# Renormalization of equation of state by hydrodynamic fluctuations within dynamical model

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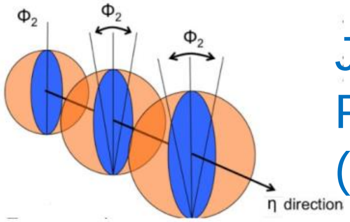
# Hydrodynamic fluctuations in dynamical models

Sakai, KM, Hirano, PRC102, 064903 (2020), NPA1005, 121969 (2021),
+ New paper will show up on arXiv in several days

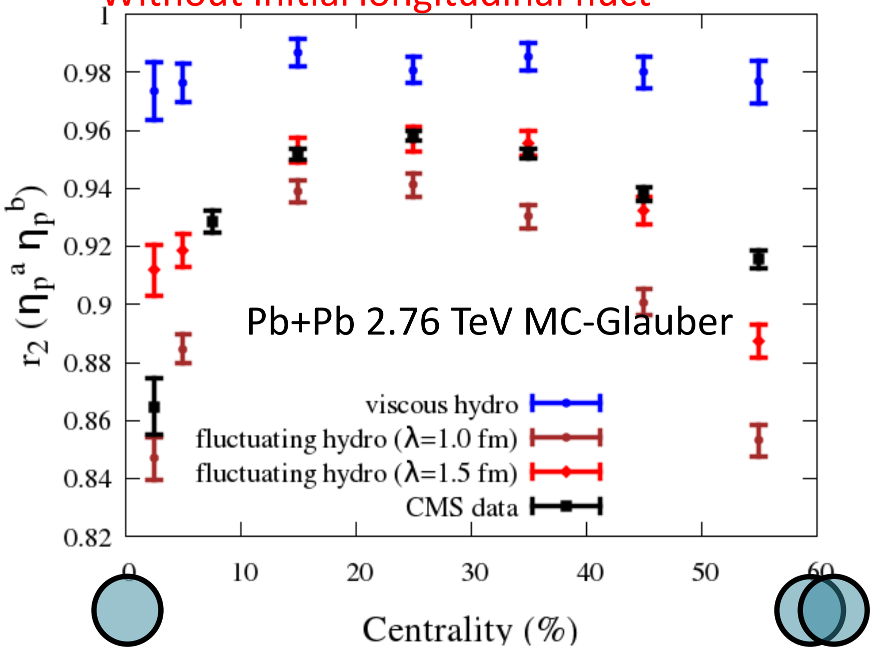
Flow correlation in longitudinal direction

$$r_n(\eta_p^a, \eta_p^b) = \frac{V_{n\Delta}(-\eta_p^a, \eta_p^b)}{V_{n\Delta}(\eta_p^a, \eta_p^b)}$$



