Covariant Spin Statistical Mechanics with Torsion

& Applications on Spin Hydrodynamics and Chiral Transports

Yu-Chen Liu
Fudan University

Collaborator: Xu-Guang Huang

2021.11.06

Table of contents

Why torsion?

Covariant vorticity in Riemann-Cartan spacetime

$$\epsilon^{\mu\nu\rho\sigma} U_\nu \widetilde{\nabla}_\rho U_\sigma = \epsilon^{\mu\nu\rho\sigma} U_\nu \partial_\rho U_\sigma + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} U_\nu \mathcal{T}_{\lambda\rho\sigma} U^\lambda \,, \tag{1}$$

where $\mathcal{T}_{\lambda\rho\sigma}$ is torsion.

Anomaly with external torsion field (Nieh-Yan anomaly (Nieh, Yan. 1982))

$$\begin{aligned} \nabla_\mu \hat{J}^\mu_{R/L} = &\pm m \hat{\mathscr{P}} \pm \frac{C_F}{4} F^{\mu\nu} \widetilde{F}_{\mu\nu} \\ &\pm C_{\mathcal{T}} \epsilon^{\mu\nu\rho\sigma} \left(\widetilde{R}_{\mu\nu\rho\sigma} + \frac{1}{2} \mathcal{T}^\lambda{}_{\mu\nu} \mathcal{T}_{\lambda\rho\sigma} \right) , \end{aligned} \tag{2}$$

where $\mathscr{P}$ is the axial condensate, $\widetilde{R}^\rho{}_{\sigma\mu\nu}$ is curvature.

In this talk, we introduce torsion as an external field. Thus, ∇_μ is torsion free.

Conservation laws

Energy-momentum tensor $\hat{\Theta}^{\mu\nu}$ and spin tensor $\hat{\mathscr{S}}^{\mu,\rho\sigma}$:

$$\nabla_\mu \hat{\Theta}^{\mu\nu} = K^{\mu\nu\lambda}\hat{\Theta}_{\lambda\mu} - \frac{1}{2}\hat{\mathscr{S}}_{\mu,\rho\sigma}\widetilde{R}^{\rho\sigma\mu\nu} + F^{\nu\lambda}\left(\hat{J}^R_\lambda + \hat{J}^L_\lambda\right) ,$$
$$\nabla_\mu \hat{\mathscr{S}}^{\mu,\rho\sigma} = \hat{\Theta}^{\sigma\rho} - \hat{\Theta}^{\rho\sigma} - 2\hat{\mathscr{S}}^{\mu,\lambda[\sigma}K^{\rho]}{}_{\mu\lambda} , \tag{3}$$

where $K^\lambda{}_{\mu\nu} \equiv \frac{1}{2}\left(\mathcal{T}^\lambda{}_{\mu\nu} + \mathcal{T}_\mu{}^\lambda{}_\nu + \mathcal{T}_\nu{}^\lambda{}_\mu\right)$ is the Contorsion tensor,
and $X^{[\mu}Y^{\nu]} \equiv \frac{1}{2}(X^\mu Y^\nu - X^\nu Y^\mu)$.

Maxwell equations:

$$\nabla_\mu F^{\mu\nu} = J^\nu_R + J^\nu_L , \qquad \nabla_\mu \widetilde{F}^{\mu\nu} = 0 . \tag{4}$$

Spin statistical mechanics

Entropy

$$S \equiv -\mathrm{Tr}\left(\hat{\rho}\ln\hat{\rho}\right) , \tag{5}$$

where $\hat{\rho}$ is density operator of the system.

Maximizing the total entropy S, we can obtain the local equilibrium density operator(LEDO) $\hat{\rho}_{LE}$.

