A new family of exact solutions of relativistic viscous hydrodynamics

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Introduction and motivation

Viscous Relativistic Hydro

Multipole solutions with temperature dependent speed of sound

Family of solutions with temperature profile

Outlook

Summary

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Context

Renowned exact solutions, reviewed in arXiv:1805.01427

- **Landau-Khalatnikov solution: dn/dy ~ Gaussian**
- **Hwa solution (1974) – Bjorken: same solution + ε_0 (1983)**
- **Chiu, Sudarshan and Wang: plateaux, Wong: Landau revisited**
- **Revival of interest:** Zimányi, Bondorf, Garpman (1978)
  - Buda-Lund model + exact solutions (1994-96)
  - Biró, Karpenko, Sinyukov, Pratt (2007)
  - Bialas, Janik, Peschanski, Borsch+Zhdanov (2007)
  - CsT, Csanád, Nagy (2007-2008)
  - CsT, Csernai, Grassi, Hama, Kodama (2004)
  - Gubser (2010-11)
  - Hatta, Noronha, Xiao (2014-16)
  - CsT, Kasza, Csanad, Jiang (2017-18)

New simple solutions → **Evaluation of measurables**

Rapidity distribution → **Advanced initial energy density (Kasza)**

Viscous solutions → **Theorems**

Goal

Need for solutions that are:

- explicit
- simple
- accelerating
- relativistic
- viscous
- realistic / compatible with the data:
  - lattice QCD EoS
  - ellipsoidal symmetry
  - multipole solution (spectra, v_2, v_3, v_4 ..., HBT)
  - finite dn/dy

Some of our recent new exact hydro solutions:

finite in y: CsT, Kasza, Csanád, Jiang (CKCJ): arXiv.org:1805.01427

viscous: Csanád, Nagy, Jiang, CsT: arxiv.org:1909.02498

This has been generalized to T-dependent speed of sound

to multipole solutions and to spatially dependent T profiles

Self-similar solutions for perfect fluid

Publication (for example):

T. Cs, L.P.Csernai, Y. Hama, T. Kodama, Heavy Ion Phys. A 21 (2004) 73

3D spherically symmetric HUBBLE flow:

$$u^\mu = \frac{x^\mu}{\tau}$$

No acceleration:

$$u^\mu \partial_\mu u_\nu = 0.$$

Define a scaling variable for self-similarly expanding ellipsoids:

$$s = \frac{r_x^2}{\dot{X}_0^2 t^2} + \frac{r_y^2}{\dot{Y}_0^2 t^2} + \frac{r_z^2}{\dot{Z}_0^2 t^2}$$

EoS: (massive)

ideal gas

$$\epsilon = mn + \kappa p\,, \qquad p = nT.$$

$$\varepsilon_Q = m_Q n_Q + \lambda_\varepsilon n_Q T + B, \qquad p_Q = \lambda_p n_Q T - B,$$

$$n(t,\mathbf{r}) = n_0 \left(\frac{\tau_0}{\tau}\right)^3 \mathcal{V}(s) \qquad T(t,\mathbf{r}) = T_0 \left(\frac{\tau_0}{\tau}\right)^{3/\kappa} \frac{1}{\mathcal{V}(s)} \Rightarrow p(t,\mathbf{r}) = p_0 \left(\frac{\tau_0}{\tau}\right)^{3+3/\kappa}$$

Scaling function $\mathcal{V}(s)$ can be chosen freely.

Shear and bulk viscous corrections in NR limit: known analytically.

New, exact solutions with viscosity

$$T^{\mu\nu} = eu^\mu u^\nu - p\Delta^{\mu\nu} + \Pi^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu},$$

$$\Pi^{\mu\nu} = \pi^{\mu\nu} + \Delta^{\mu\nu}\Pi,$$

$$e = \kappa p, \qquad p = nT,$$

$$\partial_\mu(N^\mu) = 0.$$

$$De = -(e+p+\Pi)\theta + \sigma_{\mu\nu}\pi^{\mu\nu},$$

$$(e+p+\Pi)Du^\alpha = \nabla^\alpha(p+\Pi) - \Delta^\alpha_\nu u_\mu D\pi^{\mu\nu} - \Delta^\alpha_\nu \nabla_\mu \pi^{\mu\nu},$$

$$S = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2},$$

$$u^\mu = \frac{x^\mu}{\tau} = \gamma\left(1, \frac{\dot X}{X}r_x, \frac{\dot Y}{Y}r_y, \frac{\dot Z}{Z}r_z\right),$$