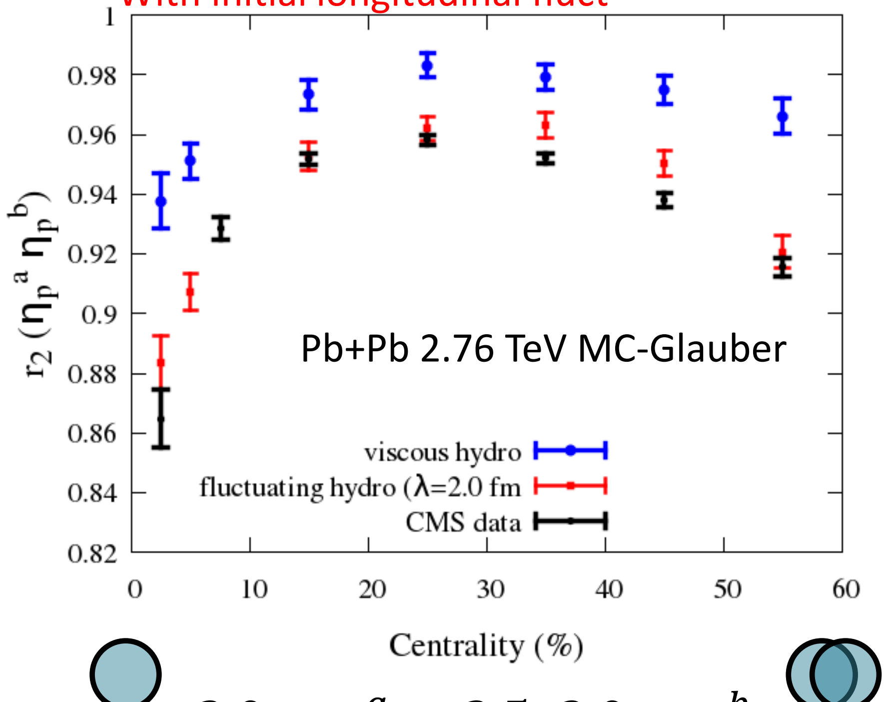

Jia and Huo, PRC90 034905 (2014)

Without initial longitudinal fluct




$2.0 < \eta_p^a < 2.5, 3.0 < \eta_p^b < 4.0$

With initial longitudinal fluct




$2.0 < \eta_p^a < 2.5, 3.0 < \eta_p^b < 4.0$

# Hydrodynamic fluctuations

## = Thermal fluctuations of *hydrodynamic description*

Noise terms ξ(x) in hydrodynamic equations (~ Langevin eq)

e.g. In the second-order theory (causal & stable)

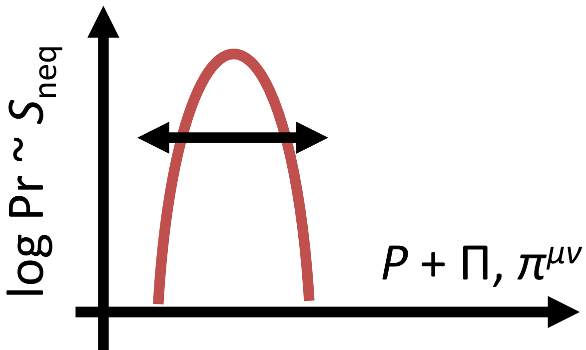
$$\pi^{\mu\nu} + \tau_\pi \Delta^{\mu\nu}{}_{\alpha\beta} \mathrm{D}\pi^{\alpha\beta} = 2\eta\partial^{\langle\mu}u^{\rangle\nu} + \cdots + \xi_\pi^{\mu\nu},$$
$$[1 + \tau_\Pi \mathrm{D}]\Pi = -\zeta\partial_\mu u^\mu + \cdots + \xi_\Pi.$$

→ "Fluctuating hydrodynamics"

## Fluctuation-dissipation relation (FDR)

For correct thermal distribution ($T$) of hydrodynamic fields,

$$\langle \xi^{\mu\nu}(x)\xi^{\alpha\beta}(x')\rangle = 4T\eta\Delta^{\mu\nu\alpha\beta}\delta^{(4)}(x-x'),$$
$$\langle \xi(x)\xi(x')\rangle = 2T\zeta\delta^{(4)}(x-x'),$$



# Cutoff parameter of fluctuating hydrodynamics

$\left\{ \begin{array}{l} \text{Noise autocorrelation} \sim \delta^{(4)}(x-x') \\ \text{Hydro equations are non-linear} \end{array} \right.$

← FDR with assumption of scale separation

→ No continuum limit.

