{d^2}{\kappa-d \tau_0} \frac{\zeta_0}{\tau_0} \right] - \frac{d^2}{\kappa-d \tau_0} \frac{\zeta_0}{\tau_0} \tau^{d+1}.$$
Bulk viscosity important at late stage, heats up
Shear viscosity effects cancel for asymptotically Hubble flows
New viscous solutions in 1+3 dim

CsT and G. Kasza

1st solution: $\mu \neq 0, \rho = nT$

Energy conservation:

$$\partial_\tau [\kappa(T)nT] + [1 + \kappa(T)]nT \frac{d}{\tau} = \frac{d^2}{\tau^2} \zeta$$

Using the continuity equation:

$$\frac{1}{T}\partial_\tau (\kappa T) + \frac{d}{\tau} = \frac{d^2}{\tau^2} \frac{\zeta}{nT}$$

In the right hand side let’s define $f(T)$:

$$f(T) = \frac{\zeta}{nT}$$

With that, we have:

$$\frac{\dot{T}}{T} \frac{d}{dT}(\kappa T) + \frac{d}{\tau} = \frac{d^2}{\tau^2} f(T)$$

where $\kappa = \kappa(T)$ and $\dot{T} = \partial_\tau T$. We introduce the $g(T)$ function as

$$g(T) = \frac{d}{dT}(\kappa T)$$

so the energy equation becomes:

$$\frac{\dot{T}}{T} g(T) + \frac{d}{\tau} = \frac{d^2}{\tau^2} f(T)$$

T-dependent speed of sound
Case 1/A: \( f = f_0 \) = const and \( \kappa(T) = \kappa_0 = \text{const} \)
\( \kappa = \kappa_0 \) is constant, so \( g(T) \) is:
\[
g(T) = \kappa_0
\]
In this case the energy equation is:
\[
\frac{\dot{T}}{T^{\kappa_0}} + \frac{d}{\tau} = \frac{d^2}{\tau^2} f_0
\]
By integration we obtain:
\[
T = T_0 \left( \frac{\tau_0}{\tau} \right)^{\frac{d}{\kappa_0}} \exp \left( \frac{f_0 d^2}{\kappa_0} \left[ \frac{1}{\tau_0} - \frac{1}{\tau} \right] \right)
\]

Case 1/B: \( f = f_0 \) = const, \( \kappa = \kappa(T) \), but \( g(T) \) = const

Let \( g(T) = g_0 \) be:
\[
g_0 = \frac{\kappa_c T_c - \kappa_f T_f}{T_c - T_f}
\]  \hspace{1cm} (15)

The energy equation is the same as the one in Case 1/A, only \( \kappa_0 \) is substituted by \( g_0 \):
\[
\frac{\dot{T}}{T} g_0 + \frac{d}{\tau} = \frac{d^2}{\tau^2} f_0
\]  \hspace{1cm} (16)

So the solution is:
\[
T = T_0 \left( \frac{\tau_0}{\tau} \right)^{\frac{d}{g_0}} \exp \left( \frac{f_0 d^2}{g_0} \left[ \frac{1}{\tau_0} - \frac{1}{\tau} \right] \right)
\]  \hspace{1cm} (17)
\[
T = T_0 \left( \frac{\tau_0}{\tau} \right)^{\frac{d(T_c - T_f)}{\kappa_c T_c - \kappa_f T_f}} \exp \left( \frac{f_0 d^2 (T_c - T_f)}{\kappa_c T_c - \kappa_f T_f} \left[ \frac{1}{\tau_0} - \frac{1}{\tau} \right] \right)
\]  \hspace{1cm} (18)

T-dependent speed of sound

Bulk viscosity effect at late time: same as rescaling \( T_{\text{init}} \) for a perfect fluid
New viscous solutions in 1+3 dim

T-dependent speed of sound

Bulk viscosity effect at late time:
same as rescaling $T_{\text{init}}$
for a perfect fluid

Case 1/B: $f = f_0 = \text{const}$, $\kappa = \kappa(T)$, but $g(T) = \text{const}$