$$\theta = \partial_\mu u^\mu = \frac{d}{\tau},$$

$$D = u^\mu \partial_\mu = \frac{\partial}{\partial\tau}.$$

$$\tau = \sqrt{t^2 - r^2}$$

$$\sigma_{\mu\nu}\pi^{\mu\nu} = 0,$$

ζ/s η/s

$$\eta_s = \frac{1}{2}\ln\left(\frac{t+r_z}{t-r_z}\right)$$

Csanád, Nagy, Jiang, CsT:
arxiv.org:1909.02498

Exact solution with viscosity

Csanád, Nagy, Jiang, CsT:
arxiv.org:1909.02498

$$u^\mu = \frac{x^\mu}{\tau} = \gamma\left(1, \frac{\dot X}{X} r_x, \frac{\dot Y}{Y} r_y, \frac{\dot Z}{Z} r_z\right),$$

$$\theta = \partial_\mu u^\mu = \frac{d}{\tau},$$

$$D = u^\mu \partial_\mu = \frac{\partial}{\partial \tau}.$$

$$\frac{de}{d\tau} + \frac{d(e+p)}{\tau} - \zeta\left(\frac{d}{\tau}\right)^2 = 0.$$

ζ/s

η/s

$$\tau = \sqrt{t^2 - r^2}$$

$$\eta_s = \frac{1}{2}\ln\left(\frac{t+r_z}{t-r_z}\right)$$

Case A: No conserved n, and constant ζ:

$$\zeta = \zeta_0 \text{ (const)}, \quad \varepsilon = \kappa p, \quad p = p_0 (T/T_0)^{\kappa+1}.$$

Case B: With conserved n, and constant ζ:

$$\zeta = \zeta_0 \text{ (const)}, \quad \varepsilon = \kappa p, \quad p = nT.$$

Case C: No conserved n, and $\zeta \propto s$:

$$\zeta = \zeta_0 (T/T_0)^\kappa, \quad \varepsilon = \kappa p, \quad p = p_0 (T/T_0)^{\kappa+1}.$$

Case D: With conserved n, and $\zeta/n =$const:

$$\zeta = \zeta_0 (n/n_0), \quad \varepsilon = \kappa p, \quad p = nT.$$

Case E: With conserved n, and "$\zeta \propto s$":

$$\zeta = \zeta_0 (T/T_0)^\kappa, \quad \varepsilon = \kappa p, \quad p = nT.$$

Case	ζ	T definition	asymptotics
(A)	ζ_0 (const)	$p = p_0 (T/T_0)^{\kappa+1}$	physical
(B)	ζ_0 (const)	$p = nT$	$T(\tau) \to \infty$
(C)	$\zeta_0 (T/T_0)^\kappa$	$p = p_0 (T/T_0)^{\kappa+1}$	physical
(D)	$\zeta_0 (n/n_0)$	$p = nT$	physical
(E)	$\zeta_0 (T/T_0)^\kappa$	$p = nT$	conditionally physical

Case A:

$$p(\tau) = \left[p_0 - \frac{d^2}{(\kappa+1)d - \kappa}\frac{\zeta_0}{\tau_0}\right]\left(\frac{\tau_0}{\tau}\right)^{d\frac{\kappa+1}{\kappa}} + \\ + \frac{d^2}{(\kappa+1)d - \kappa}\frac{\zeta_0}{\tau}.$$

Case C:

$$p(\tau) = p_0\left\{\left(1 + \frac{d^2}{(\kappa+1)(\kappa-d)}\frac{\zeta_0}{p_0 \tau_0}\right)\left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa}} - \\ - \frac{d^2}{(\kappa+1)(\kappa-d)}\frac{\zeta_0}{p_0}\frac{1}{\tau}\right\}^{\kappa+1},$$

Case D:

$$p(\tau) = \left[p_0 + \frac{d^2}{\kappa-d}\frac{\zeta_0}{\tau_0}\right]\left(\frac{\tau_0}{\tau}\right)^{\frac{\kappa+1}{\kappa}d} - \frac{d^2}{\kappa-d}\frac{\zeta_0}{\tau_0}\frac{\tau_0^{d+1}}{\tau^{d+1}},$$