Physical cutoff Λ (i.e. hydro validity limit) needed

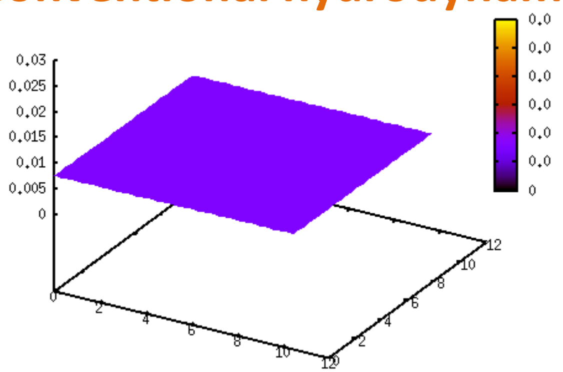
## Cutoff dependence of macroscopic quantities

e.g. Decomposition of energy density of **Global equilibrium**

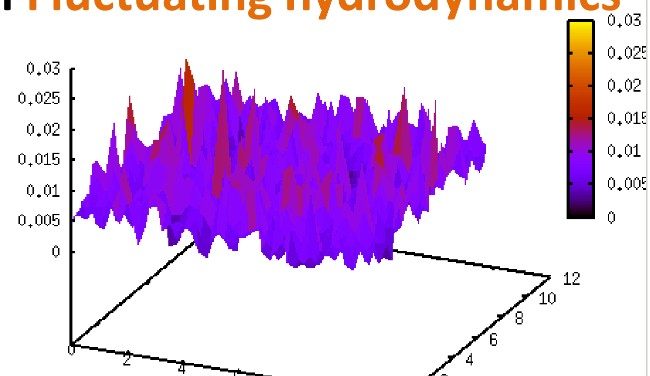
$$\langle T^{00} \rangle = e = \langle e_\Lambda \rangle + \langle (e_\Lambda + P_\Lambda) \boldsymbol{u}_\Lambda^2 + \pi_\Lambda^{00} \rangle$$

internal energy in ordinary sense — "internal" energy — "kinetic" energy

**in Conventional hydrodynamics**




**in Fluctuating hydrodynamics**



# Cutoff parameter of fluctuating hydrodynamics

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← FDR with assumption of scale separation

→ No continuum limit.
Physical cutoff $\Lambda$ (i.e. hydro validity limit) needed

## Cutoff dependence of macroscopic quantities

✓ Every "macroscopic" quantities ($e_\Lambda$, $u_\Lambda$, etc.)
are *redefined for each cutoff* $\Lambda$.

= equation of state

✓ Macroscopic relations such as viscosity, EoS, etc. should be
modified not to change the bulk properties

"Renormalization"

**Studies on the lowest order wrt noise contributions**

P. Kovtun, et al., Phys. Rev. D68, 025007 (2003); P. Kovtun, et al., Phys. Rev. D84, 025006 (2011); P. Kovtun, J. Phys. A45, 473001 (2012); Y. Akamatsu, et al., Phys. Rev. C95, 014909 (2017); Y. Akamatsu, et al., Phys. Rev. C97, 024902 (2018)

# Motivation for EoS Renormalization

**The situation is opposite between EoS and transport properties**

$\Lambda = 0$ (IR, long-range) ⟷ large $\Lambda$ (UV, short scale)

We **know** this!

Lattice EoS
$p(e)$

← well-defined, straight forward

→ inverse problem non-trivial

**Focus on this!**

Bare EoS
for fluctuating
hydrodynamics
$p_\Lambda(e_\Lambda)$

We **want** this!

Transport
properties
$\eta/s$, $\zeta/s$

← well-defined, straight forward

Bare transport
coefficients
$\eta_\Lambda$, $\zeta_\Lambda$

# Perturbative results

P. Kovtun, et al., Phys. Rev. D84, 025006 (2011); Y. Akamatsu, et al., Phys. Rev. C95, 014909 (2017);

$$e_0 = e_\Lambda + \frac{T\Lambda^3}{2\pi^2} \qquad p_0 = p_\Lambda + \frac{T\Lambda^3}{6\pi^2}$$

conformal e=3p, the first-order hydro

Y. Akamatsu, et al., Phys. Rev. C97, 024902 (2018)

$$e_0 = e_\Lambda + \frac{T\Lambda^3}{2\pi^2} \qquad p_0 = p_\Lambda + \left(1 + \frac{T}{2}\frac{dc_s^2}{dT}\right)\frac{T\Lambda^3}{6\pi^2}$$

