Renormalization of equation of state by hydrodynamic fluctuations within dynamical model

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Hydrodynamic fluctuations in dynamical models

Sakai, KM, Hirano, PRC102, 064903 (2020), NPA1005, 121969 (2021),
+ New paper will show up on arXiv in several days

Flow correlation in longitudinal direction

\[ r_n(\eta^a_p, \eta^b_p) = \frac{V_n\Delta (-\eta^a_p, \eta^b_p)}{V_n\Delta (\eta^a_p, \eta^b_p)} \]

Without initial longitudinal fluct

With initial longitudinal fluct

Jia and Huo, PRC90 034905 (2014)
Hydrodynamic fluctuations

= Thermal fluctuations of *hydrodynamic description*

Noise terms $\xi(x)$ in hydrodynamic equations (~*Langevin eq*)

\[
\pi^{\mu\nu} + \tau_\pi \Delta^{\mu\nu}_{\alpha\beta} D\pi^{\alpha\beta} = 2\eta \partial^{(\mu} u^{\nu)} + \cdots + \xi^{\mu\nu}_\pi,
\]
\[
[1 + \tau_\Pi D] \Pi = -\zeta \partial_\mu u^\mu + \cdots + \xi_\Pi.
\]

→ “Fluctuating hydrodynamics”

*Fluctuation-dissipation relation (FDR)*

For correct thermal distribution ($T$) of hydrodynamic fields,

\[
\langle \xi^{\mu\nu}(x) \xi^{\alpha\beta}(x') \rangle = 4T \eta \Delta^{\mu\nu}_{\alpha\beta} \delta^{(4)}(x - x'),
\]
\[
\langle \xi(x) \xi(x') \rangle = 2T \zeta \delta^{(4)}(x - x'),
\]
Cutoff parameter of fluctuating hydrodynamics

\{ \begin{align*}
\text{Noise autocorrelation} & \sim \delta^{(4)}(x-x') & \xrightarrow{\text{FDR with assumption of scale separation}} \\
\text{Hydro equations are non-linear} & \end{align*} \}

\rightarrow \text{No continuum limit.}

Physical cutoff $\Lambda$ (i.e. hydro validity limit) needed

Cutoff dependence of macroscopic quantities

e.g. Decomposition of energy density of \textbf{Global equilibrium}

\[ \langle T^{00} \rangle = e = \langle e_\Lambda \rangle + \langle (e_\Lambda + p_\Lambda) u_\Lambda^2 + \pi_\Lambda^{00} \rangle \]

\begin{align*}
\text{internal energy} & \quad \text{“internal” energy} & \text{“kinetic” energy} \\
\text{in ordinary sense} & \quad \text{energy} & \\
\text{in Conventional hydrodynamics} & \quad \text{in Fluctuating hydrodynamics} & \\
\end{align*}
Cutoff parameter of fluctuating hydrodynamics

\{ \begin{align*}
\text{Noise autocorrelation } & \sim \delta^{(4)}(x-x') \\
\text{Hydro equations are non-linear} 
\end{align*} \}

\rightarrow \text{No continuum limit.}

Physical cutoff \( \Lambda \) (i.e. hydro validity limit) needed

**Cutoff dependence of macroscopic quantities**

\checkmark \text{Every “macroscopic” quantities (} e_\Lambda, u_\Lambda, \text{etc.})

\text{are redefined for each cutoff} \( \Lambda \).

\checkmark \text{Macroscopic relations such as viscosity, EoS, etc. should be}

\text{modified not to change the bulk properties}

\text{“Renormalization”}

Studies on the lowest order wrt noise contributions

Motivation for EoS Renormalization

The situation is opposite between EoS and transport properties

\[ \Lambda = 0 \text{ (IR, long-range)} \quad \text{large } \Lambda \text{ (UV, short scale)} \]

We know this!

- Lattice EoS \( p(e) \)
  - well-defined, straight forward

We want this!

- Transport properties \( \eta/s, \zeta/s \)
  - well-defined, straight forward

Focus on this!

- Bare EoS for fluctuating hydrodynamics \( p_\Lambda(e_\Lambda) \)

- Bare transport coefficients \( \eta_\Lambda, \zeta_\Lambda \)

We know this!

- Bare transport coefficients \( \eta_\Lambda, \zeta_\Lambda \)
  - inverse problem non-trivial

Focus on this!

- Lattice EoS \( p(e) \)
  - well-defined, straight forward
Perturbative results


\[ e_0 = e_\Lambda + \frac{T \Lambda^3}{2 \pi^2} \quad p_0 = p_\Lambda + \frac{T \Lambda^3}{6 \pi^2} \]

conformal e=3p, the first-order hydro


\[ e_0 = e_\Lambda + \frac{T \Lambda^3}{2 \pi^2} \quad p_0 = p_\Lambda + \left( 1 + \frac{T}{2} \frac{d c_s^2}{dT} \right) \frac{T \Lambda^3}{6 \pi^2} \]

general EoS, the first-order hydro

The case for the second-order hydro (used in dynamical models)

general EoS, the second-order hydro (non-zero relaxation time \( \tau_\pi \))

(omit calculations) \( \rightarrow \) **Exactly the same as the first-order case for EoS**
What we can show non-perturbatively