CNJC viscous solutions in 1+3 dim

Csanád, Nagy, Jiang, CsT:
arxiv.org:1909.02498

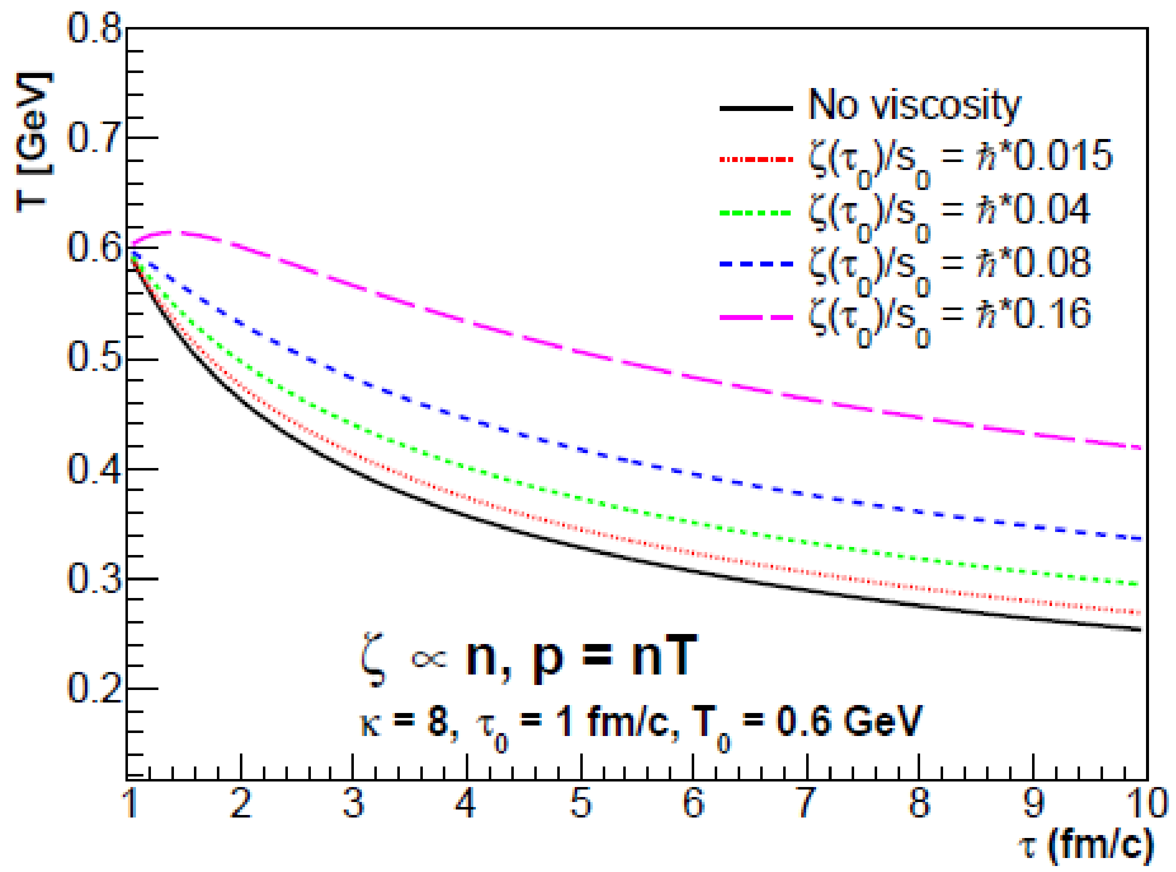

FIG. 3: (Color online) Temperature evolution in Case D: non-vanishing conserved n, and $\zeta = \zeta_0(n/n_0)$, proportional to n.

Bulk viscosity important at late stage, heats up
Shear viscosity effects cancel for asymptotically Hubble flows

New viscous solutions in 1+3 dim

CsT and G. Kasza

1st solution: $\mu \neq 0$, $p = nT$

Energy conservation:

$$\partial_\tau \left[\kappa(T) n T\right] + \left[1 + \kappa(T)\right] n T \frac{d}{\tau} = \frac{d^2}{\tau^2}\zeta$$

Using the continuity equation:

$$\frac{1}{T}\partial_\tau \left(\kappa T\right) + \frac{d}{\tau} = \frac{d^2}{\tau^2}\frac{\zeta}{nT}$$

In the right hand side let's define $f(T)$:

$$f(T) = \frac{\zeta}{nT}$$

With that, we have:

$$\frac{\dot{T}}{T}\frac{d}{dT}\left(\kappa T\right) + \frac{d}{\tau} = \frac{d^2}{\tau^2} f(T)$$

where $\kappa = \kappa(T)$ and $\dot{T} = \partial_\tau T$. We introduce the $g(T)$ function as

$$g(T) = \frac{d}{dT}\left(\kappa T\right)$$

so the energy equation becomes:

$$\frac{\dot{T}}{T} g(T) + \frac{d}{\tau} = \frac{d^2}{\tau^2} f(T)$$

T-dependent speed of sound

New viscous solutions in 1+3 dim

Case 1/A: $f = f_0$ =const and $\kappa(T) = \kappa_0$ =const

$\kappa = \kappa_0$ is constant, so $g(T)$ is:

$$g(T) = \kappa_0$$

In this case the energy equation is:

$$\frac{\dot{T}}{T}\kappa_0 + \frac{d}{\tau} = \frac{d^2}{\tau^2} f_0$$

By integration we obtain:

$$T = T_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \exp\left(\frac{f_0 d^2}{\kappa_0}\left[\frac{1}{\tau_0} - \frac{1}{\tau}\right]\right)$$

T-dependent speed of sound

Bulk viscosity effect at late time:
same as rescaling T_{init}
for a perfect fluid