Let $g(T) = g_0$ be:

$$g_0 = \frac{\kappa_c T_c - \kappa_f T_f}{T_c - T_f}$$

(15)

The energy equation is the same as the one in Case 1/A, only $\kappa_0$ is substituted by $g_0$:

$$\frac{\dot{T}}{T} g_0 + \frac{d}{\tau} = \frac{d^2}{\tau^2} f_0$$

(16)

So the solution is:

$$T = T_0 \left( \frac{\tau}{\tau_0} \right) \frac{d}{g_0} \exp \left( \frac{f_0 d^2}{g_0} \left[ \frac{1}{\tau_0} - \frac{1}{\tau} \right] \right)$$

(17)

$$T = T_0 \left( \frac{\tau}{\tau_0} \right) \frac{d(T_c - T_f)}{\kappa_c T_c - \kappa_f T_f} \exp \left( \frac{f_0 d^2 (T_c - T_f)}{\kappa_c T_c - \kappa_f T_f} \left[ \frac{1}{\tau_0} - \frac{1}{\tau} \right] \right)$$

(18)
New viscous solutions in 1+3 dim

2nd solution: $\mu = 0$, $p = T\sigma/(1 + \kappa)$

Energy conservation:

$$\partial_\tau [\kappa(T)p] + T\sigma \frac{d}{\tau} = \frac{d^2}{\tau^2} \zeta$$

Since $dp = \sigma d\tau$:

$$\dot{T}\sigma \left[ \frac{T}{1+\kappa} \frac{\partial \kappa}{\partial T} + \kappa \right] + T\sigma \frac{d}{\tau} = \frac{d}{\tau}$$

In the right hand side let’s define $f(T)$:

$$f(T) = \frac{\zeta}{T\sigma}$$

With that, we have:

$$\left[ (1 + \kappa) \frac{d}{d\tau} \left( \frac{\kappa T}{1 + \kappa} \right) \right] \frac{\dot{T}}{T} + \frac{d}{\tau} = \frac{d^2}{\tau^2} f(T)$$

where $\kappa = \kappa(T)$ and $\dot{T} = \partial_\tau T$. We introduce the $g(T)$ function as

$$g(T) = (1 + \kappa) \frac{d}{d\tau} \left( \frac{\kappa T}{1 + \kappa} \right)$$

so the energy equation becomes:

$$\frac{\dot{T}}{T}g(T) + \frac{d}{\tau} = \frac{d^2}{\tau^2} f(T)$$

**T-dependent speed of sound**

Now for $\mu_B = 0$ case

**Bulk viscosity effect at late time:**

same as rescaling $T_{\text{init}}$

for a perfect fluid

**Case 2/A:** $f = f_0 =$ const and $\kappa(T) = \kappa_0 =$ const

The result is completely the same as the one we saw in case 1/A. The only difference is the definition of $f_0$:

$$f_0^{(2/A)} = \frac{\zeta}{T\sigma} \leftrightarrow f_0^{(1/A)} = \frac{\zeta}{\pi T} \quad (25)$$

**Case 2/B:** $f = f_0 =$ const, $\kappa = \kappa(T)$, but $g(T) =$ const

The result is completely the same as the one we saw in case 1/B. The only difference is the definition of $f_0$ and $g_0$:

$$g_0^{(2/B)} = \kappa_0 \leftrightarrow g_0^{(1/B)} = \frac{\kappa_0 T \tau - \kappa f T_f}{T_e - T_f} \quad (26)$$
Conclusions

Explicit solutions of a very difficult problem

New exact solutions
for Hubble flows
for arbitrary EOS with const e/p
with T dependent e/p
for temperature profiles
for multipole solutions*

shear effects cancel at late time
bulk viscosity heats up matter
Late time evolution $\sim$ perfect fluid, with rescaled $T_{\text{init}}$

A lot to do ...
E.g. rotating viscous solutions
Thank you for your attention

Questions and Comments?
Consider a 1+1 dimensional, finite, expanding fireball

Assume: $\Omega = \Omega(\eta_x)$

Notation T. Cs., G. Kasza, M. Csanád, Z. Jiang, \url{arXiv.org:1805.01427}
Hydro in Rindler coordinates, new sol

\[ \partial_v T^{\mu\nu} = 0, \]
\[ \partial_{\mu}(\sigma u^\mu) = 0, \]

Assumptions of TCs, Kasza, Csanád and Jiang, [arXiv.org:1805.01427]:

\[ \Omega = \Omega(\eta_x), \quad \varepsilon = \kappa \rho, \quad p = \frac{T\sigma}{1 + \kappa}. \]

For the entropy density, the continuity equation is solved.

