Vacuum birefringence and dichroism in strong magnetic fields

- KH and Kaz. Itakura, “Vacuum birefringence in strong magnetic fields”:
  (I) Photon polarization tensor with all the Landau levels,” [1209.2663]
  (II) Complex refractive index from the lowest Landau level,” [1212.1897]

- KH, Hidetoshi Taya, Shinsuke Yoshida, “Di-lepton production from a single photon in strong magnetic fields: vacuum dichroism”, [2010.13492]

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ATHIC 2021, Inha U., Nov. 5-9

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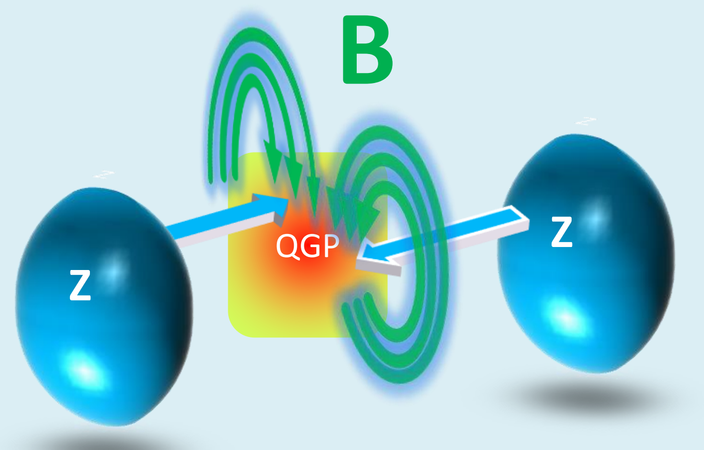
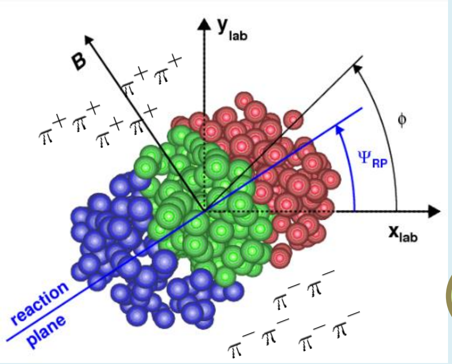
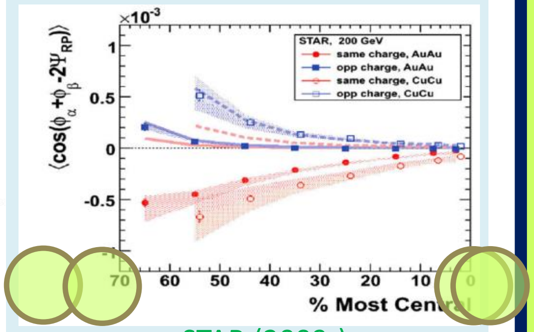
Peripheral collisions



Novel transport phenomena in QGP

- Chiral magnetic effect
- magnetohydrodynamics
- thermal radiations in B
- etc.

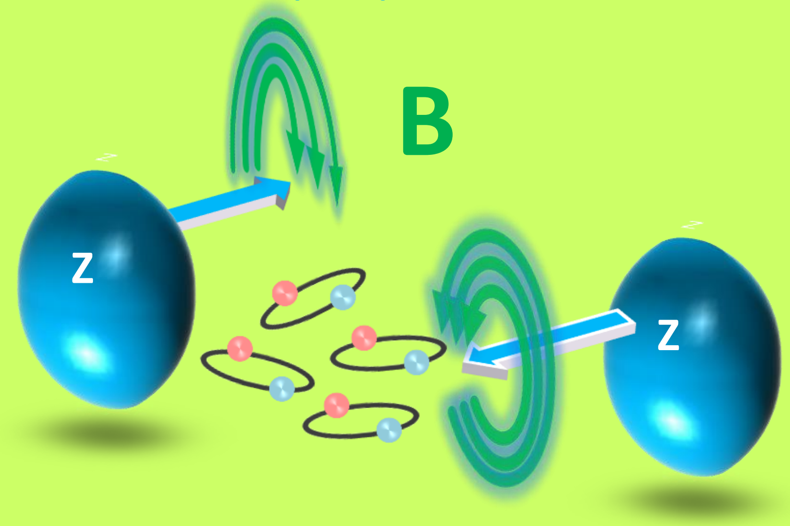




STAR (2009-)
ALICE (2014-)

Cf. new results on isobaric collisions (2021)

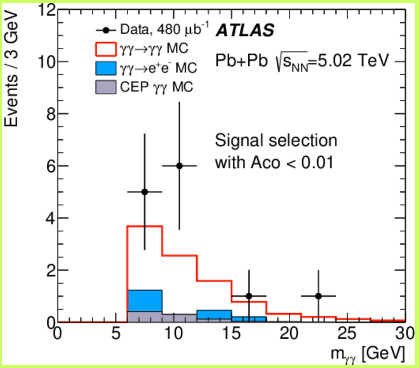
Ultraperipheral collisions

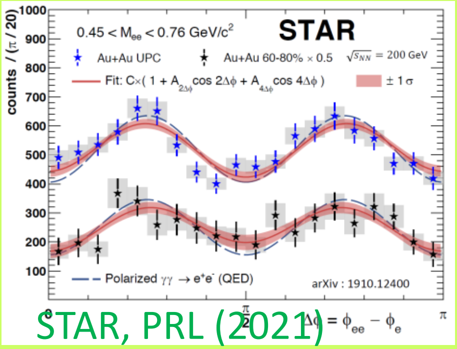


"Strong-field QED" without QGP

--- Vacuum physics in strong fields

- Photon-photon interactions
- Vacuum fluctuations in B
- etc.

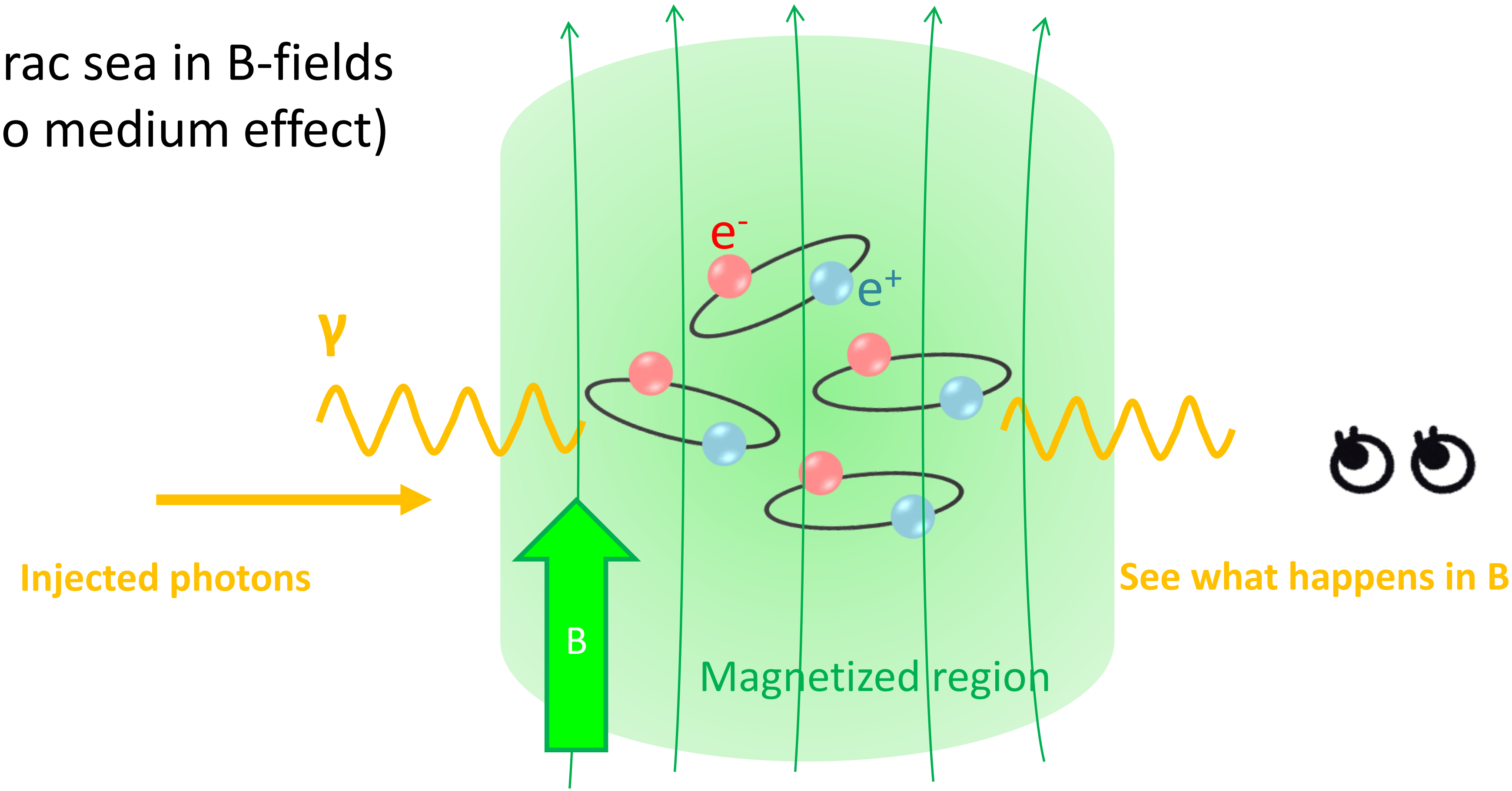




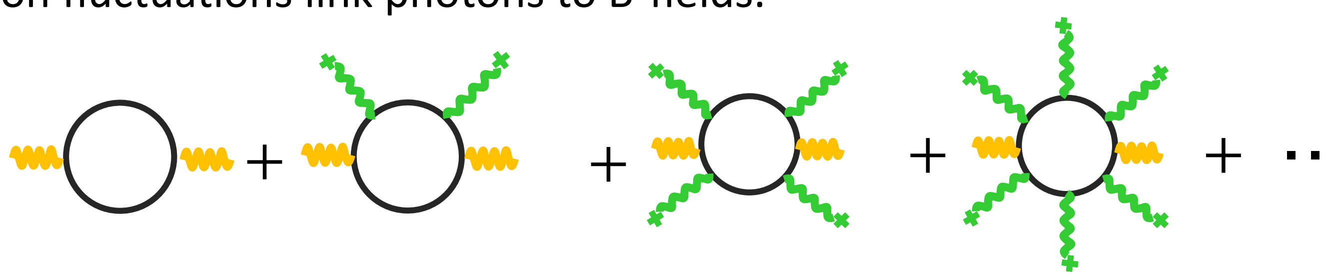
STAR, PRL (2021)

Photon propagation in B

Dirac sea in B-fields
(no medium effect)



Vacuum fermion fluctuations link photons to B-fields.