Case 1/B: $f = f_0$ =const, $\kappa = \kappa(T)$, but $g(T)$ =const

Let $g(T) = g_0$ be:

$$g_0 = \frac{\kappa_c T_c - \kappa_f T_f}{T_c - T_f} \tag{15}$$

The energy equation is the same as the one in Case 1/A, only κ_0 is substituted by g_0:

$$\frac{\dot{T}}{T} g_0 + \frac{d}{\tau} = \frac{d^2}{\tau^2} f_0 \tag{16}$$

So the solution is:

$$T = T_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{g_0}} \exp\left(\frac{f_0 d^2}{g_0}\left[\frac{1}{\tau_0} - \frac{1}{\tau}\right]\right) \tag{17}$$

$$T = T_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{d(T_c - T_f)}{\kappa_c T_c - \kappa_f T_f}} \exp\left(\frac{f_0 d^2 (T_c - T_f)}{\kappa_c T_c - \kappa_f T_f}\left[\frac{1}{\tau_0} - \frac{1}{\tau}\right]\right) \tag{18}$$

New viscous solutions in 1+3 dim

T-dependent speed of sound

Bulk viscosity effect at late time:
same as rescaling T_{init}
for a perfect fluid

Case 1/B: $f = f_0$ =const, $\kappa = \kappa(T)$, but $g(T)$ =const

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The energy equation is the same as the one in Case 1/A, only κ_0 is substituted by g_0:

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$$T = T_0 \left(\frac{\tau_0}{\tau} \right)^{\frac{d(T_c - T_f)}{\kappa_c T_c - \kappa_f T_f}} \exp\left(\frac{f_0 d^2 (T_c - T_f)}{\kappa_c T_c - \kappa_f T_f} \left[\frac{1}{\tau_0} - \frac{1}{\tau} \right] \right) \tag{18}$$

New viscous solutions in 1+3 dim

2nd solution: $\mu = 0$, $p = T\sigma/(1+\kappa)$

Energy conservation:

$$\partial_\tau\left[\kappa(T)p\right] + T\sigma\frac{d}{\tau} = \frac{d^2}{\tau^2}\zeta$$

Since $dp = \sigma dT$:

$$\dot{T}\sigma\left[\frac{T}{1+\kappa}\frac{\partial\kappa}{\partial T} + \kappa\right] + T\sigma\frac{d}{\tau} = \frac{d^2}{\tau^2}\zeta \longrightarrow \frac{\dot{T}}{T}\left[\frac{T}{1+\kappa}\frac{\partial\kappa}{\partial T} + \kappa\right] + \frac{d}{\tau} = \frac{d^2}{\tau^2}\frac{\zeta}{T\sigma}$$

In the right hand side let's define $f(T)$:

$$f(T) = \frac{\zeta}{T\sigma}$$

T-dependent speed of sound

Now for μ_B = 0 case

Bulk viscosity effect at late time: same as rescaling T_{init} for a perfect fluid

With that, we have:

$$\left[(1+\kappa)\frac{d}{dT}\left(\frac{\kappa T}{1+\kappa}\right)\right]\frac{\dot{T}}{T} + \frac{d}{\tau} = \frac{d^2}{\tau^2}f(T)$$

where $\kappa = \kappa(T)$ and $\dot{T} = \partial_\tau T$. We introduce the $g(T)$ function as

$$g(T) = (1+\kappa)\frac{d}{dT}\left(\frac{\kappa T}{1+\kappa}\right)$$

so the energy equation becomes:

$$\frac{\dot{T}}{T}g(T) + \frac{d}{\tau} = \frac{d^2}{\tau^2}f(T)$$

Case 2/A: $f = f_0$ =const and $\kappa(T) = \kappa_0$ =const

The result is completely the same as the one we saw in case 1/A. The only difference is the definition of f_0:

$$f_0^{(2/A)} = \frac{\zeta}{T\sigma} \longleftrightarrow f_0^{(1/A)} = \frac{\zeta}{nT} \tag{25}$$

Case 2/B: $f = f_0$ =const, $\kappa = \kappa(T)$, but $g(T)$ =const

The result is completely the same as the one we saw in case 1/B. The only difference is the definition of f_0 and g_0:

$$g_0^{(2/B)} = \kappa_q \longleftrightarrow g_0^{(1/B)} = \frac{\kappa_c T_c - \kappa_f T_f}{T_c - T_f}, \tag{26}$$

Conclusions

Explicit solutions of a very difficult problem

New exact solutions
for Hubble flows
for arbitrary EOS with const e/p
with T dependent e/p
for temperature profiles
for multipole solutions*

