# Stability and thermodynamic consistency in dissipative relativistic hydrodynamics

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*"The law that entropy always increases holds, I think, the supreme position among the laws of Nature. If someone points out to you that your pet theory of the universe is in disagreement with Maxwell's equations - then so much the worse for Maxwell's equations. If it is found to be contradicted by observation - well, these experimentalists do bungle things sometimes. But if your theory is found to be against the Second Law of Thermodynamics I can give you no hope; there is nothing for it to collapse in deepest humiliation."*

Arthur Eddington, New Pathways in Science

Question :
What is local equilibrium ?

Answer:
A flow-frame with thermodynamics

# Relativistic dissipative fluids

- Thermodynamics is about stability.
- Local rest frame: time and space separation.
  Fow-frames: Eckart, Landau-Lifshitz, thermo.

**Problems**

1. Thermoframe ($\beta^\mu = \beta u^\mu$) is used together with other ones e.g. Eckart.
2. Consistency with temperature of moving bodies. The enigma of covariant thermodynamics.
3. What is ideal fluid? A spacelike freedom.
4. Persistent instabilities. (presentation of Michal Heller).

# Fluid basics

$$T^{\mu\nu} = eu^\mu u^\nu + q^\mu u^\nu + u^\mu q^\nu + P^{\mu\nu},$$
$$N^\mu = nu^\mu + j^\mu,$$
$$T^{\mu\nu} = \begin{pmatrix} e & q^j \\ q^i & P^{ij} \end{pmatrix}, \quad N^a = \begin{pmatrix} n \\ j^i \end{pmatrix}$$

$P^{\mu\nu} u_\mu = 0^\nu, \quad q^\mu u_\mu = 0, \quad j^\mu u_\mu = 0.$
$diag(1, -1, -1, -1)$, $\Delta^{\mu\nu} = \delta^{\mu\nu} - u^\mu u^\nu$.

- arbitrary four-velocity
- energy-momentum density
- particle number density
- momentum density and energy flux

Energy-momentum conservation:

$$\partial_\nu T^{\mu\nu} = u^\mu(\dot{e} + e\partial_\nu u^\nu + \partial_\nu q^\nu + u_\nu \dot{q}^\nu - P^{\mu\nu}\partial_\nu u_\mu) + e\dot{u}^\mu + \Delta^{\mu\nu}\dot{q}_\nu + q^\mu \partial_\nu u^\nu + q^\nu \partial_\nu u^\mu + \Delta^{\mu\nu}\partial_\nu P^{\mu\nu} = 0.$$

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# Equilibrium and non-equilibrium

Thermodynamics: time-space separation and stability

$$\partial_\mu S^\mu = \dot{s} + s\partial_\mu u^\mu + \partial_\mu J^\mu \geq 0$$

u-EOS and entropy flux:

$$ds = \beta de - \alpha dn, \qquad \beta = \frac{1}{T}, \quad \alpha = \frac{\mu}{T} \qquad J^\mu = \beta q^\mu (!?)$$

$$\beta(e+p) = s + \mu n. \quad \text{Euler relation}$$

u-energy is not conserved:

$$\dot{e} + e\partial_\nu u^\nu + \partial_\nu q^\nu + u_\nu \dot{q}^\nu - P^{\mu\nu}\partial_\nu u_\mu = 0$$

It is a constraint:

$$\beta\dot{e} + s\partial_\mu u^\mu + \partial_\mu\left(\beta q^\mu\right) = \ldots =$$
$$(s-\beta e)\partial_\nu u^\nu + q^\mu \partial_\mu \beta - \beta u_\nu \dot{q}^\nu - \beta P^{\mu\nu}\partial_\nu u^\mu =$$
$$\beta(P^{\mu\nu} + \beta p \Delta^{\mu\nu})\partial_\nu u_\mu + \boxed{q^\mu}(\partial_\mu \beta + \underline{\beta \dot{u}_\mu}) \geq 0$$

# Complete Eckart theory

Solving the inequality:

$$\boxed{\beta(P^{\mu\nu} + \beta p\Delta^{\mu\nu})\partial_\nu u_\mu + q^\mu(\partial_\mu\beta + \beta\dot{u}_\mu) \geq 0}$$

$$\begin{aligned} P^{\mu\nu} + \beta p\Delta^{\mu\nu} &= \eta_{shear}\Delta^\nu_\gamma\partial^{<\gamma}u^{\mu>} + \eta_{bulk}\partial_\gamma u^\gamma\Delta^{\mu\nu} \\ q^\mu &= \lambda\Delta^{\mu\nu}(\partial_\nu\beta + \beta\dot{u}_\nu) \end{aligned}$$

$$\dot{e} + e\partial_\nu u^\nu + \partial_\nu q^\nu + u_\nu\dot{q}^\nu - P^{\mu\nu}\partial_\nu u_\mu = 0$$

$$e\dot{u}^\mu + \Delta^{\mu\nu}\dot{q}_\nu + q^\mu\partial_\nu u^\nu + q^\nu\partial_\nu u^\mu + \Delta^{\mu\nu}\partial_\nu P^{\mu\nu} = 0^\mu.$$