**general EoS**, the first-order hydro

The case for the second-order hydro (used in dynamical modesl)

**general EoS**, the second-order hydro (non-zero relaxation time $\tau_\pi$)

(omit calculations) → **Exactly the same as the first-order case for EoS**

# What we can show non-perturbatively

**Consider a global equilibrium as background**

$$\mathrm{diag}(e_0, p_0, p_0, p_0) = \langle T^{\mu\nu} \rangle = \langle e_\Lambda u_\Lambda^\mu u_\Lambda^\nu - p_\Lambda \Delta_\Lambda^{\mu\nu} + \pi_\Lambda^{\mu\nu} \rangle$$

**Define Kinetic energy density *$K_\Lambda$***

$$\begin{aligned} \langle T^{00} \rangle &= \langle e_\Lambda \gamma_\Lambda^2 - p_\Lambda (1 - \gamma_\Lambda^2) + \pi_\Lambda^{00} \rangle \\ &= \langle e_\Lambda \rangle + \langle (e_\Lambda + p_\Lambda) \boldsymbol{u}_\lambda^2 + \pi_\Lambda^{00} \rangle \\ &=: \langle e_\Lambda \rangle + \langle K_\Lambda \rangle . \end{aligned}$$

Bare energy is smaller than equilibrium energy

$K_\Lambda > 0 \rightarrow \langle e_\Lambda \rangle < e_0$

**Consider trace**

$$e_0 - 3p_0 = \langle g_{\mu\nu} T^{\mu\nu} \rangle = \langle e_\Lambda - 3p_\Lambda \rangle,$$

$$p_0 = \langle p_\Lambda \rangle + \frac{1}{3} \langle K_\Lambda \rangle .$$

Difference of $p_0$ & $\langle p_\Lambda \rangle$ are exactly **1/3** of that of $e$

# What we can show non-perturbatively

**Expand $\langle p_\Lambda \rangle$ using EoS$_\Lambda$: $p_\Lambda(e_\Lambda)$**

$$p_0 = \langle p_\Lambda(e_\Lambda) \rangle + \frac{1}{3}\langle K_\Lambda \rangle = p_\Lambda(e_\Lambda) + \frac{1}{3}\langle K_\Lambda \rangle + \sum_{k=2}^{\infty} \frac{1}{k!}\frac{\partial^k p_\Lambda}{\partial e_\Lambda^k}\langle [e - \langle e_\Lambda \rangle]^k \rangle$$

The *k = 2* term corresponds to the existing result via

Fluctuation-dissipation theorem of the first kind:

$$\langle [e - \langle e_\Lambda \rangle]^2 \rangle = \frac{T^2}{V}\frac{\partial e}{\partial T}$$

and Equipartition

$$\frac{1}{3}\langle K_\Lambda \rangle = T\int_0^\Lambda \frac{d^3k}{(2\pi)^3} = \frac{T\Lambda^3}{6\pi^2} \sim \frac{T}{V_{\mathrm{eff}}}$$

Note: *k > 2 cases can also be written in terms of higher-order susceptibilities*

✔ **General formula for EoS renormalization**

NB: Implicit assumption Onsager's regression hypothesis, non-Gaussian noises with higher-order susceptibility, …

# Inverse problem

**We have a formula of $p_0$ as a functional of $p_\Lambda$**

$$p_0 = p_\Lambda + \left(1 + \frac{T}{2}\frac{dc_s^2}{dT}\right)\frac{T\Lambda^3}{6\pi^2}$$

**To invert the relation, we need to solve 2nd order differential eq.**

$$\ddot{p}_\Lambda = \frac{2\dot{e}_\Lambda(T)}{T}\left[\frac{p_0(T) - p_\Lambda(T)}{T\Lambda^3/6\pi^2} - 1\right] + \frac{\ddot{e}_\Lambda(T)\dot{p}_\Lambda(T)}{\dot{e}_\Lambda(T)}$$