The exact form of LEDO:

$$\begin{aligned} \hat{\rho}_{LE} &\equiv \frac{1}{Z_{LE}} exp\left[-\int d\Xi_\mu \hat{I}^\mu\right], \\ \hat{I}^\mu &\equiv \hat{\Theta}^{\mu\nu}\beta_\nu + \hat{\mathscr{S}}^{\mu,\lambda\nu}\Omega_{\lambda\nu} - \bar{\mu}_R\hat{J}^\mu_R - \bar{\mu}_L\hat{J}^\mu_L , \end{aligned} \tag{6}$$

where $Z_{LE} \equiv Tr\left[e^{-\int d\Xi_\mu \hat{I}^\mu}\right]$ is the partition function and Ξ_μ is a spacelike Cauchy hypersurface.

Physical constraints for the Lagrange multiplier β^μ, $\Omega^{\mu\nu}$ and $\bar{\mu}_{R/L}$:

$$\Xi_\mu\left(\Theta^{\mu\nu}_{LE}, \mathscr{S}^{\mu\rho\sigma}_{LE}, J^{\mu(LE)}_{R/L}\right) = \Xi_\mu\left(\Theta^{\mu\nu}, \mathscr{S}^{\mu\rho\sigma}, J^\mu_{R/L}\right) , \tag{7}$$

where $O_{LE} \equiv \mathrm{Tr}\left[\hat{\rho}_{LE}\hat{O}\right]$ and $O \equiv \mathrm{Tr}\left[\hat{\rho}\hat{O}\right]$.

(Becattini, Buzzegoli, Grossi. 2019)

Spin statistical mechanics

We suppose local equilibrium entropy is
an integral of an entropy current s^μ:

$$S_{LE} = -\mathrm{Tr}\left[\hat{\rho}_{LE} \ln \hat{\rho}_{LE}\right] = \int d\boldsymbol{\Xi}_\mu s^\mu(x)\,. \tag{8}$$

Divergence of entropy current

$$\begin{aligned}\nabla_\mu s^\mu = {}& \Delta\Theta^{\mu\nu}_{sy}\nabla_\mu\beta_\nu + \Delta\Theta^{\mu\nu}_{as}\left(\nabla_\mu\beta_\nu - K_{\mu\lambda\nu}\beta^\lambda - 2\Omega_{\mu\nu}\right) \\ & + \Delta\mathscr{S}^{\mu,\lambda\nu}\left(\nabla_\mu\Omega_{\lambda\nu} - 2K^{\rho}{}_{\mu\lambda}\Omega_{\rho\nu} - \frac{1}{2}\widetilde{R}_{\lambda\nu\mu\alpha}\beta^\alpha\right) \\ & - \Delta J^\mu_R\left(F_{\mu\nu}\beta^\nu + \nabla_\mu\bar{\mu}_R\right) - \Delta J^\mu_L\left(F_{\mu\nu}\beta^\nu + \nabla_\mu\bar{\mu}_L\right) \\ & - m\Delta\mathscr{P}\left(\bar{\mu}_R - \bar{\mu}_L\right)\end{aligned} \tag{9}$$

where $\Delta O \equiv O - O_{LE}$, $\Theta^{\mu\nu}_{sy/as}$ is the symmetric/antisymmetric part of the stress-energy tensor.

Spin Hydrodynamics

$\nabla_\mu s^\mu \geqslant 0$:

$$\Delta J^\mu_{R/L} = -\sigma_{R/L}\left(F_{\mu\nu}\beta^\nu + \nabla^\perp_\mu \bar{\mu}_{R/L}\right), \tag{10}$$

$$\begin{aligned}\Delta\Theta^{\mu\nu} &= \eta\left(\nabla^\mu_\perp U^\nu + \nabla^\nu_\perp U^\mu\right) - \left(\frac{2}{3}\eta - \zeta\right)\Delta^{\mu\nu}\nabla_\lambda U^\lambda \\ &\quad -\chi\left(\Delta^{\mu\alpha}U^\nu + \Delta^{\nu\alpha}U^\mu\right)G_\alpha + \kappa\Delta^{\mu\alpha}\Delta^{\nu\beta}H_{\alpha\beta} \\ &\quad -\lambda\left(\Delta^{\mu\alpha}U^\nu - \Delta^{\nu\alpha}U^\mu\right)K_\alpha\,,\end{aligned} \tag{11}$$