Closed system, violent instability of the static equilibrium:
$u^\mu = 0$, $q^\mu = 0$, $e = const.$, $P^{\mu\nu} = -\beta p\Delta^{\mu\nu} = const.$ .

- Eliminating acceleration term? Does not help.
- Israel-Stewart: suppression with price. $q^\mu$ dependent EOS
- Landau-Lifshitz frame $q^\mu = 0^\mu$? Stability!

# More covariant thermodynamics

Eckart four-entropy:

$$S^\mu = s(e)u^\mu + \beta q^\mu = s(e)u^\mu + \partial_e s(e) q^\mu$$

Temperature is a four-vector. Energy-momentum density vector should be a state variable, $s(E^\mu)$:

$$ds = Y_\mu dE^\mu, \qquad Y_\mu = \beta(u_\mu + w_\mu),$$
$$S^\mu - Y_\nu T^{\nu\mu} = pY^\mu \quad \text{four-Euler relation}$$

$$ds = \beta(de + w_\mu dq^\mu + (ew_\mu - q_\mu)du^\mu)$$
$$s = \beta(p + e + w_\mu q^\mu), \qquad J^\mu = \beta(q^\mu + w_\nu P^{\mu\nu} + w^\mu p)$$

Thermoframe (EOS): $w_\mu = 0_\mu$

# Thermodynamics in thermoframe: $w_\mu = 0$

Simplified thermodynamic relations:

$$ds = \beta de + \beta u_\mu dq^\mu$$
$$s = \beta(p+e), \qquad J^\mu = \beta q^\mu$$

Energy balance:

$$\dot{e} + e\partial_\nu u^\nu + \partial_\nu q^\nu + u_\nu \dot{q}^\nu - P^{\mu\nu}\partial_\nu u_\mu = 0$$

It is a constraint:

$$\beta(\dot{e} + u_\nu \dot{q}^\nu) + s\partial_\mu u^\mu + \partial_\mu\left(\beta q^\mu\right) = \ldots =$$
$$(s - \beta e)\partial_\nu u^\nu + q^\mu \partial_\mu \beta - \beta P^{\mu\nu}\partial_\nu u^\mu =$$

$$\boxed{\beta(P^{\mu\nu} + \beta p \Delta^{\mu\nu})\partial_\nu u_\mu + q^\mu \partial_\mu \beta \geq 0}$$

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$$\boxed{\beta(P^{\mu\nu} + \beta p \Delta^{\mu\nu})\partial_\nu u_\mu + q^\mu \partial_\mu \beta \geq 0}$$

It is not enough ...

# Thermodynamics in kinetic frame: $w_\mu = q^\mu/h$

Momentum balance with particle number conservation requires further modifications.
Velocity may be a state variable, too, $s(E^\mu, u^\mu)$:

$$ds = Y_\mu dE^\mu + p w_\mu du^\mu, \qquad Y_\mu = \beta(u_\mu + w_\mu)$$

$$ds = \beta(de + w_\mu dq^\mu + (h w_\mu - q_\mu) du^\mu)$$

Kinetic frame: $w_\mu = q_\mu/h$

$$ds = \beta de + \beta \frac{q_\mu}{h} dq^\mu$$

Energy and momentum balances are both constraints for the entropy production:

$$\boxed{\beta(P^{\mu\nu} + \beta p \Delta^{\mu\nu})\partial_\nu u_\mu + q^\mu \left(\partial_\mu \beta + \frac{\beta}{h}(h\dot{u}_\mu + \dot{q}^\mu + q^\mu \partial_\nu u^\nu)\right) \geq 0}$$

It is stable.

# Summary

- The equilibrium of dissipative relativistic hydrodynamics is linearly stable if the constitutive relations are defined in kinetic frame.
  Then they can be transformed to other frames.
- It is consistent with relativistic temperature.

# Thank you for the attention!


**Our related works:**

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VP, Biró, TS., PLB, 709(1-2):106–110, 2012, (arXiv:1109.0985).
VP, CMT, 29/2, 133/151, 2017 (arXiv:1510.03900)
VP-Pavelka-Grmela, JNET, 42/2, 133-142, 2017 (arXiv:1508.00121)

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# What is ideal?