From energy-momentum conservation, the Euler and temperature equations are obtained:

\[ \partial_{\eta_x} \Omega + \kappa (\tau \partial_\tau + \tanh(\Omega - \eta_x) \partial_{\eta_x}) \ln(T) = 0, \]
\[ \partial_{\eta_x} \ln(T) + \tanh(\Omega - \eta_x) (\tau \partial_\tau \ln(T) + \partial_{\eta_x} \Omega) = 0. \]
A New Family of Exact Solutions of Hydro

\[ \eta_x(H) = \Omega(H) - H, \]
\[ \Omega(H) = \frac{\lambda}{\sqrt{\lambda - 1}\sqrt{\kappa - \lambda}} \arctan \left( \sqrt{\frac{\kappa - \lambda}{\lambda - 1}} \tanh(H) \right), \]
\[ \sigma(\tau, H) = \sigma_0 \left( \frac{\tau_0}{\tau} \right)^\lambda \mathcal{V}_\sigma(s) \left[ 1 + \frac{\kappa - 1}{\lambda - 1} \sinh^2(H) \right]^{-\frac{\lambda}{2}}, \]
\[ T(\tau, H) = T_0 \left( \frac{\tau_0}{\tau} \right)^{\frac{\lambda}{\kappa}} \mathcal{T}(s) \left[ 1 + \frac{\kappa - 1}{\lambda - 1} \sinh^2(H) \right]^{-\frac{\lambda}{2\kappa}}, \]
\[ \mathcal{T}(s) = \frac{1}{\mathcal{V}_\sigma(s)}, \]
\[ s(\tau, H) = \left( \frac{\tau_0}{\tau} \right)^{\lambda - 1} \sinh(H) \left[ 1 + \frac{\kappa - 1}{\lambda - 1} \sinh^2(H) \right]^{-\lambda/2}. \]
A New Family of Exact Solutions of Hydro

New: not discovered before, as far as we know ... 

Family: For each positive scaling function \(\tau(s)\), a different solution, with same \(T_0, s_0, \kappa, \lambda\)

Not self-similar: Coordinate dependenc NOT on scaling variable \(s\) ONLY, but additional dependence on \(H = H(\eta_x)\) too.

Explicit and Exact: Fluid rapidity, temperature, entropy density explicitly given by formulas

\[
\begin{align*}
\eta_x(H) &= \Omega(H) - H, \\
\Omega(H) &= \frac{\lambda}{\sqrt{\lambda - 1} \sqrt{\kappa - \lambda}} \arctan\left(\sqrt{\frac{\kappa - \lambda}{\lambda - 1}} \tanh(H) \right), \\
\sigma(\tau, H) &= \sigma_0 \left(\frac{\tau_0}{\tau}\right)^\lambda \mathcal{V}_\sigma(s) \left[1 + \frac{\kappa - 1}{\lambda - 1} \sinh^2(H)\right]^{-\frac{1}{2}},
\end{align*}
\]

\[
\begin{align*}
T(\tau, H) &= T_0 \left(\frac{\tau_0}{\tau}\right)^\frac{\lambda}{\kappa} \mathcal{T}(s) \left[1 + \frac{\kappa - 1}{\lambda - 1} \sinh^2(H)\right]^{-\frac{1}{2\kappa}}, \\
\mathcal{T}(s) &= \frac{1}{\mathcal{V}_\sigma(s)}, \\
s(\tau, H) &= \left(\frac{\tau_0}{\tau}\right)^{\lambda - 1} \sinh(H) \left[1 + \frac{\kappa - 1}{\lambda - 1} \sinh^2(H)\right]^{-\lambda/2}.
\end{align*}
\]