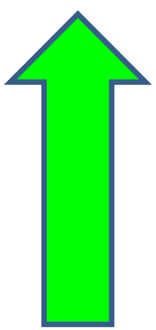


Photon propagation in magnetic fields

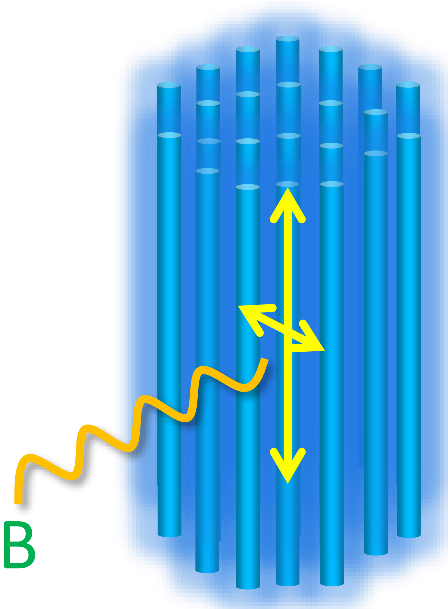
(in four dimensions)

~~Lorentz~~ & gauge symmetries → $n \neq 1$ in general

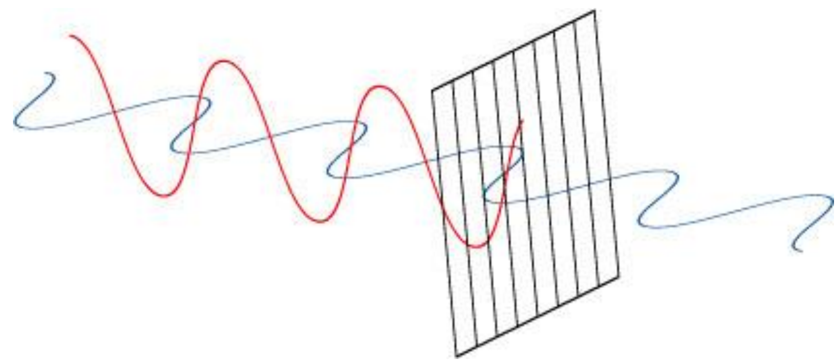
Preferred orientation provided by an external B



Strong B



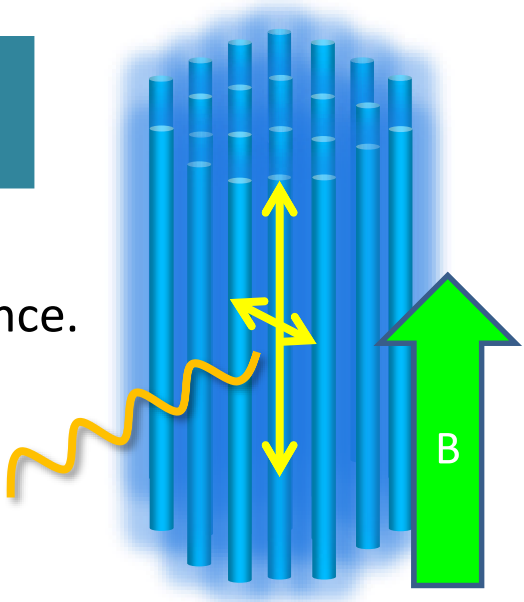
Cyclotron motion in B field



Polarizer

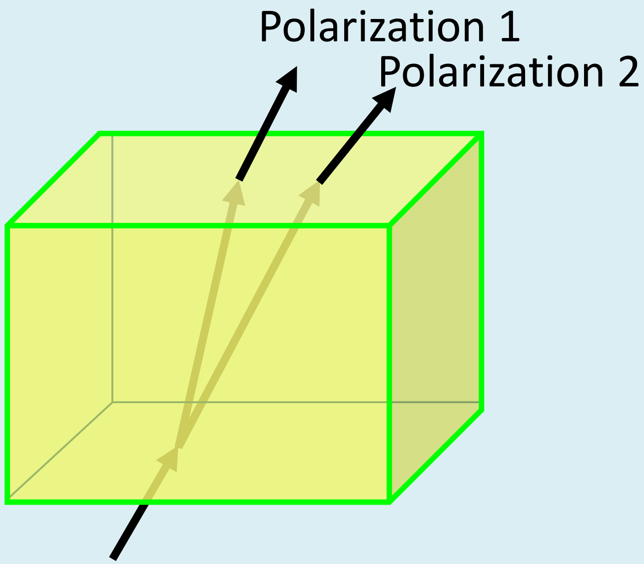
What is birefringence？

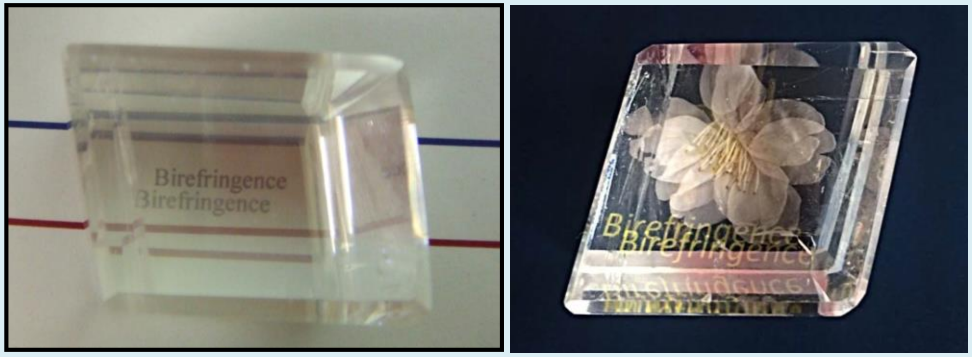
Birefringence = Polarization-dependent refractive indices



Anisotropic response of the Dirac sea may induce birefringence.

→ “Vacuum birefringence”

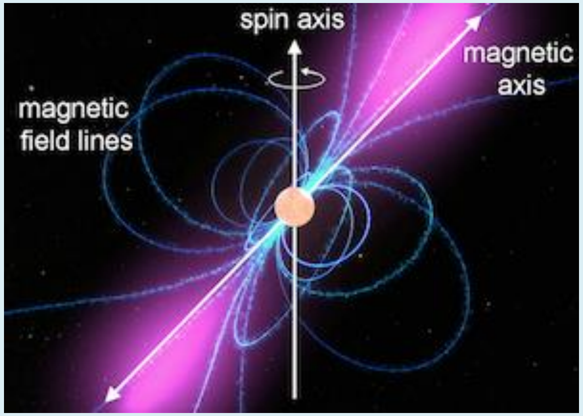




Birefringent substance “Calcite”

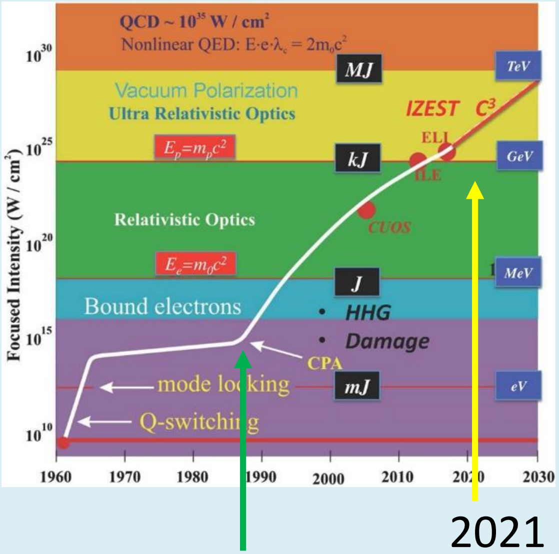
Experimental searches for the vacuum birefringence after the work by J. Toll (1952)

Magnetospheres of neutron stars/magnetars



EM processes in the magnetosphere

High-intensity laser field



Mourou & Strickland (2018 Nobel laureates)

https://www.nobelprize.org/prizes/physics/2018/mourou/lecture/

No confirmed results yet in the last half century.

- *Old but unsolved problem*

→ *Feasibility with HIC?*

Can we put a milestone in Strong-field QED/Nonlinear QED?

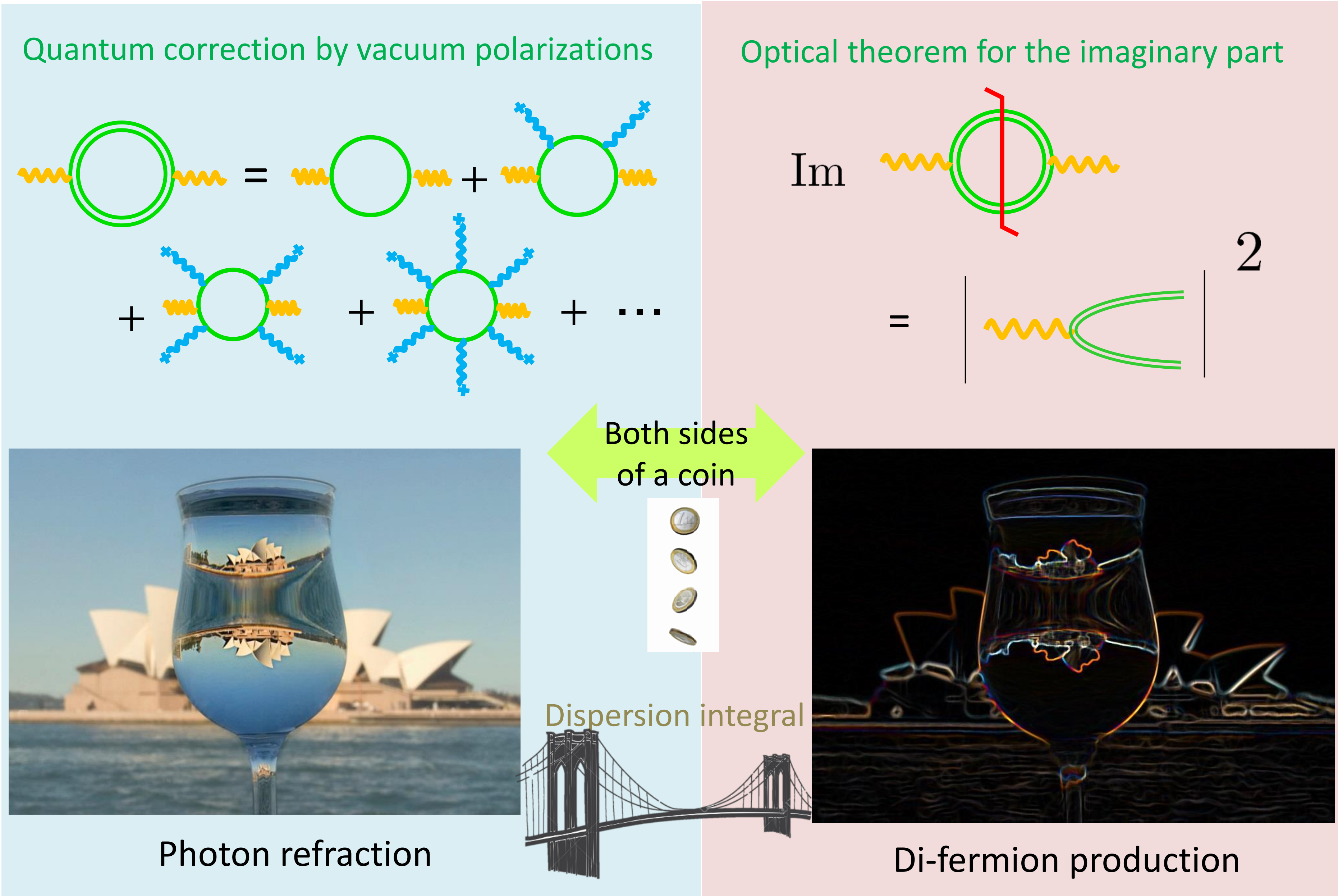
(1) Refractive index of photon in strong B-fields

KH and Kaz. Itakura, "Vacuum birefringence in strong magnetic fields":