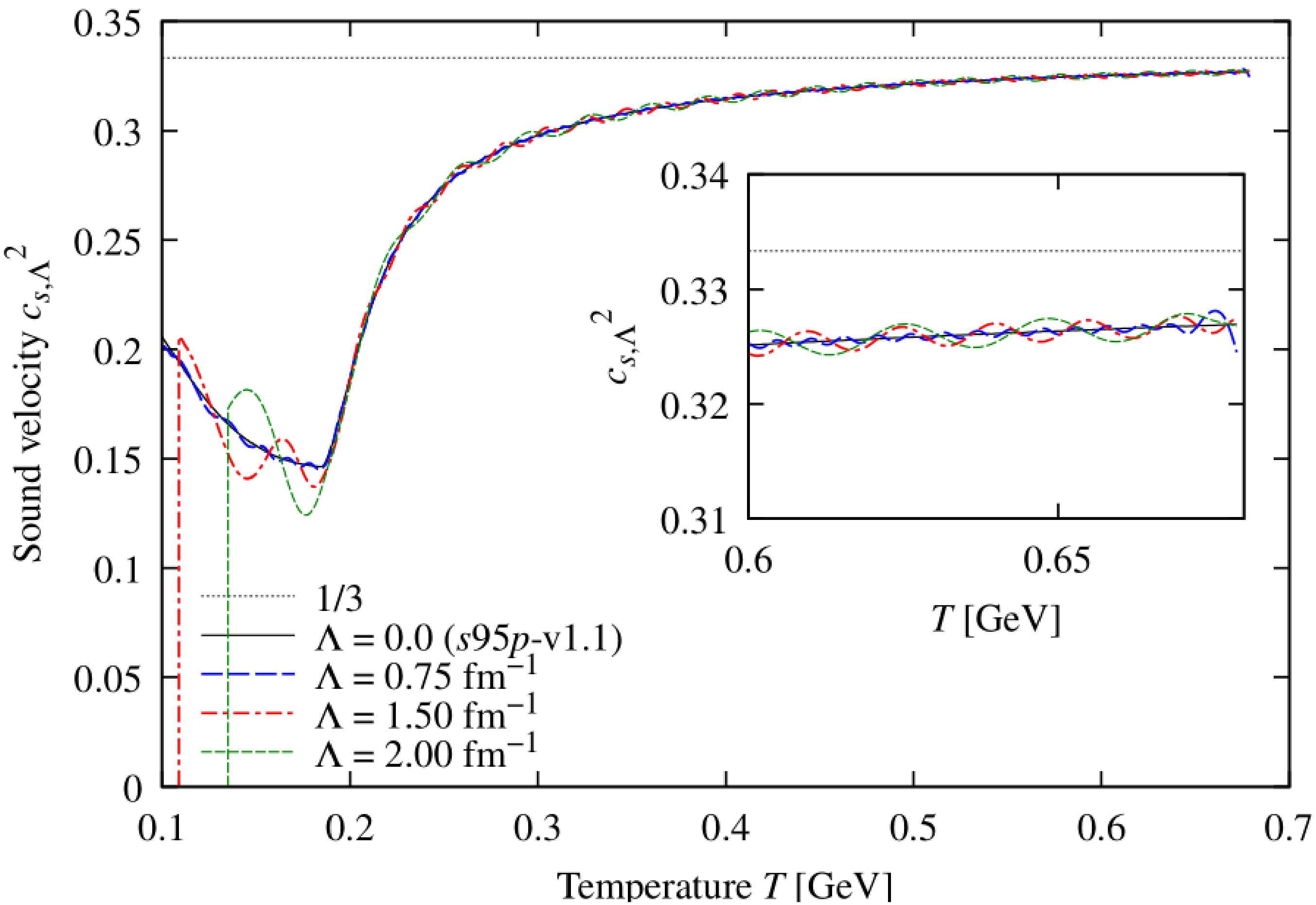
dots represents *d*/*dT*

*2 integral constants* → **The solution is not-unique**

**Assumption: Minimize sound velocity variations**

$$I = \int_{T_1}^{T_2} dT \left(\frac{dc_{s,\Lambda}^2}{dT}\right)^2$$