New feature: Solution is given as \textit{parametric curves of} \(H\) in \(\eta_x\):

\[
(\eta_x(H), \Omega(H, \tau))
\]

Simplification, for now: limit the solution in \(\eta_x\) where parametric curves correspond to \textit{functions}
Illustration: results for $T$

Figure 1. Temperature maps in the forward light cone from our new, longitudinally finite solutions for $\kappa = \varepsilon/p = 3$ (left column) and for $\kappa = 7$ (right column) evaluated for the acceleration parameters $\lambda = 1.08$ (top row) corresponding to a broader rapidity distribution and for $\lambda = 1.16$ (bottom row) corresponding to a narrower rapidity distribution, approximately corresponding to heavy ion collisions at $\sqrt{s_{NN}} = 200$ GeV RHIC and $\sqrt{s_{NN}} = 2.76$ TeV LHC energies.
Figure 3. Fluid rapidity $\Omega$ maps in the forward light cone from our new, longitudinally finite solutions are shown for $\kappa = \varepsilon/p = 3$ (left column) and for $\kappa = 7$ (right column) evaluated for the acceleration parameters $\lambda = 1.01, 1.02, 1.04$ and 1.08 (from top to bottom rows) corresponding to nearly flat and with increasing $\lambda$, gradually narrowing rapidity distributions.
Perfect fluid hydrodynamics

Energy-momentum tensor:

\[ T_{\mu\nu} = w u_\mu u_\nu - p g_{\mu\nu} \]

\[ w = \varepsilon + p \]

\[ \partial_\nu T^{\mu\nu} = 0 \]

Relativistic Euler equation:

\[ w u^\nu \partial_\nu u^\mu = (g^{\mu\rho} - u^\mu u^\rho) \partial_\rho p \]

Energy conservation:

\[ w \partial_\mu u^\mu = -u^\mu \partial_\mu \varepsilon \]

Charge conservation:

\[ \sum \mu_i \partial_\mu (n_i w^\mu) = 0 \]

Consequence is entropy conservation:

\[ \partial_\mu (\sigma u^\mu) = 0. \]
New, exact solution with bulk viscosity

\[ u^\mu = \frac{x^\mu}{\tau} = \gamma \left( 1, \frac{\dot{X}}{X} r_x, \frac{\dot{Y}}{Y} r_y, \frac{\dot{Z}}{Z} r_z \right), \]

\[ \frac{de}{d\tau} + \frac{d(e + p)}{\tau} - \zeta \left( \frac{d}{\tau} \right)^2 = 0. \]

\[ e = e_0 \left( \frac{\tau_0}{\tau} \right)^{(1 + \frac{1}{\kappa}) d} + \zeta \frac{d}{\tau} \left( 1 + \frac{1}{\kappa} \right) d - 1, \]

\[ v = \frac{r}{t} \quad \text{or} \quad u^\mu = \frac{x^\mu}{\tau}, \]

\[ n = n_0 \left( \frac{\tau_0}{\tau} \right)^3 \mathcal{V}(s), \]

\[ p = p_0 \left( \frac{\tau_0}{\tau} \right)^3 \left( 1 + \frac{1}{\kappa} \right) + \zeta \frac{3}{\kappa \tau} \frac{3}{\left( 1 + \frac{1}{\kappa} \right)^3 - 1}, \]

\[ T = T_0 \left( \frac{\tau_0}{\tau} \right)^{\frac{3}{3}} + \zeta \frac{3}{\kappa n_0 \tau} \frac{3}{\tau_0} \frac{3}{\left( 1 + \frac{1}{\kappa} \right)^3 - 1} \mathcal{T}(s), \]

\[ \theta = \partial_\mu u^\mu = \frac{d}{\tau}, \]

\[ D = u^\mu \partial_\mu = \frac{\partial}{\partial \tau}. \]

\[ \tau = \sqrt{t^2 - r^2}, \]

\[ \eta_s = \frac{1}{2} \ln \left( \frac{t + r_z}{t - r_z} \right). \]