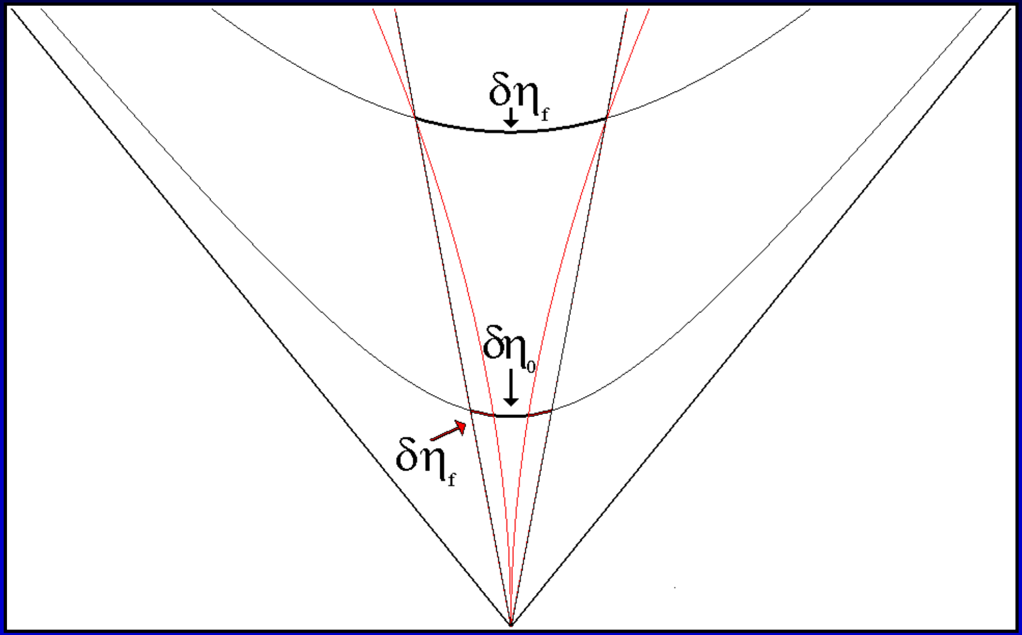
shear effects cancel at late time
bulk viscosity heats up matter
Late time evolution ~ perfect fluid, with rescaled T_{init}

A lot to do ...
E.g. rotating viscous solutions

Thank you for your attention

Questions and Comments ?

Auxiliary variables: η_x, τ, Ω , η_p, y



$$\tau = \sqrt{t^2 - r_z^2},$$

$$\eta_x = \frac{1}{2}\ln\left(\frac{t + r_z}{t - r_z}\right),$$

$$\Omega = \frac{1}{2}\ln\left(\frac{1 + v_z}{1 - v_z}\right).$$

$$u^\mu = (\cosh(\Omega), \sinh(\Omega)),$$

$$v_z = \tanh(\Omega).$$

$$\eta_p = \frac{1}{2}\ln\left(\frac{p + p_z}{p - p_z}\right),$$

$$y = \frac{1}{2}\ln\left(\frac{E + p_z}{E - p_z}\right),$$

Consider a 1+1 dimensional, finite, expanding fireball

Assume: $\Omega = \Omega(\eta_x)$

Notation T. Cs., G. Kasza, M. Csanád, Z. Jiang, arXiv.org:1805.01427

Hydro in Rindler coordinates, new sol

$$\partial_\nu T^{\mu\nu} = 0,$$
$$\partial_\mu(\sigma u^\mu) = 0,$$

Assumptions of TCs, Kasza, Csanád and Jiang, arXiv.org:1805.01427 :

$$\Omega = \Omega(\eta_x), \qquad \varepsilon = \kappa p, \qquad p = \frac{T\sigma}{1+\kappa}.$$

For the entropy density, the continuity equation is solved.

From energy-momentum conservation, the
Euler and temperature equations are obtained:

$$\begin{aligned} \partial_{\eta_x}\Omega + \kappa\left(\tau\partial_\tau + \tanh(\Omega-\eta_x)\partial_{\eta_x}\right)\ln(T) &= 0, \\ \partial_{\eta_x}\ln(T) + \tanh(\Omega-\eta_x)\left(\tau\partial_\tau\ln(T) + \partial_{\eta_x}\Omega\right) &= 0. \end{aligned}$$