# Result: $c_s^2 = \partial p_\Lambda / \partial e_\Lambda$

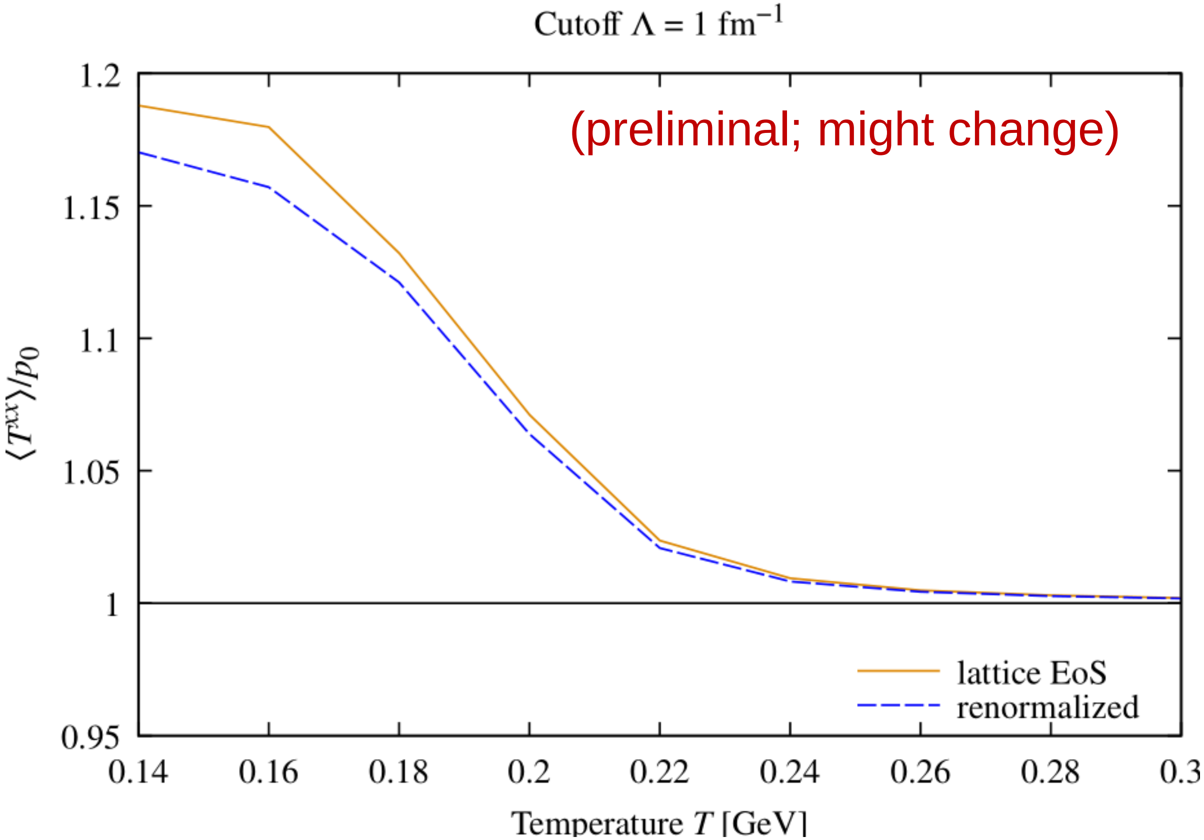



Some oscillatory behavior (artifact of truncation?)

Nevertheless $p(e)$ is smooth enough

# Result: global 〈$T^{\mu\nu}$〉 reproduced?

**Global pressure / expected pressure (must be 1)**




Only slightly improved (??)