(I) Photon polarization tensor with all the Landau levels," [1209.2663]

(II) Complex refractive index from the lowest Landau level," [1212.1897]

Complex refractive indices



Maxwell eq. w/ quantum corrections: $[q^2 g^{\mu\nu} - q^\mu q^\nu - \Pi^{\mu\nu}]A_\nu(q) = 0$

U(1) gauge symmetry constrains possible tensor structures $q_\mu \Pi^{\mu\nu} = 0$

$$\Pi^{\mu\nu} = -[\chi_0 P_0^{\mu\nu} + \chi_1 P_1^{\mu\nu} + \chi_2 P_2^{\mu\nu}]$$

$$P_0^{\mu\nu} = q^2\eta^{\mu\nu} - q^\mu q^\nu$$

$$P_1^{\mu\nu} = q_\parallel^2 \eta_\parallel^{\mu\nu} - q_\parallel^\mu q_\parallel^\nu$$
$$P_2^{\mu\nu} = q_\perp^2 \eta_\perp^{\mu\nu} - q_\perp^\mu q_\perp^\nu$$

B-induced structures

Preferred orientation in B

$$B = (0, 0, B)$$

$$\eta_\parallel^{\mu\nu} = \mathrm{diag}(1, 0, 0, -1)$$
$$\eta_\perp^{\mu\nu} = \mathrm{diag}(0, -1, -1, 0)$$
$$q_\parallel^\mu = (q^0, 0, 0, q^3)$$
$$q_\perp^\mu = (0, q^1, q^2, 0)$$

(Boost invariance along B)

Refractive indices from the Maxwell eq.

$$n = \frac{|\boldsymbol{q}|}{\omega}$$

Two (physical) polarization modes in || and ⊥ to B.

$$n_\parallel^2 = \frac{1 + \chi_0 + \chi_1}{1 + \chi_0 + \chi_1 \cos^2\theta} \to 1$$

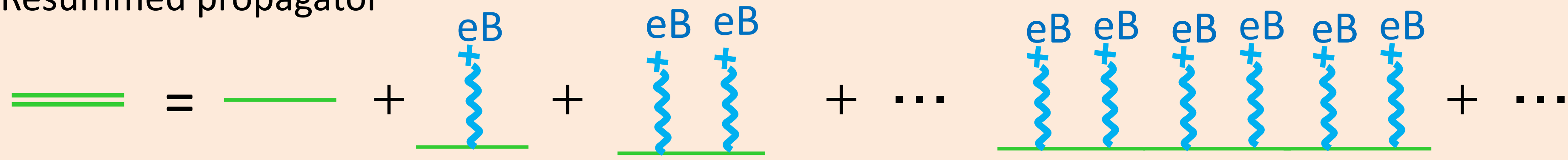
$$n_\perp^2 = \frac{1 + \chi_0}{1 + \chi_0 + \chi_2 \sin^2\theta} \to 1$$

Direct consequence of the gauge symmetry and the breaking of one spatial rotational symmetry.

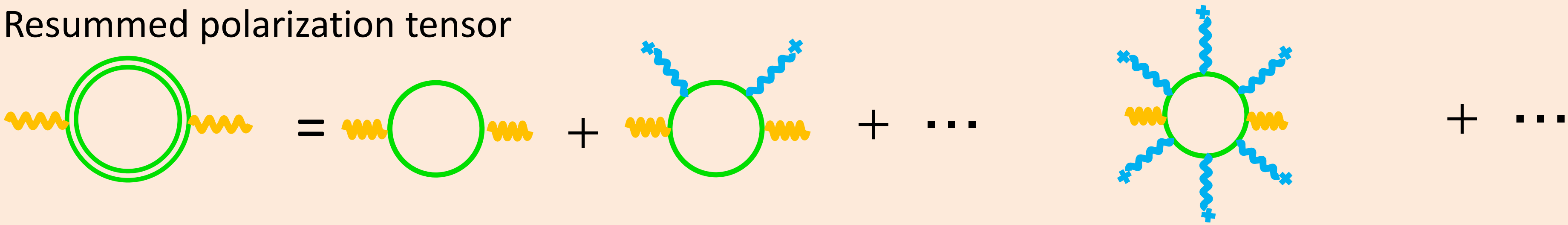
Vanishing B limit: $\chi_0 \to \Pi_{\text{vac}}\ ,\ \ \chi_{1,2} \to 0$

Resummation for external-field insertions

Resummed propagator



Resummed polarization tensor



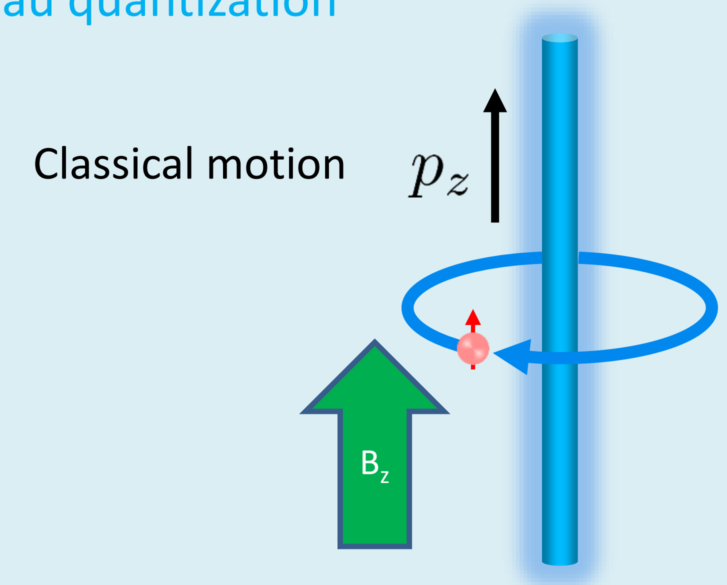
The resummation can be performed
with the proper-time method analytically.

Fock (1937), Nambu, Feynman (1950), Schwinger (1951)

Fermion spectrum in a magnetic field

Fermion spectra are crucial to understand how fermions response to photons.

Landau quantization

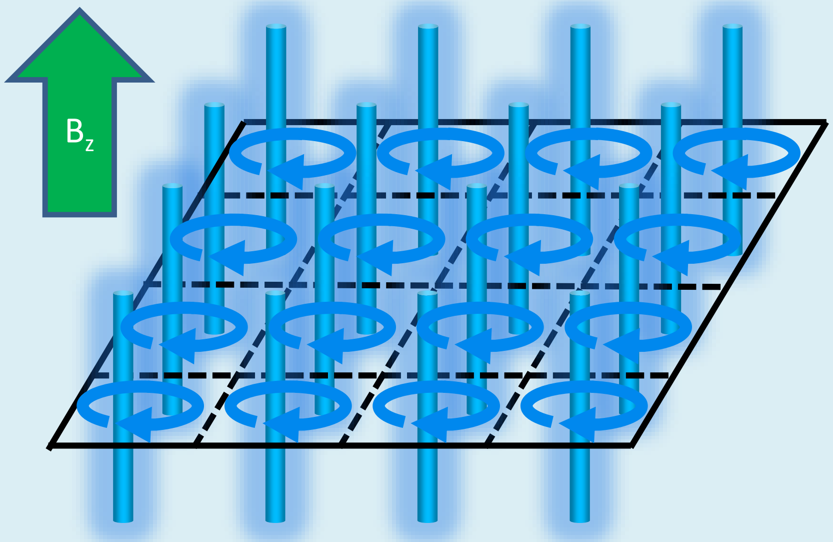


1. No effects on the longitudinal motion
2. Cyclotron motion ∼ Harmonic oscillation
3. Spin polarization (Zeeman effect)

Energy spectrum

$$\epsilon_n^2 = p_z^2 + m^2 + (2n+1)|eB| \pm \frac{g}{2}|eB|$$

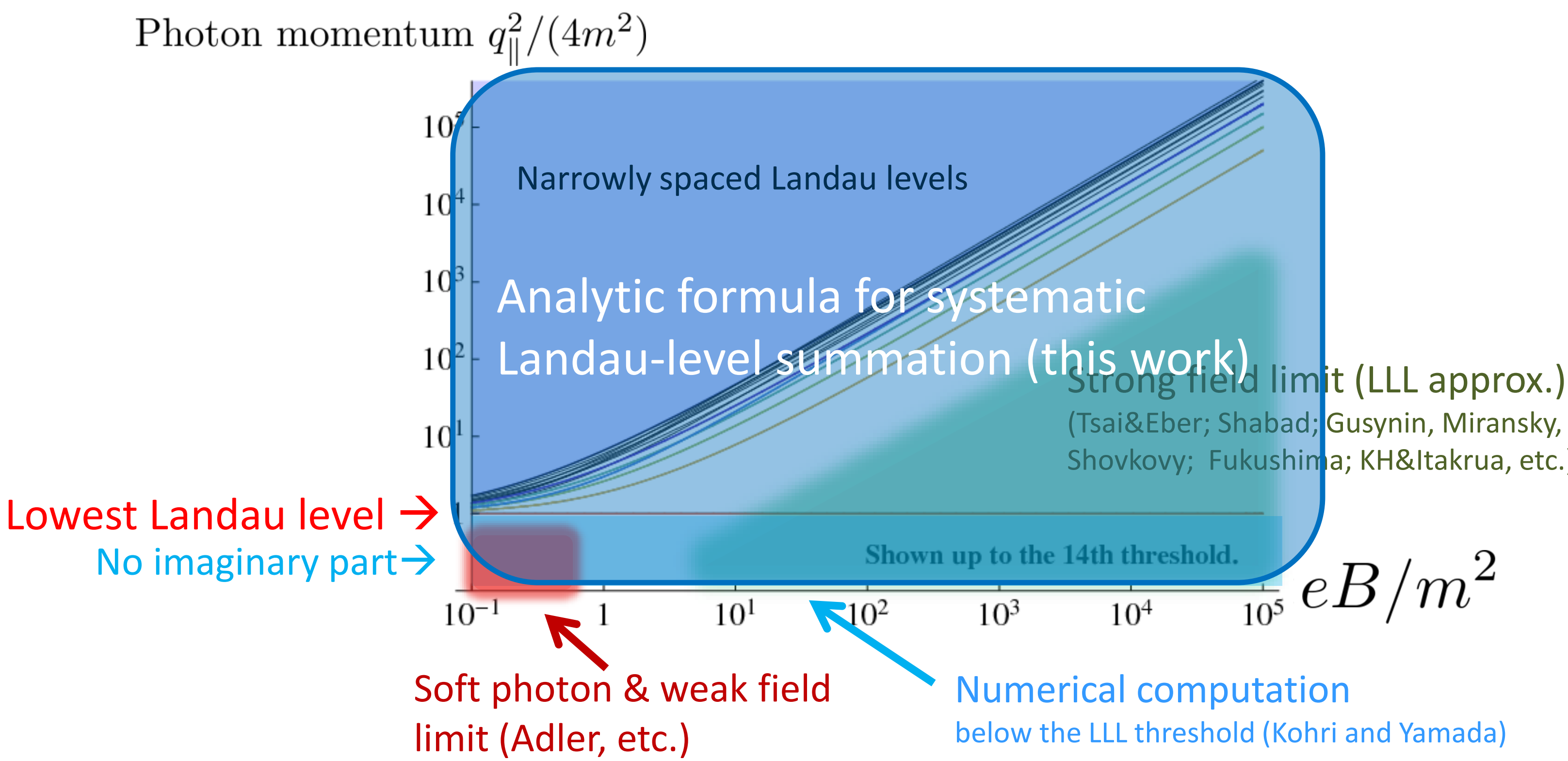
Landau degeneracy



No energetically favored position.
→ Degeneracy

$$\text{Landau degeneracy } \rho_B = \frac{eB}{2\pi}$$

Relevant scales and Landau levels



KH, Itakura, 1209.2663, 1212.1897

2. Differential dilepton spectrum

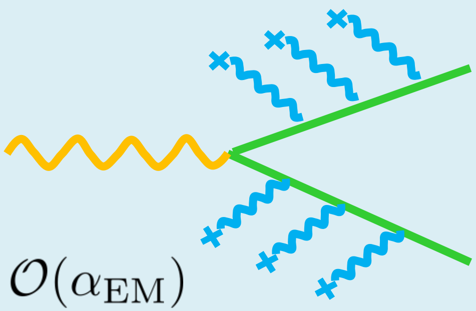
--- Better accessibility than the photon polarization

KH, Hidetoshi Taya, Shinsuke Yoshida, “Di-lepton production from a single photon in strong magnetic fields: vacuum dichroism”, [2010.13492]

Photo-induced dilepton production in a magnetic field

LO w/ B-fields

$$\gamma \to (f\bar{f})_B$$
$$\gamma^* \to (f\bar{f})_B$$



$$|\mathcal{M}|^2 \sim \mathcal{O}(\alpha_{\mathrm{EM}})$$

Nonperturbatively dressed fermions

Both on-shell and off-shell photons can decay in B-fields.