**Landau-Lifshitz:**

$$N^a = \hat{n}\,\hat{u}^a + \hat{j}^a$$
$$T^{ab} = \hat{e}\,\hat{u}^b\,\hat{u}^a + \hat{P}^{ab} = \hat{e}\,\hat{u}^b\,\hat{u}^a - \hat{p}\,\hat{\Delta}^{ab} + \hat{\Pi}^{ab}$$

$$\Longleftrightarrow$$

**Eckart:**

$$N^a = n u^a$$
$$T^{ab} = e u^b u^a + q^b u^a + q^a u^b - p\Delta^{ab} + \Pi^{ab}$$

**Transformation:** $\boxed{\hat{u}^a = \dfrac{u^a + w^a}{\zeta}}$

**What is ideal?**

$$N_0^a = n u^a$$
$$T_0^{ab} = e u^b u^a - p\Delta^{ab}$$

$$\Longrightarrow$$

$$N_0^a = \hat{n}\,\hat{u}^a + j^a$$
$$T_0^{ab} = \hat{e}\,\hat{u}^b\,\hat{u}^a + q^b\,\hat{u}^a + q^a\,\hat{u}^b - p\,\hat{\Delta}^{ab} + \Pi^{ab}$$

$$\hat{n} = \frac{n}{\zeta},\quad j^a = n\frac{\hat{w}^a}{\zeta},\quad \hat{e} = \frac{h}{\zeta^2} - p,\quad q^a = h\hat{w}^a,\quad \Pi^{ab} = \frac{\hat{w}^a\hat{w}^b}{h}$$

Ideal fluid is a class of $N^a, T^{ab}$

Dissipation leads to homogenization? Equilibrium is a submanifold?

*Fields:*

| | |
|---|---|
| $N^a$ | 4 |
| $T^{ba}$ | 10 |
| $u^a$ | 3 |
| | $\Sigma 17$ |

| | |
|---|---|
| $j^a$ | 3 |
| $q^a$ | 3 |
| $\Pi^{ab}$ | 6 |
| | $\Sigma 12$ |

$q^a u_a = j^a u_a = 0, \quad \Pi^{ba} u_a = \Pi^{ab} u_a = 0^b$

*Equations:*

| | |
|---|---|
| $\partial_a N^a = 0,$ | 1 |
| $\partial_b T^{ab} = 0^a,$ | 4 |

$N^a$ – particle number vector
$T^{ab}$ – energy-momentum density
$u^a$ – velocity field

$j^a$ – particle flux/current
$q^a$ – energy flux and momentum density
$\Pi^{ab}$ – viscous pressure

$n, e, u^a$ – basic fields

Flow-frames, non-equilibrium thermodynamics, second law

# Paradox solved? Second order

$$\partial_a S^a = \dot{s} + s\,\partial_a u^a + \partial_a J^a \geqslant 0$$

Eckart (1940), theory and flow-frame:

$$S^a(T^{ab}, N^a) = s(e,\ n)u^a + \frac{q^a}{T}$$

(Müller)-Israel-Stewart (1969-72) theory in Eckart flow-frame:

$$S^a(T^{ab}, N^a) = \left(s(e,\ n) - \frac{\beta_0}{2\mathrm{T}}\Pi^2 - \frac{\beta_1}{2\mathrm{T}} q_b q^b - \frac{\beta_2}{2\mathrm{T}} \pi^{bc}\pi_{bc}\right)u^a + \frac{1}{T}\left(q^a + \alpha_0 \Pi q^a + \alpha_1 \pi^{ab} q_b\right)$$

isotropic, Grad compatible

## Paradox solved? Divergence type

(Müller)-Israel-Stewart theory: The linearized version is conditionally hyperbolic in Eckart frame.

**Classical structure, normal convariant entropy inequality:**

$$\partial_\mu N^\mu = 0; \qquad \partial_\mu T^{\mu\nu} = 0^\nu;$$
$$\partial_\mu S^\mu + \alpha \partial_\mu N^\mu + \beta_\nu \partial_\mu T^{\mu\nu} \geq 0.$$

**Divergence, type theories: hyperbolic by construction (Geroch).**

$$\partial_\mu N^\mu = 0; \qquad \partial_\mu T^{\mu\nu} = 0^\nu; \qquad \partial_\mu A^{abc} = I^{bc}$$
$$\partial_\mu S^\mu + \chi \partial_\mu N^\mu + \chi_\nu \partial_\mu T^{\mu\nu} + \chi_{bc} \partial_\mu A^{abc} = 0,$$
$$\partial_\mu S^\mu = -\chi_{bc} I^{bc} \geq 0.$$