where $U^\mu = T\beta^\mu$, $\Delta^{\mu\nu} = g^{\mu\nu} - U^\mu U^\nu$, $\nabla^\perp_\mu \equiv \Delta^{\mu\nu}\nabla_\nu$, and

$$\begin{aligned} G_\alpha &= \nabla_\alpha T - TU^\lambda\nabla_\lambda U_\alpha\,, \\ H_{\alpha\beta} &= -4T\Omega_{\alpha\beta} + \left(\partial_\alpha U_\beta - \partial_\beta U_\alpha\right) - 2K_{\alpha\lambda\beta}U^\lambda\,, \\ K_\alpha &= \nabla_\alpha T + TU^\lambda\nabla_\lambda U_\alpha + 2TK_{\alpha\lambda\beta}U^\lambda U^\beta \\ &\quad + 4T^2\Omega_{\alpha\beta}U^\beta\,. \end{aligned} \tag{12}$$

The coefficients $\sigma_{R/L}$, η, ζ, χ, κ and λ are positive.

(Hattori, Hongo, Huang, Matsuo, Taya. 2019.)

The above results are anomaly independent.

Global equilibrium conditions

$\nabla_\mu s^\mu = 0$:

$$
\begin{aligned}
\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu &= b(x) g_{\mu\nu}\,, && (13)\\
\nabla_{[\mu}\beta_{\nu]} - K_{\mu\lambda\nu}\beta^\lambda - 2\Omega_{\mu\nu} &= 0\,, && (14)\\
\nabla_\mu \Omega_{\lambda\nu} + 2K^\rho{}_{\mu[\lambda}\Omega_{\nu]\rho} - \frac{1}{2}\widetilde{R}_{\lambda\nu\mu\alpha}\beta^\alpha &= 0\,, && (15)\\
T\nabla_\mu \bar{\mu}_{R/L} = -E_\mu\,, \qquad m\left(\bar{\mu}_R - \bar{\mu}_L\right) &= 0\,, && (16)
\end{aligned}
$$

where $b(x)$ can be finite if the system is conformal invariant($\Theta^\mu{}_\mu = 0$).

The above global equilibrium conditions are also anomaly independent.

Spin thermodynamics

We suppose the logarithm of partition function is an integral of a thermal potential current ϕ^μ:

$$\ln Z_{GE} = \int d\Xi_\mu \phi^\mu(x)\,. \tag{17}$$

Currents, stress-energy tensor and spin tensor are related to ϕ^μ as follows (Becattini. 2012.):

$$\begin{aligned} J^\mu_{R/L} &= \left(\frac{\delta\phi^\mu}{\delta\bar{\mu}_{R/L}}\right)_{\beta^\mu,\bar{\mu}_{L/R},\Omega^{\mu\nu}}\,, \\ \Theta^{\mu\nu} &= -\left(\frac{\delta\phi^\mu}{\delta\beta_\nu}\right)_{\bar{\mu}_R,\bar{\mu}_L,\Omega^{\mu\nu}}\,, \\ \mathscr{S}^{\mu,\rho\sigma} &= -\left(\frac{\delta\phi^\mu}{\delta\Omega_{\rho\sigma}}\right)_{\beta^\mu,\bar{\mu}_R,\bar{\mu}_L}\,. \end{aligned} \tag{18}$$

Derive spin thermodynamical relations

Global equilibrium spin current:

$$s^\mu = \phi^\mu + \Theta^{\mu\nu}\beta_\nu + \mathscr{S}^{\mu,\lambda\nu}\Omega_{\lambda\nu} - \bar{\mu}_R J_R^\mu - \bar{\mu}_L J_L^\mu. \quad (19)$$

Define

- Energy density and pressure: $(\epsilon, p) \equiv TU_\mu \left(\Theta^{\mu\nu}\beta_\nu, \phi^\mu\right)$;
- Entropy density, spin density and particle number densities: $(s, \mathscr{S}^{\rho\sigma}, n_R, n_L) \equiv U_\mu \left(s^\mu, \mathscr{S}^{\mu,\rho\sigma}, J_R^\mu, J_L^\mu\right)$.