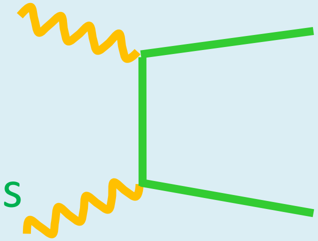
LO w/o B-fields

No kinematical window for a single on-shell photon when B = 0.

→ Starts only from 2 photons

$$|\mathcal{M}|^2 \sim \mathcal{O}(\alpha_{\mathrm{EM}}^2)$$

E.g., Breit-Wheeler process



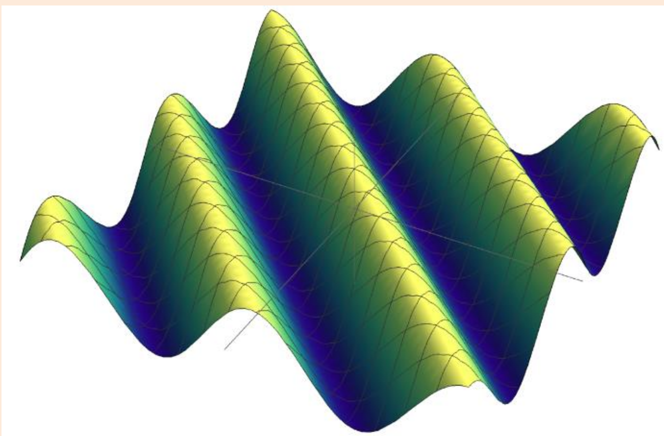
Ritus basis formalism

See a review part in [2010.13492]

= Perturbation theory with the eigenmodes in B-fields.

Free Dirac eq. → Plane wave solutions

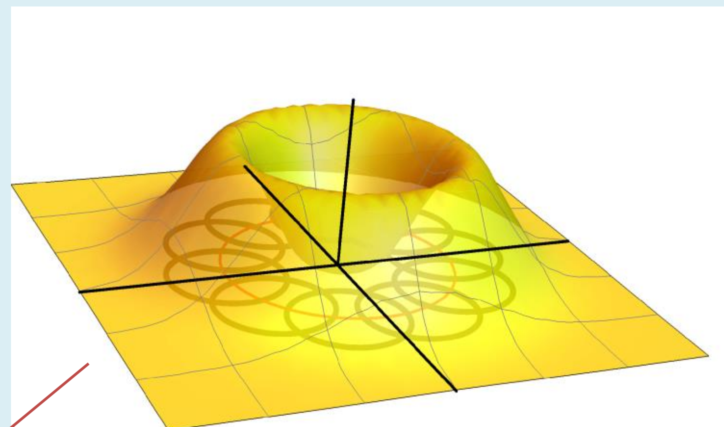
$$(i\partial\!\!\!/ - m)\psi = 0$$



Dirac eq. in B-field → Localized wave functions

$$(iD\!\!\!\!/_{\rm ext} - m)\psi = 0$$

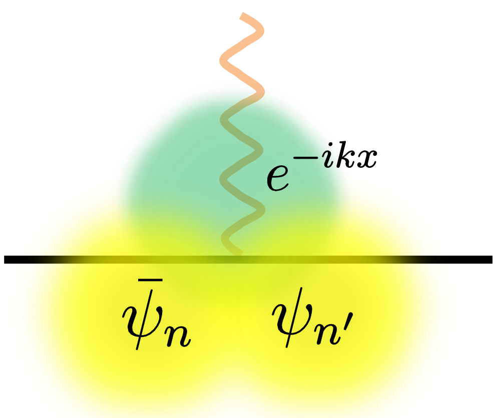
$$D^\mu_{\rm ext} = \partial^\mu + ieA^\mu_{\rm ext} \qquad {\rm rot}\,\boldsymbol{A}_{\rm ext} = \boldsymbol{B}$$



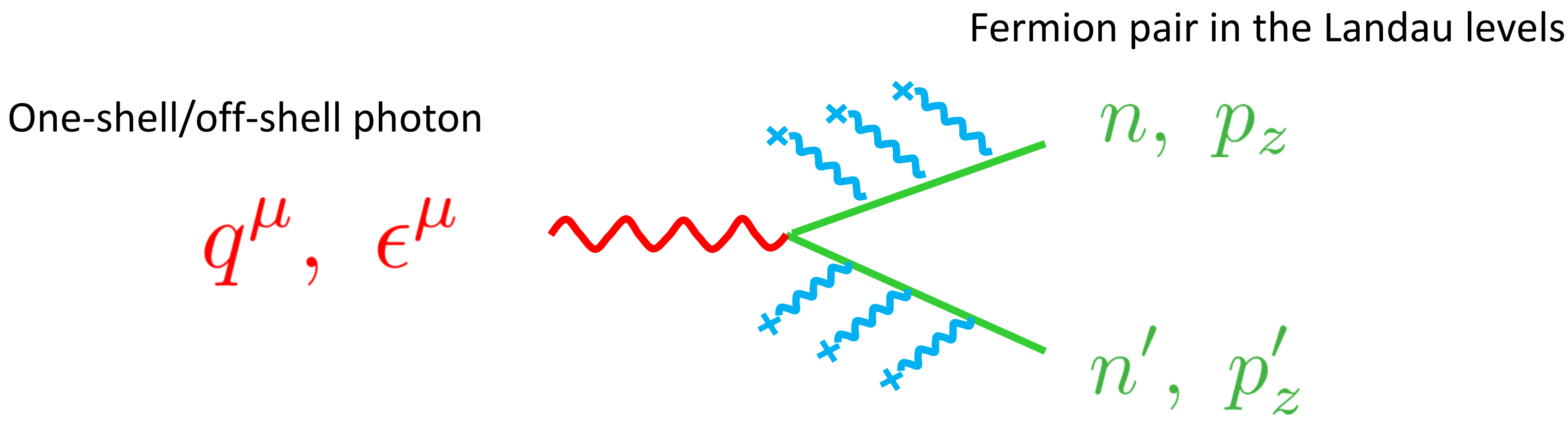
Mode expansion with the eigen-basis in B-fields

- Fermion propagator (= Inverse of D) is given in a simple form.
- Price: The vertex function is no longer a simple delta function.

  Convolution among different bases at the vertices.



Pair production rate for general kinematics



Differential information includes

- Photon polarization (ε^μ)
- Photon momentum (q^μ) with a general direction and invariant mass
- Landau levels (n, n') and continuous momentum (p_z, p_z') along B

Consistency checks done; Ward identity, etc. in [2010.13492]

Muon-pair excess over electron pairs

KH, Taya, Yoshida [2010.13492]

"Helicity suppression" in pion decay

PDG

$\pi^- \to \mu^- \bar{\nu}_\mu \quad 99.9877\ \%$

$\pi^- \to e^- \bar{\nu}_e \quad 10.23 \times 10^{-4}\ \%$

Opposite momentum

Opposite spin   Same helicity

$\ell^- \quad S_\ell = \pm\frac{1}{2} \quad h_\ell = \pm\frac{1}{2}$

$\pi^- \quad S = 0$

Opposite chirality
→ Needs <u>chirality mixing by a finite mass</u>.

$\bar{\nu} \quad S_{\bar\nu} = \mp\frac{1}{2} \quad h_{\bar\nu} = \pm\frac{1}{2}$

The LLL (= soft photon)

$$\frac{N_{\mu^+\mu^-}}{N_{e^+e^-}} \propto \frac{m_\mu^2}{m_e^2} \sim 4.4 \times 10^4$$

Opposite spin
<u>along B-field</u>   Same helicity

$S_{\ell-} = -\frac{1}{2} \quad h_{\ell-} = -\frac{1}{2}$

$S_\gamma = 0$

Superposition of $(\pm, L)$

Opposite chirality

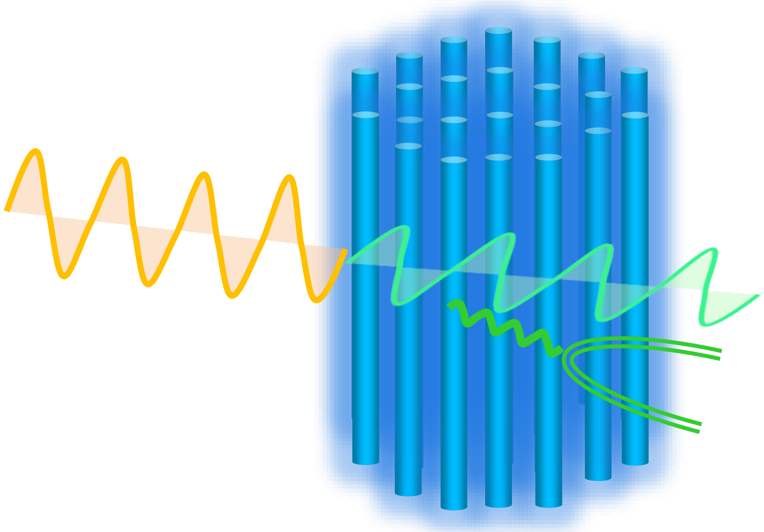
$S_{\ell+} = +\frac{1}{2} \quad h_{\ell+} = -\frac{1}{2}$

Especially, R neutrino (L antineutrino) does not exist at the QCD scale.
(However, this is not an essential reason for the helicity suppression.)

Summary

- Magnetic fields give rise to refractive indices and (on-shell) photon decay.

When photons go through B-fields



- Photon polarization rotates.
- Some of the photons decay into fermion pairs.