A New Family of Exact Solutions of Hydro

$$\eta_x(H) = \Omega(H) - H,$$

$$\Omega(H) = \frac{\lambda}{\sqrt{\lambda-1}\sqrt{\kappa-\lambda}}\arctan\left(\sqrt{\frac{\kappa-\lambda}{\lambda-1}}\tanh(H)\right),$$

$$\sigma(\tau,H) = \sigma_0\left(\frac{\tau_0}{\tau}\right)^{\lambda}\mathcal{V}_\sigma(s)\left[1+\frac{\kappa-1}{\lambda-1}\sinh^2(H)\right]^{-\frac{\lambda}{2}},$$

$$T(\tau,H) = T_0\left(\frac{\tau_0}{\tau}\right)^{\frac{\lambda}{\kappa}}\mathcal{T}(s)\left[1+\frac{\kappa-1}{\lambda-1}\sinh^2(H)\right]^{-\frac{\lambda}{2\kappa}},$$

$$\mathcal{T}(s) = \frac{1}{\mathcal{V}_\sigma(s)},$$

$$\boxed{u_\mu\partial^\mu s = 0.}$$

$$s(\tau,H) = \left(\frac{\tau_0}{\tau}\right)^{\lambda-1}\sinh(H)\left[1+\frac{\kappa-1}{\lambda-1}\sinh^2(H)\right]^{-\lambda/2}.$$

A New Family of Exact Solutions of Hydro

$$
\begin{aligned}
\eta_x(H) &= \Omega(H) - H, \\
\Omega(H) &= \frac{\lambda}{\sqrt{\lambda-1}\sqrt{\kappa-\lambda}} \arctan\left(\sqrt{\frac{\kappa-\lambda}{\lambda-1}} \tanh(H)\right), \\
\sigma(\tau, H) &= \sigma_0 \left(\frac{\tau_0}{\tau}\right)^{\lambda} \mathcal{V}_\sigma(s) \left[1 + \frac{\kappa-1}{\lambda-1} \sinh^2(H)\right]^{-\frac{\lambda}{2}}, \\
T(\tau, H) &= T_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{\lambda}{\kappa}} \mathcal{T}(s) \left[1 + \frac{\kappa-1}{\lambda-1} \sinh^2(H)\right]^{-\frac{\lambda}{2\kappa}}, \\
\mathcal{T}(s) &= \frac{1}{\mathcal{V}_\sigma(s)}, \\
s(\tau, H) &= \left(\frac{\tau_0}{\tau}\right)^{\lambda-1} \sinh(H) \left[1 + \frac{\kappa-1}{\lambda-1} \sinh^2(H)\right]^{-\lambda/2}.
\end{aligned}
$$

$$\boxed{u_\mu \partial^\mu s = 0.}$$

New feature:
Solution is given
as *parametric curves of H* in η_x :
$(\eta_x(H), \Omega(H, \tau))$

Simlification, for now:
limit the solution in η_x
where parametric curves
correspond to *functions*

New: not discovered before,
as far as we know …

Family:
For each positive scaling
function $\mathcal{T}(s)$, a different
solution, with same T_0, s_0, κ, λ

Not self-similar:
Coordinate dependenc NOT on
scaling variable s ONLY,
butadditional dependence on
$H = H(\eta_x)$ too.

Explicit and Exact:
Fluid rapidity, temperature,
entropy density explicitly given
by formulas