We derive the following thermodynamical relations:

$$\epsilon + p = T\left(s - \mathscr{S}^{\mu\nu}\Omega_{\mu\nu} + \bar{\mu}_R n_R + \bar{\mu}_L n_L\right), \quad (20)$$

$$dp = \frac{(\epsilon + p)}{T} dT - T\left(\mathscr{S}^{\mu\nu} d\Omega_{\mu\nu} + n_R d\bar{\mu}_R + n_L d\bar{\mu}_L\right), \quad (21)$$

Work out the exact form of ϕ^μ

Divergence of entropy current at global equilibrium:

$$
\begin{aligned}
0 &= \nabla_\mu s^\mu \\
&= \nabla_\mu \phi^\mu - \left[C_F E^\mu B_\mu + C_{\mathcal{T}} \epsilon^{\mu\nu\rho\sigma} \left(\widetilde{R}_{\mu\nu\rho\sigma} + \frac{1}{2} \mathcal{T}^\lambda{}_{\mu\nu} \mathcal{T}_{\lambda\rho\sigma} \right) \right] (\bar{\mu}_R - \bar{\mu}_L) \ .
\end{aligned} \tag{22}
$$

For massless fermions, $(\bar{\mu}_R - \bar{\mu}_L) \neq 0$. Thus, ϕ^μ must be anomaly dependent.

The general form of thermal potential current:

$$
\phi^\mu = p\beta^\mu + \vartheta_B B^\mu + \vartheta_\omega \omega^\mu + \vartheta_{\mathcal{T}} \epsilon^{\mu\nu\rho\sigma} \mathcal{T}_{\nu\rho\sigma} \ . \tag{23}
$$

where ϑ_B, ϑ_ω and $\vartheta_{\mathcal{T}}$ are coefficients.

Chiral torsional effect

Solutions of ϑ_B, ϑ_ω and $\vartheta_\mathcal{T}$:

$$\vartheta_B = -\frac{C_F T}{2}\left(\bar{\mu}_R^2 - \bar{\mu}_L^2\right), \quad (24)$$

$$\vartheta_\omega = -\frac{C_F T^2}{3}\left(\bar{\mu}_R^3 - \bar{\mu}_L^3\right) + \frac{T^2}{12}\left(\bar{\mu}_R - \bar{\mu}_L\right), \quad (25)$$

$$\vartheta_\mathcal{T} = C_\mathcal{T}\left(\bar{\mu}_R - \bar{\mu}_L\right). \quad (26)$$

Current derived from ϕ^μ:

$$J^\mu_{(GE)R/L} = n_{R/L}U^\mu \mp C_F \mu_{R/L} B^\mu \mp \left(C_F \mu_{R/L}^2 - \frac{T^2}{12}\right)\omega^\mu \pm C_\mathcal{T}\epsilon^{\mu\nu\rho\sigma}\mathcal{T}_{\nu\rho\sigma}\,. \quad (27)$$

where the last three terms corresponding to chiral magnetic effect, chiral vortical effect and chiral torsional effect(Khaidukov, Zubkov. 2018; Imaki, Yamamoto. 2019.) respectively.

Summary and outlook

- We introduce spin statistical mechanics with external torsion field by using local equilibrium density operator.
- We derive spin hydrodynamics, spin thermodynamics and chiral torsional effect.
- In the future: non-equilibrium density operator with torsion.

Thank you!