Prospects for the UPC

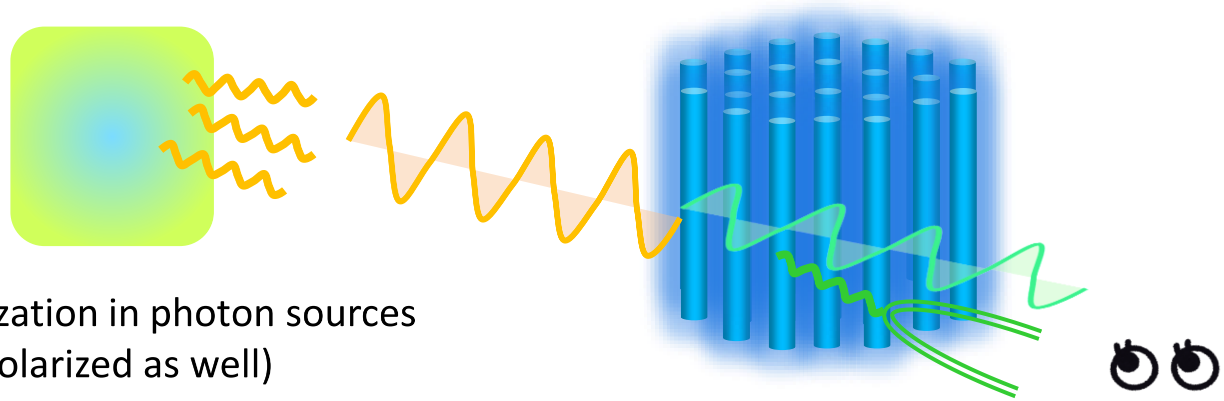
- Time dependence of a magnetic field
- Needs convolution with the photon distribution function

KH, Xu-Guang Huang, Hidetoshi Taya, Shinsuke Yoshida, In progress.

Back-up slides

Feasibility with HIC?

Ideal set-up

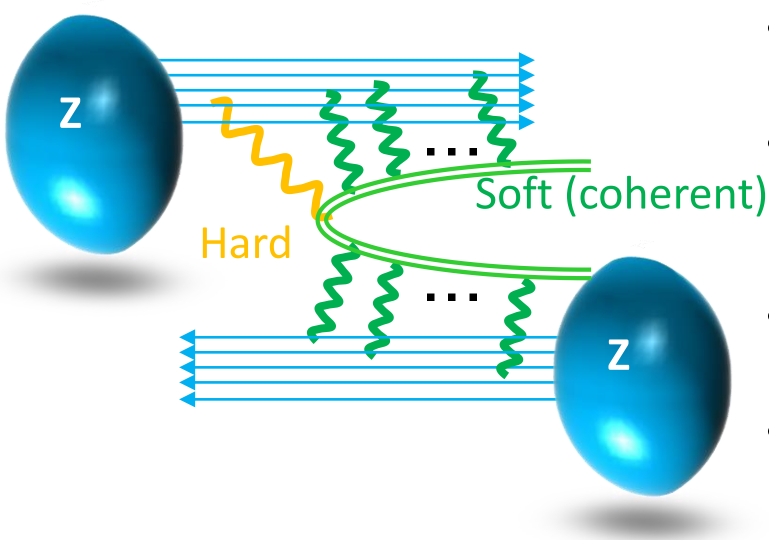


Inborn polarization in photon sources
(Possibly unpolarized as well)

- Acquired polarization due to the birefringence
- Some of photons decay into fermion pairs

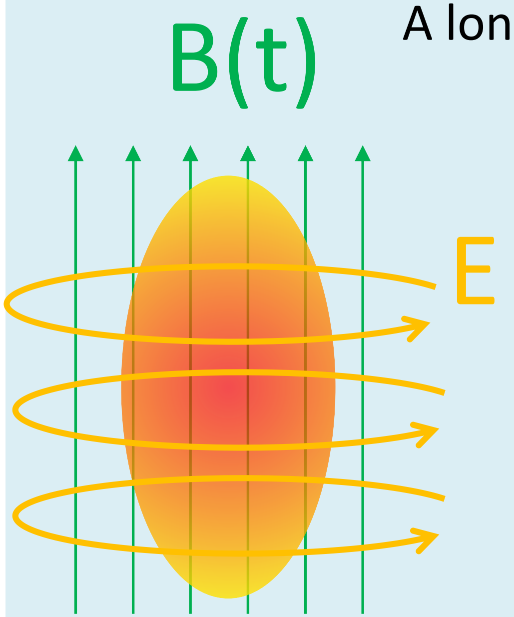
UPC events

What is a photon source and what is a strong B field in HIC?



- There is no a priori difference: Both are Coulomb electric fields in the nucleus rest frame.
- The photon source has a momentum distribution: Fourier transform of the charge distribution, e.g., the Woods-Saxon profile.
- The hard and soft components of the distribution may be regarded as "photons" and a "magnetic field".
- Needs quantitative estimates with a separation scale.

Lifetime of the B-field after the collisions



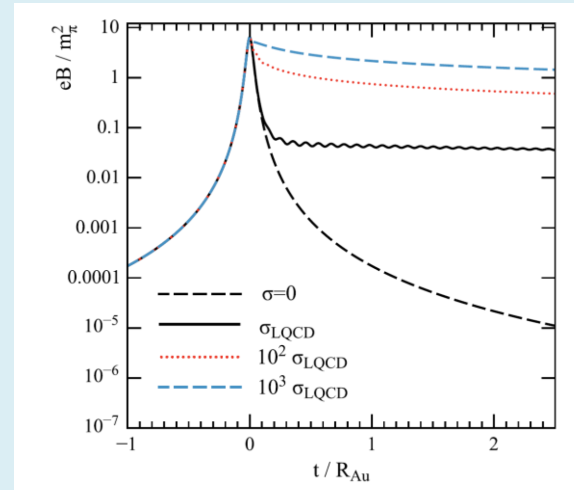
A longer lifetime due to the Lenz's law?

Tuchin

Time dependent B induces E.

E induces J if QGP is conducting.

→ Induced J sustains B.



McLerran & Skokov

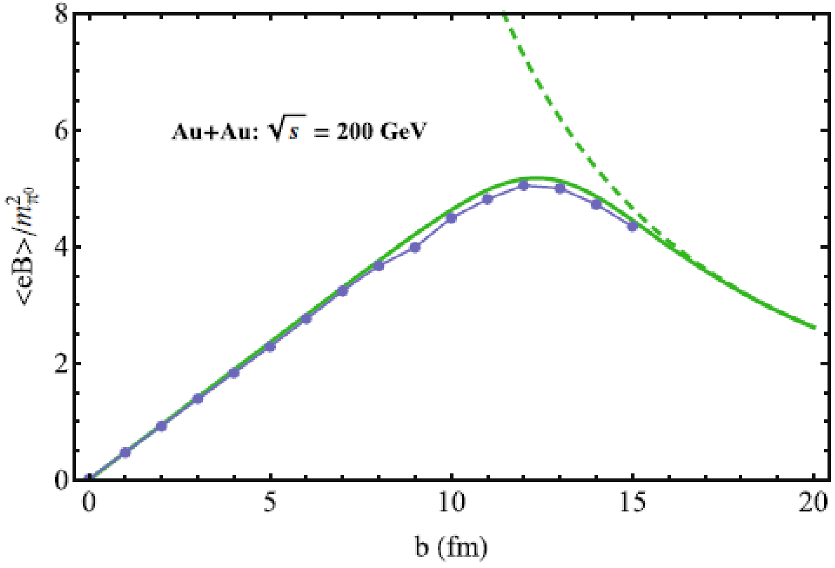
Important to know the conductivity of QGP in magnetic fields. KH & Satow; KH, Li, Satow, Yee; Fukushima & Hidaka

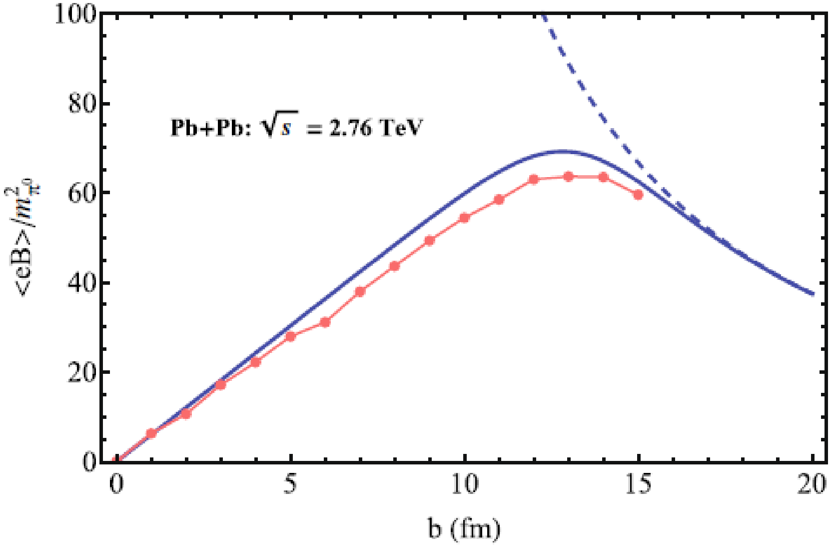
Isobaric collisions will help us to understand the backgrounds and to extract the magnetic-field effects.

$$\Delta Z \sim 10\ \% \ \rightarrow \Delta B \sim 10\ \%,$$
but little difference in the flow effects

Cf. Deng, Huang, Ma, Wang for recent estimates

Analytic and numerical estimates of the strong B
-- Impact parameter dependences

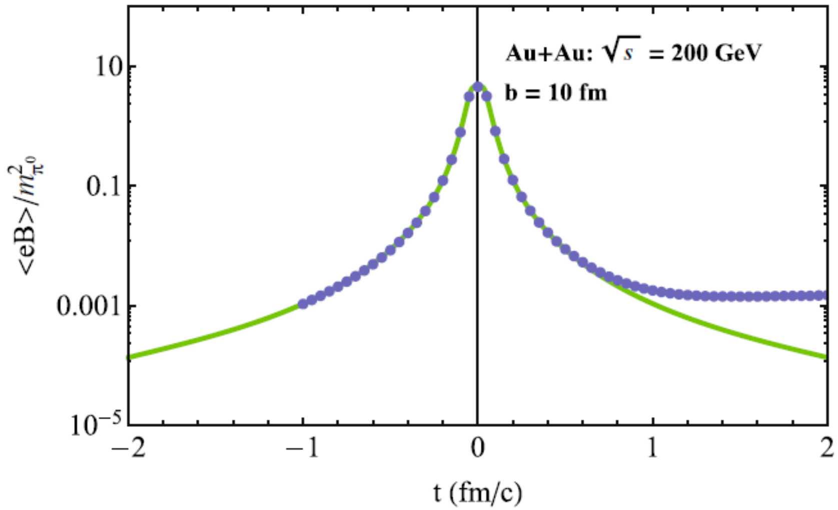


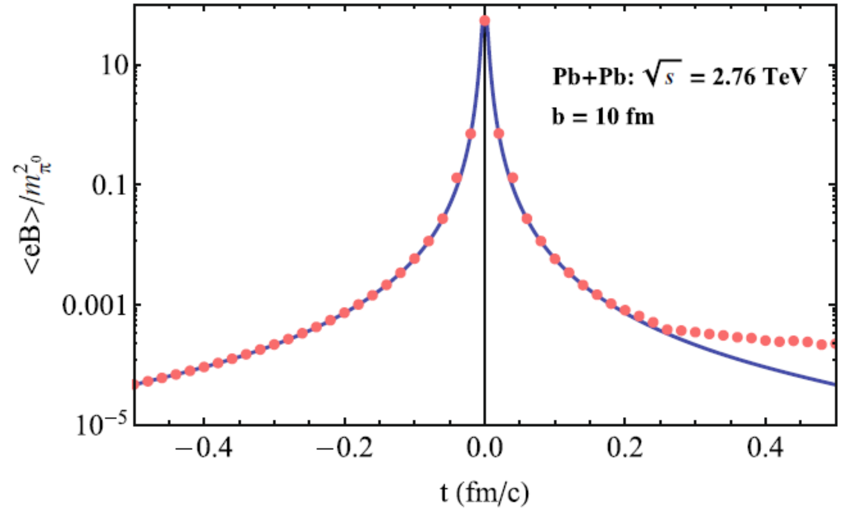


t = 0 (at the collision)