Illustration: results for T

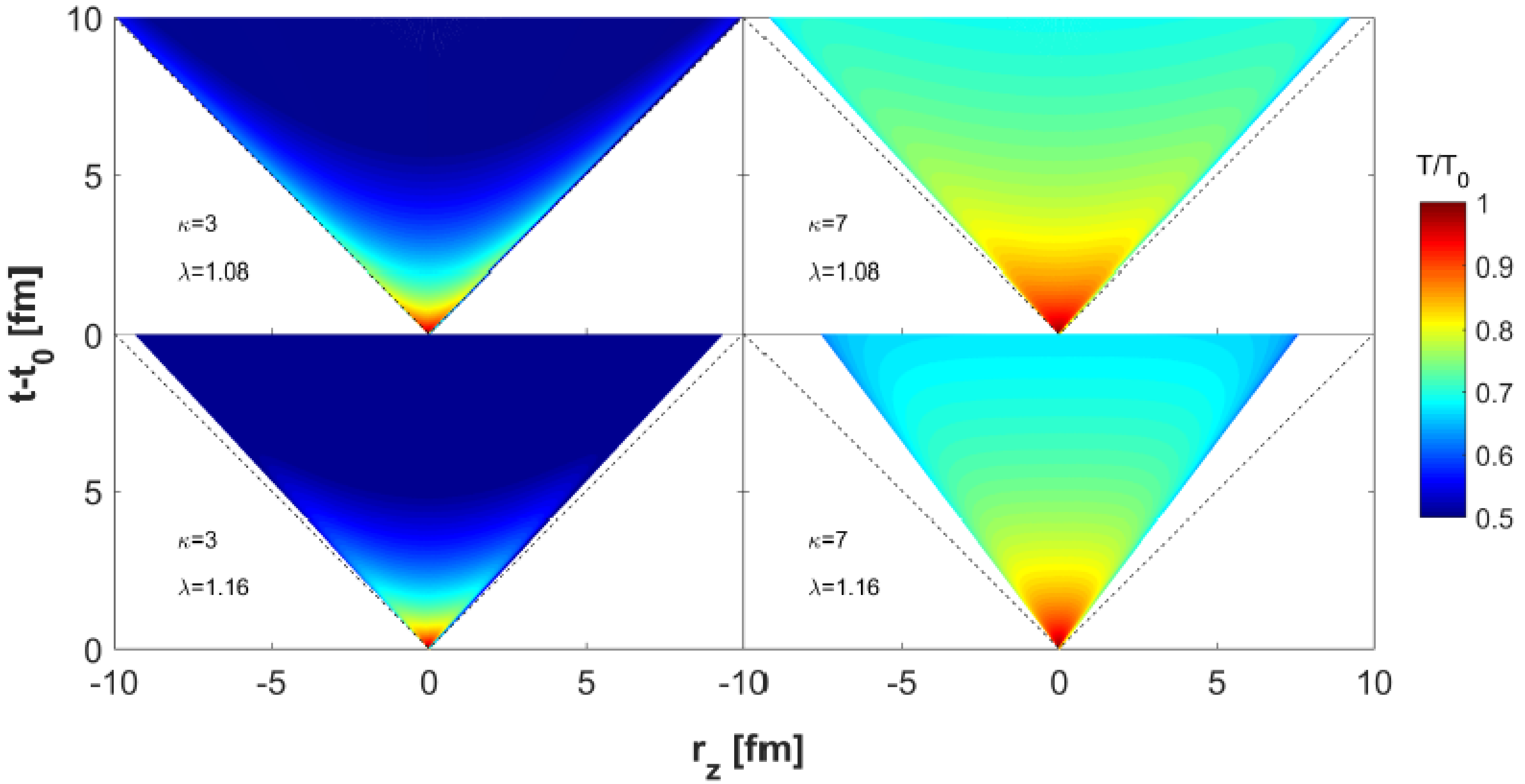


Figure 1. Temperature maps in the forward light cone from our new, longitudinally finite solutions for $\kappa = \varepsilon / p = 3$ (left column) and for $\kappa = 7$ (right column) evaluated for the acceleration parameters $\lambda = 1.08$ (top row) corresponding to a broader rapidity distribution and for $\lambda = 1.16$ (bottom row) corresponding to a narrower rapidity distribution, approximately corresponding to heavy ion collisions at $\sqrt{s_{NN}} = 200$ GeV RHIC and $\sqrt{s_{NN}} = 2.76$ TeV LHC energies.