Consider a global equilibrium as background

\[
\text{diag}(e_0, p_0, p_0, p_0) = \langle T^{\mu \nu} \rangle = \langle e_\Lambda u_\Lambda^\mu u_\Lambda^\nu - p_\Lambda \Delta_\Lambda^{\mu \nu} + \pi_\Lambda^{\mu \nu} \rangle
\]

Define Kinetic energy density \( K_\Lambda \)

\[
\langle T^{00} \rangle = \langle e_\Lambda \gamma_\Lambda^2 - p_\Lambda (1 - \gamma_\Lambda^2) + \pi_\Lambda^{00} \rangle = \langle e_\Lambda \rangle + \langle (e_\Lambda + p_\Lambda) u_\Lambda^2 + \pi_\Lambda^{00} \rangle =: \langle e_\Lambda \rangle + \langle K_\Lambda \rangle.
\]

Consider trace

\[
e_0 - 3p_0 = \langle g_{\mu \nu} T^{\mu \nu} \rangle = \langle e_\Lambda - 3p_\Lambda \rangle,
\]

\[
p_0 = \langle p_\Lambda \rangle + \frac{1}{3} \langle K_\Lambda \rangle.
\]

Bare energy is smaller than equilibrium energy

\( K_\Lambda > 0 \Rightarrow \langle e_\Lambda \rangle < e_0 \)

Difference of \( p_0 \) & \( \langle p_\Lambda \rangle \) are exactly \( 1/3 \) of that of \( e \)
What we can show non-perturbatively

Expand $\langle p_\Lambda \rangle$ using EoS$_\Lambda$: $p_\Lambda(e_\Lambda)$

$$p_0 = \langle p_\Lambda(e_\Lambda) \rangle + \frac{1}{3} \langle K_\Lambda \rangle = p_\Lambda(e_\Lambda) + \frac{1}{3} \langle K_\Lambda \rangle + \sum_{k=2}^{\infty} \frac{1}{k!} \frac{\partial^k p_\Lambda}{\partial e_\Lambda^k} \langle [e - \langle e_\Lambda \rangle]^k \rangle$$

The $k = 2$ term corresponds to the existing result via

Fluctuation-dissipation theorem of the first kind:

$$\langle [e - \langle e_\Lambda \rangle]^2 \rangle = \frac{T^2}{V} \frac{\partial e}{\partial T}$$

and Equipartition

$$\frac{1}{3} \langle K_\Lambda \rangle = T \int_{0}^{\Lambda} \frac{d^3 k}{(2\pi)^3} = \frac{T \Lambda^3}{6\pi^2} \sim \frac{T}{V_{\text{eff}}}$$

✔ General formula for EoS renormalization

NB: Implicit assumption Onsager’s regression hypothesis, non-Gaussian noises with higher-order susceptibilities, ...
Inverse problem

We have a formula of \( p_0 \) as a functional of \( p_\Lambda \)

\[
p_0 = p_\Lambda + \left( 1 + \frac{T}{2} \frac{dc_s^2}{dT} \right) \frac{T\Lambda^3}{6\pi^2}
\]

To invert the relation, we need to solve 2\textsuperscript{nd} order differential eq.

\[
\ddot{p}_\Lambda = \frac{2\dot{\varepsilon}_\Lambda(T)}{T} \left[ \frac{p_0(T) - p_\Lambda(T)}{T\Lambda^3/6\pi^2} - 1 \right] + \frac{\ddot{\varepsilon}_\Lambda(T)\dot{p}_\Lambda(T)}{\dot{\varepsilon}_\Lambda(T)}
\]

dots represents \( d/dT \)

2 integral constants \( \rightarrow \) The solution is not-unique

**Assumption:** Minimize sound velocity variations

\[
I = \int_{T_1}^{T_2} dT \left( \frac{dc_{s,\Lambda}^2}{dT} \right)^2
\]
Result: $c_s^2 = \frac{\partial \rho_\Lambda}{\partial e_\Lambda}$

Some oscillatory behavior (artifact of truncation?)
Nevertheless $\rho(e)$ is smooth enough
Result: global $\langle T^{\mu\nu} \rangle$ reproduced?

Global pressure / expected pressure (must be 1)

Cutoff $\Lambda = 1 \text{ fm}^{-1}$

(preliminary; might change)

Only slightly improved (??)

$\rightarrow$ non-linear contributions are significant with this scale
$\rightarrow$ non-perturbative approach with simulations required
Result: Iterative updates based on simulation

Numerically adjust EoS to reproduce global $T^{\mu\nu}$ (experimental)

$$e_\Lambda(T) \rightarrow e_\Lambda(T) + \epsilon(\langle T_\Lambda \rangle - T) \frac{\partial e_\Lambda}{\partial T}$$

Cutoff $\Lambda = 1$ fm$^{-1}$
Summary (Current status)

**Hydrodynamic fluctuations**
- important in longitudinal dynamics

**Fluctuating hydodynamics**
  - cutoff parameter \( \rightarrow \) EoS / transport coefficients renormalization

**Perturbative result for the second-order hydro (linear)**
  - Unmodified from the first-order hydro case

**Inverting the perturbative expression**
- not sufficient? non-linear interactions?

**Numerical adjustment of EoS**
- currently trying to get robust results