W.-T. Deng & X.-G. Huang, KH and X.-G. Huang [1609.00747]

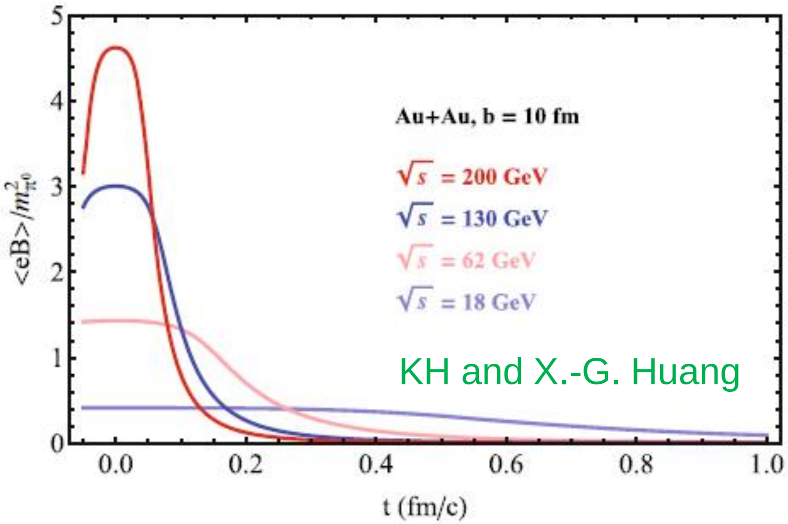
Time dependences



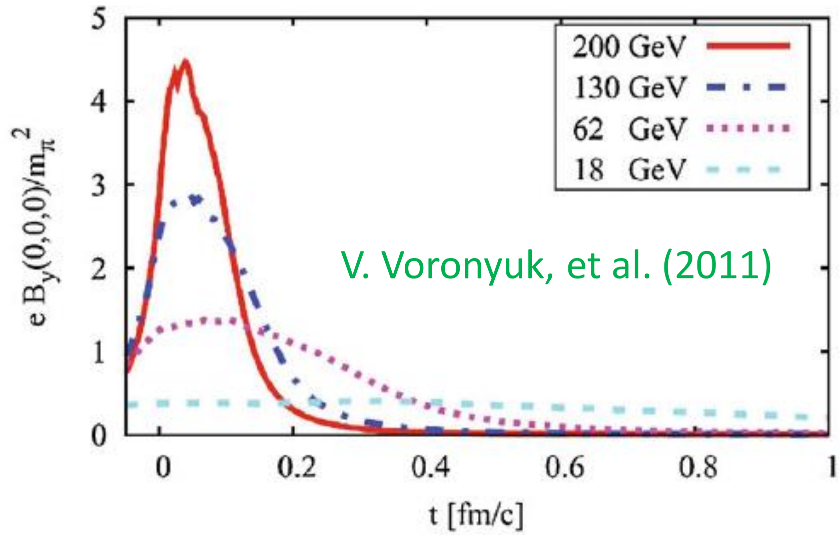


KH and X.-G. Huang, [1609.00747](1609.00747)

Collision-energy dependences



KH and X.-G. Huang



V. Voronyuk, et al. (2011)

Proper-time method

Fock (1937), Schwinger (1951)

$$S_A(p) = \frac{i}{\not{D} - m + i\epsilon}$$

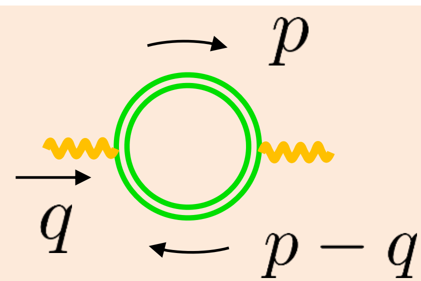
A^μ for external (constant) fields

$$= i(\not{D} + m) \times i^{-1} \int_0^\infty d\tau e^{i\tau(\not{D}\not{D} - m^2 + i\epsilon)}$$

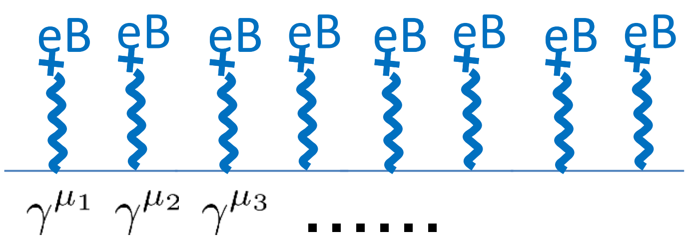
τ : proper-time

No p and A in the denominator → Gaussian form

$$i\Pi^{\mu\nu} = e^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}[\gamma^\mu S_A(p) \gamma^\nu S_A(p-q)]$$



Nonlinear wrt the external fields



Technically demanding.
Ex) One needs to perform the Dirac trace with an "infinite" number of gamma matrices.

Peskin & Schoeder tell us only tr[γ γ γ γ] or a little more…

Fermion pair spectrum in the imaginary part
--- Thresholds at the Landau levels

Only one possible source of the imaginary part:

$$\frac{1}{\sqrt{4ac-b^2}}\left[\arctan\left(\frac{b+2a}{\sqrt{4ac-b^2}}\right)-\arctan\left(\frac{b-2a}{\sqrt{4ac-b^2}}\right)\right]$$

$$a=q_{\parallel}^2/(4m^2),\ \ b=-(n-\ell)eB/m^2, c=(1-a)+(\ell+n)eB/m^2$$

Polarization tensor acquires an imaginary part when $4ac-b^2\leq 0$

Threshold condition

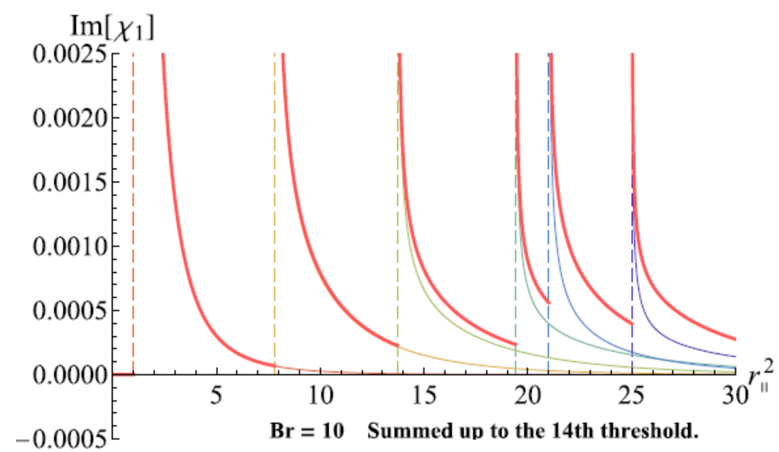
$$q_{\parallel}^2 \geq (\epsilon_\ell+\epsilon_n)^2 \qquad \epsilon_n=\sqrt{p_z^2+2neB+m^2}$$

The integers are identified with the Landau levels.

- Compatible with the on-shell condition $q^2=0$.

$$\begin{aligned} q_{\parallel}^2 &= \epsilon_\gamma^2-q_z^2 \\ q^2 &= \epsilon_\gamma^2-q_z^2-|\boldsymbol{q}_\perp|^2 \end{aligned}$$

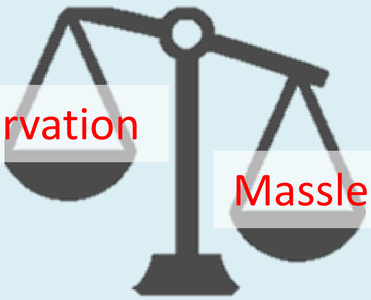
Photon transverse momentum works like a "photon mass" in the (1+1)-d kinematics.



Kinematics w/o magnetic fields

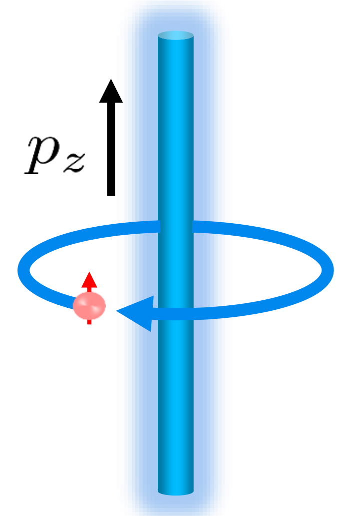
Not compatible with each other without B.

$$q_\gamma^2 = (\epsilon_f + \epsilon_{\bar{f}})^2_{\mathrm{CoM}} \neq 0$$



E-m conservation

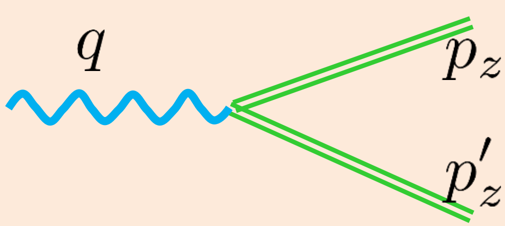
Massless on-shell condition



Kinematics with the Landau levels

Fermion transverse momentum is NOT a good quantum number.

No transverse momentum conservation between fermions and photons (due to momentum supply from external B-fields).



$$\epsilon_\gamma = \epsilon_f + \epsilon_{\bar{f}}$$

$$q_z = p_z + p'_z$$

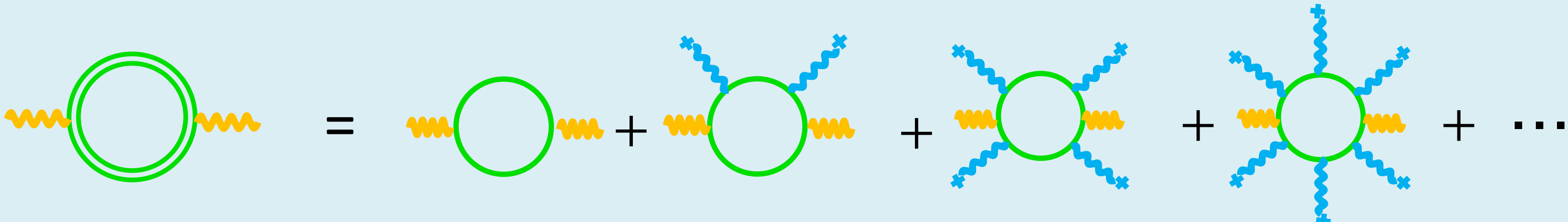
In B-field, $q_\parallel^2 \geq (\epsilon_\ell + \epsilon_n)^2$

- $q_\parallel^2 = \omega^2 - q_z^2$: Photon energy in the frame where $q_z = 0$.
  NB) Boost invariance along constant B-fields.
- Real photon can decay in B.
  NB) Cyclotron radiation is another 1-to-2 process in B-fields.