Illustration: results for fluid rapidity Ω

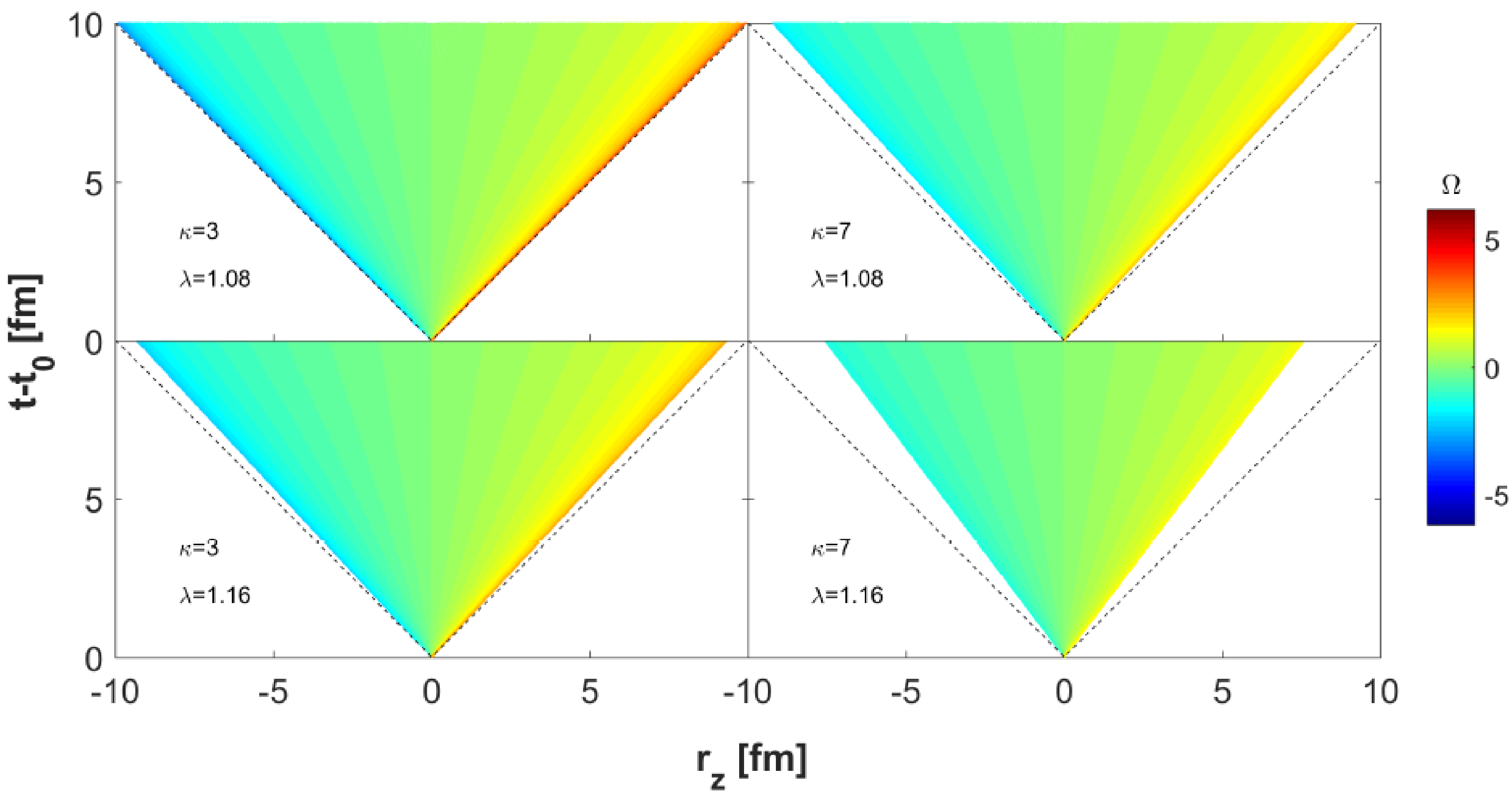


Figure 3. Fluid rapidity Ω maps in the forward light cone from our new, longitudinally finite solutions are shown for for $\kappa = \varepsilon/p = 3$ (left column) and for $\kappa = 7$ (right column) evaluated for the acceleration parameters $\lambda = 1.01, 1.02, 1.04$ and 1.08 (from top to bottom rows) corresponding to nearly flat and with increasing λ, gradually narrowing rapidity distributions.

Perfect fluid hydrodynamics

Energy-momentum tensor:

$$T_{\mu\nu} = w u_\mu u_\nu - p g_{\mu\nu}$$ $$w = \varepsilon + p$$

$$\partial_\nu T^{\mu\nu} = 0$$

Relativistic Euler equation:

$$w u^\nu \partial_\nu u^\mu = \left(g^{\mu\rho} - u^\mu u^\rho\right) \partial_\rho p$$

Energy conservation:

$$w \partial_\mu u^\mu = -u^\mu \partial_\mu \varepsilon$$

Charge conservation:

$$\sum \mu_i \partial_\mu \left(n_i u^\mu\right) = 0$$

Consequence is entropy conservation:

$$\partial_\mu \left(\sigma u^\mu\right) = 0.$$

New, exact solution with bulk viscosity

$$u^\mu = \frac{x^\mu}{\tau} = \gamma\left(1, \frac{\dot X}{X} r_x, \frac{\dot Y}{Y} r_y, \frac{\dot Z}{Z} r_z\right),$$

$$\frac{de}{d\tau} + \frac{d(e+p)}{\tau} - \zeta\left(\frac{d}{\tau}\right)^2 = 0.$$

$$e = e_0\left(\frac{\tau_0}{\tau}\right)^{\left(1+\frac{1}{\kappa}\right)d} + \zeta\frac{d}{\tau}\frac{d}{\left(1+\frac{1}{\kappa}\right)d - 1},$$

ζ/s

η/s

$$\mathbf{v} = \frac{\mathbf{r}}{t} \quad \text{or} \quad u^\mu = \frac{x^\mu}{\tau},$$

$$n = n_0\left(\frac{\tau_0}{\tau}\right)^3 \mathcal{V}(s),$$

$$p = p_0\left(\frac{\tau_0}{\tau}\right)^{3\left(1+\frac{1}{\kappa}\right)} + \frac{\zeta}{\kappa}\frac{3}{\tau}\frac{3}{\left(1+\frac{1}{\kappa}\right)3 - 1},$$

$$T = T_0\left(\frac{\tau_0}{\tau}\right)^{\frac{3}{\kappa}} + \frac{\zeta}{\kappa n_0}\frac{3}{\tau}\left(\frac{\tau}{\tau_0}\right)^3\frac{3}{\left(1+\frac{1}{\kappa}\right)3 - 1}\mathcal{T}(s),$$

$$\theta = \partial_\mu u^\mu = \frac{d}{\tau},$$

$$D = u^\mu\partial_\mu = \frac{\partial}{\partial\tau}.$$

$$\tau = \sqrt{t^2 - r^2}$$

$$\eta_s = \frac{1}{2}\ln\left(\frac{t+r_z}{t-r_z}\right)$$