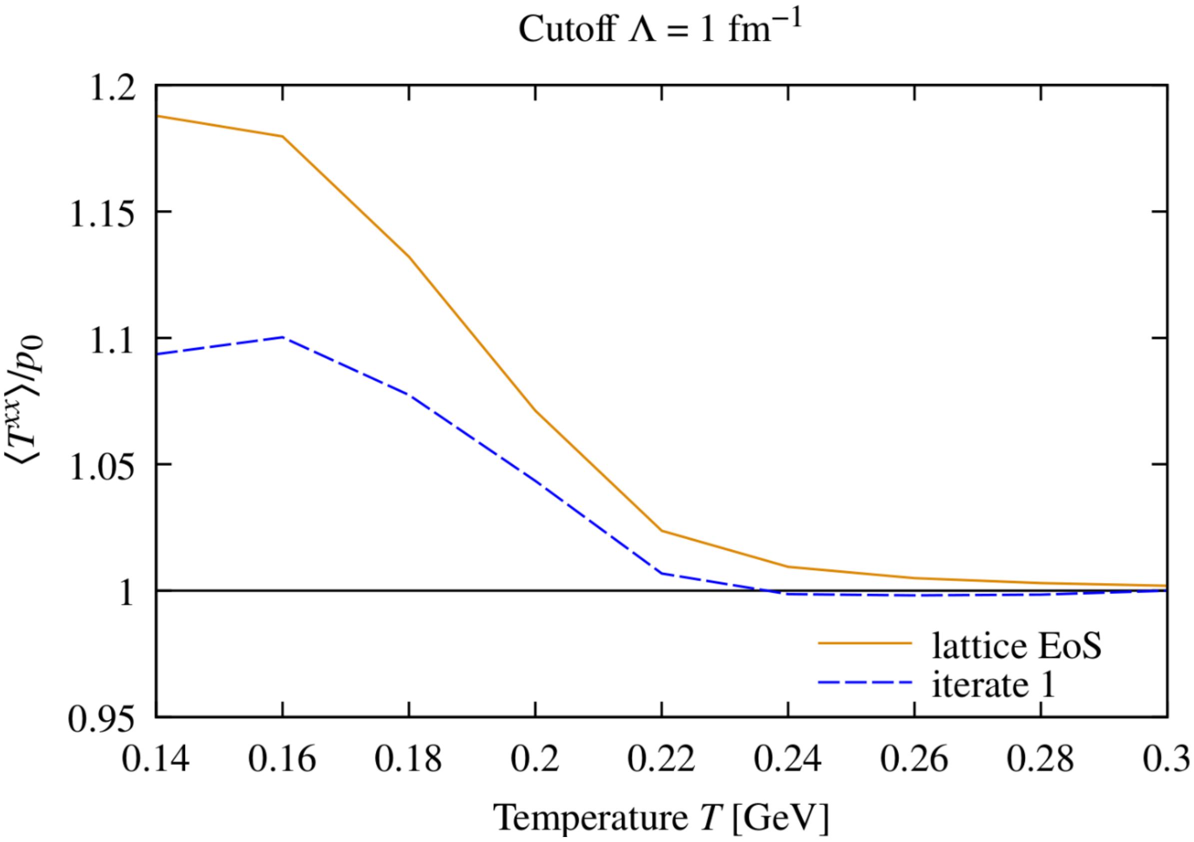
→ non-linear contributions are significant with this scale

→ non-perturbative approach with simulations required

# Result: Iterative updates based on simulation

**Numerically adjust EoS to reproduce global $T^{\mu\nu}$** (experimental)

$$e_\Lambda(T) \to e_\Lambda(T) + \epsilon(\langle T_\Lambda \rangle - T)\frac{\partial e_\Lambda}{\partial T}$$



# Summary (Current status)

**Hydrodynamic fluctuations**

→ important in longitudinal dynamics

**Fluctuating hydodynamics**

cutoff parameter → EoS / transport coefficients renormalization

**Perturbative result for the second-order hydro (linear)**

Unmodified from the first-order hydro case

**Inverting the perturbative expression**

→ not sufficient? non-linear interactions?

**Numeical adjustment of EoS**

→ currently trying to get robust results