Vacuum polarization tensor in two different series representations

1. Naïve perturbative series when eB is small.

Naïve perturbation breaks down when eB is large!



$\mathcal{O}\left((eB)^0\right)$ $\mathcal{O}\left((eB)^2\right)$ $\mathcal{O}\left((eB)^4\right)$ $\mathcal{O}\left((eB)^6\right)$ $\mathcal{O}\left((eB)^{2n}\right)$

No modification

Weak-field approximation
Adler, etc.

2. Landau level representation

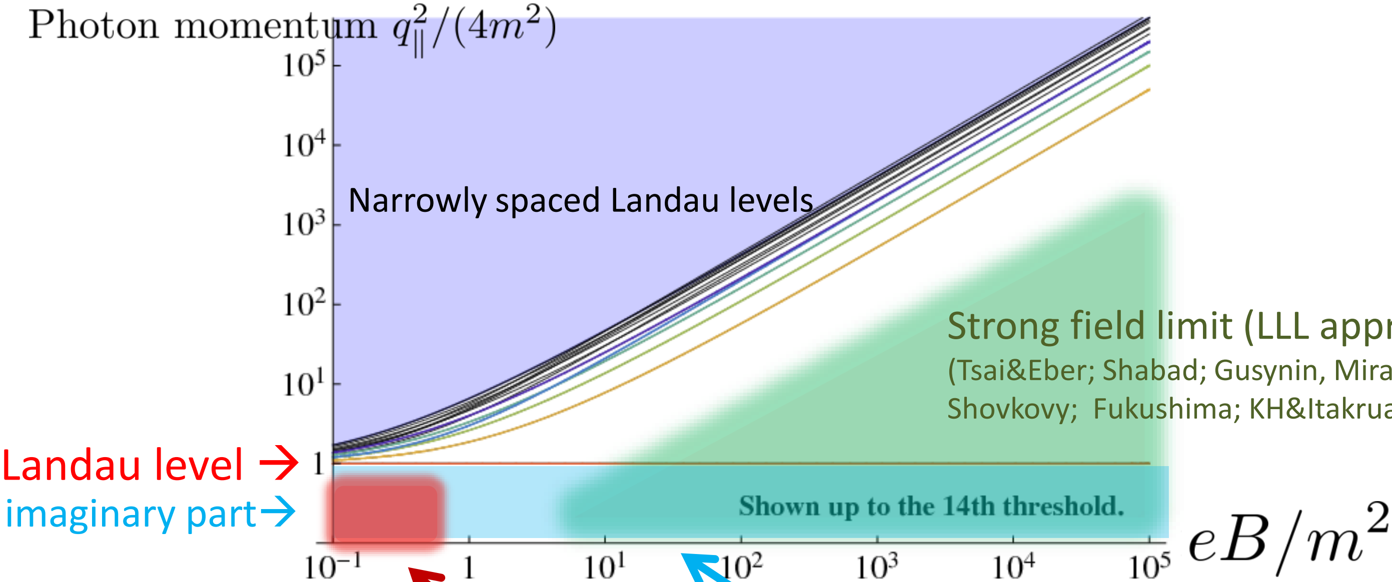
$$= \frac{eB}{2\pi} \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} F_{\ell,n}(q_{\parallel}^2, q_{\perp}^2)$$

Lowest Landau approximation
(ℓ = n = 0) when eB is large.

KH&Itakrua [1209.2663, 1212.1897]

Cf. For the Landau level representation of the HE effective action,
see KH, Itakura, Ozaki [2001.06131].

Summary of relevant scales and preceding calculations



Lowest Landau level →

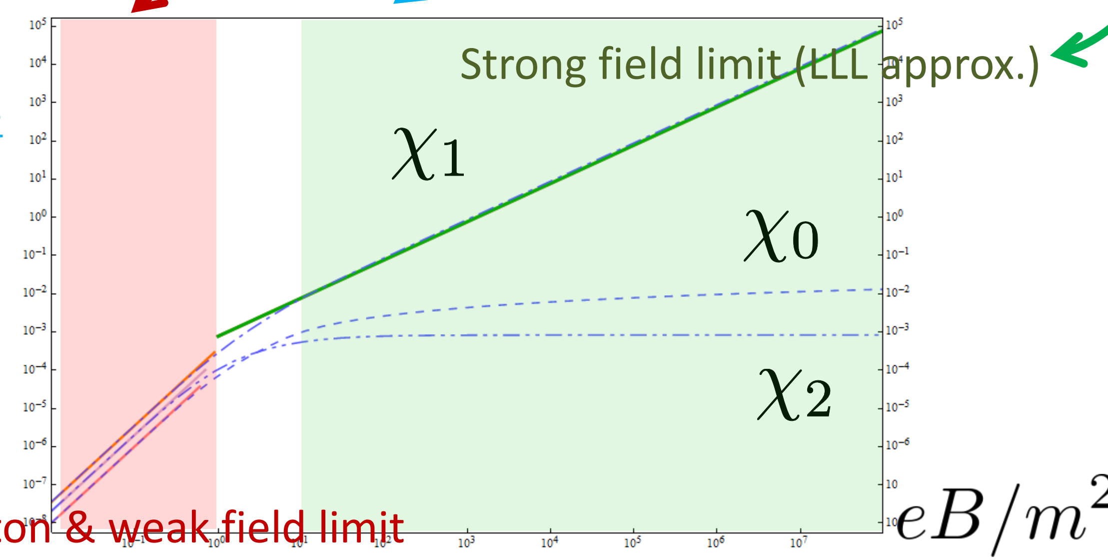
No imaginary part→

Soft photon & weak field limit (Adler, etc.)

Numerical computation

below the LLL threshold (Kohri and Yamada)

Numerical computation in the <u>soft photon regime</u>



Soft photon & weak field limit

Refractive indices and decay rate

$$\Pi^{\mu\nu} \to n_{\parallel,\perp}$$

Refractive indices with the LLL fluctuations

Refractive indices at the LLL(ℓ=n=0)

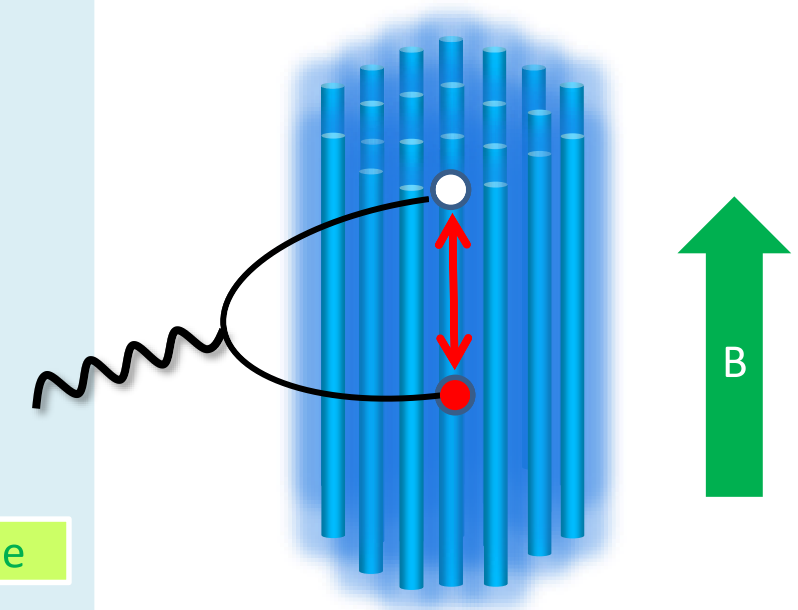
Polarization excites only along the magnetic field
``Vacuum birefringence''

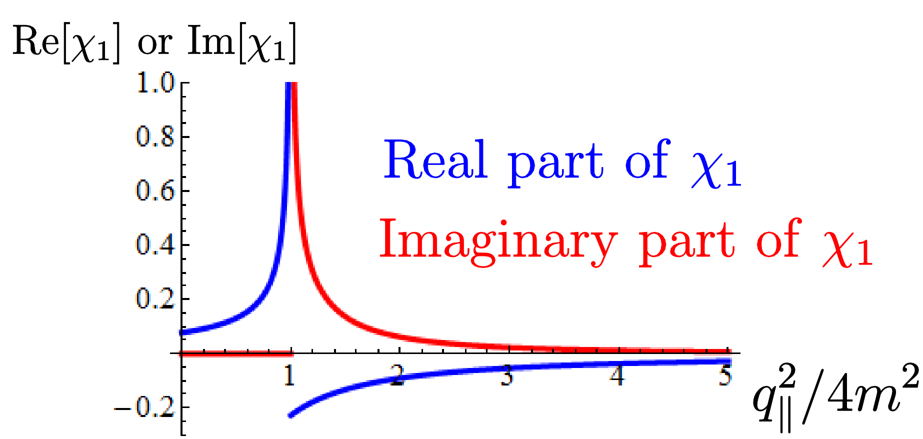
$$\chi_1 \neq 0, \quad \chi_0 = \chi_2 = 0$$

$$\begin{cases} n_{\parallel}^2 = \dfrac{1+\chi_1}{1+\chi_1 \cos^2\theta} \\ n_{\perp}^2 = 1 \end{cases}$$

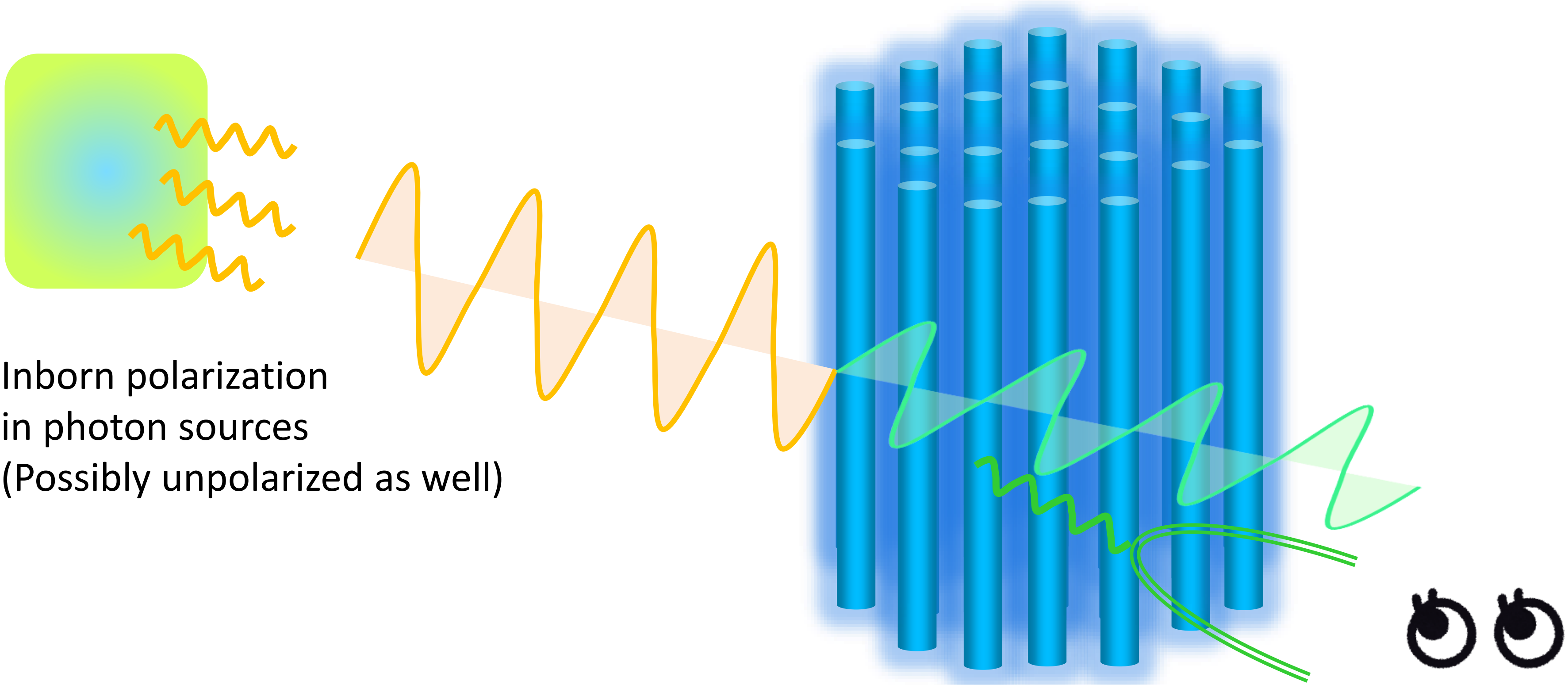
← No modification in the ⊥ mode

(1+1)-dimensional fluctuations





Rotating polarizations of transmitted ray

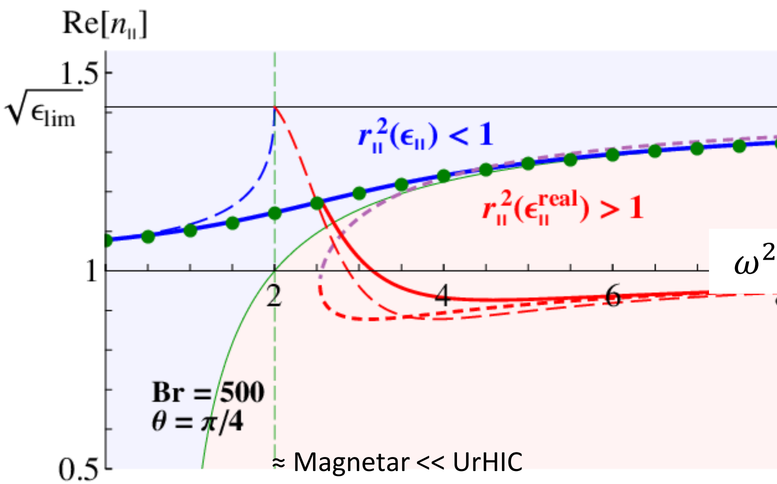


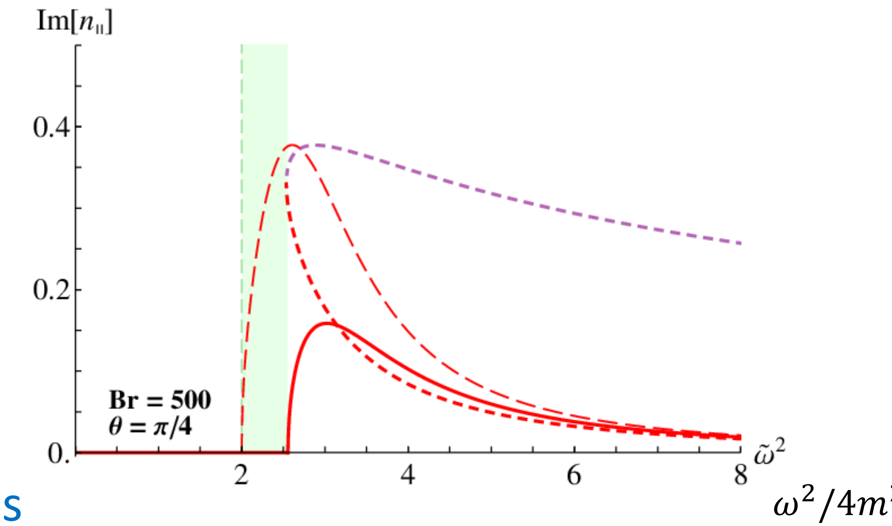
Inborn polarization
in photon sources
(Possibly unpolarized as well)

- Acquired polarization due to the birefringence
- Some of photons decay into fermion pairs

Cf. Cotton-Mouton effect, Faraday effect, etc in optics

Complex refractive indices





Final results shown by solid lines

cf. air n = 1.0003, water n = 1.3, prism n = 1.5

Refraction

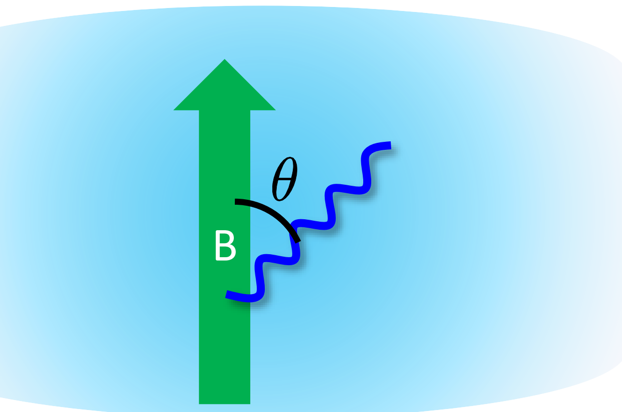
Difermion production

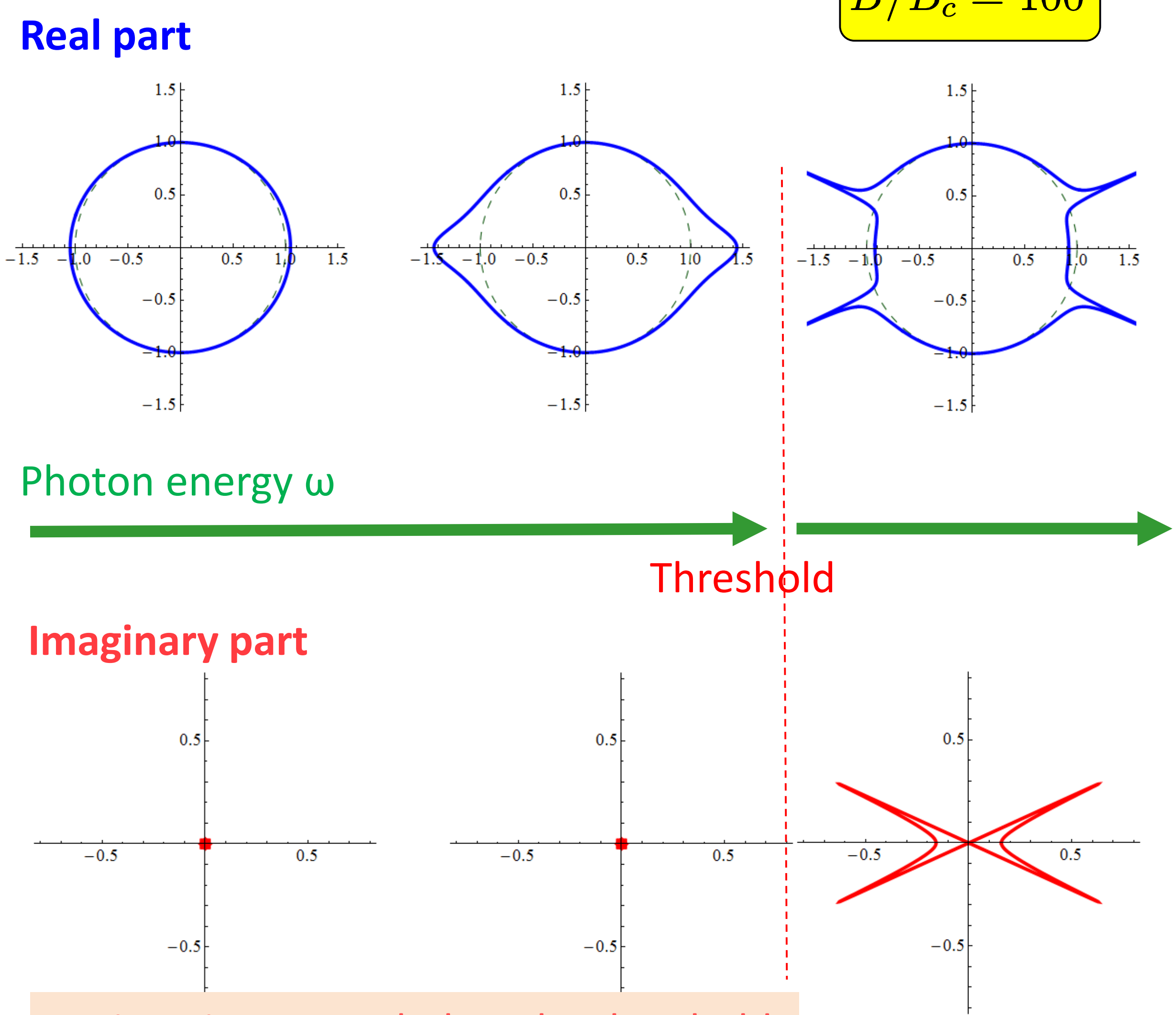
Anisotropy of the refraction index

Angle : Direction of the photon propagation

Radius : Magnitude of the refraction index

$B/B_c = 100$





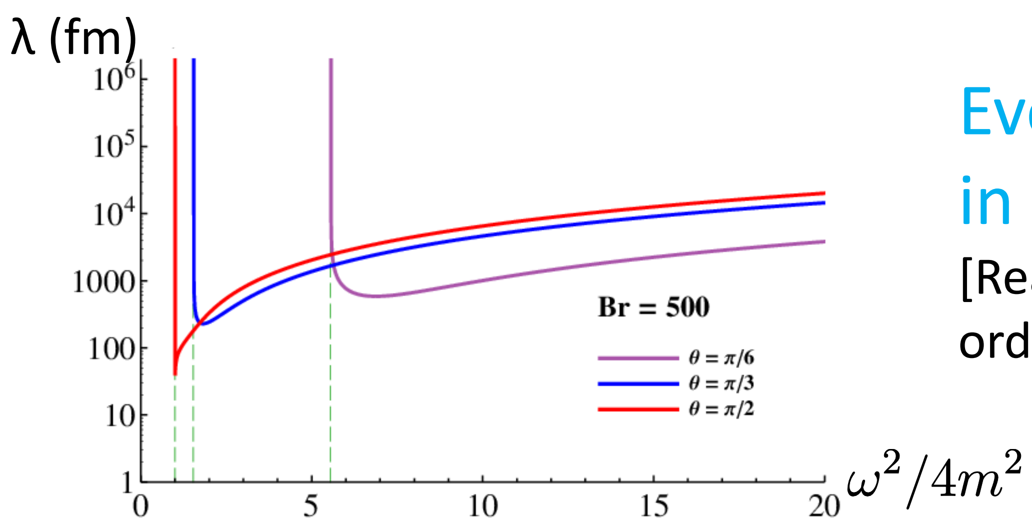
No imaginary part below the threshold

"Mean-free-path" of photons in B-fields

When the refractive index has an imaginary part,

$$\text{Photon flux}: \quad I \propto \exp\{-\lambda^{-1}\hat{\boldsymbol{q}}\cdot\boldsymbol{x}\}$$

"Mean-free-path" $$\lambda = \frac{1}{2\omega n_{\rm imag}}$$



Even real photons decay in a microscopic scale!

[Real photons never decay in ordinary vacuum without B-field.]

$\omega \sim 1\ \text{GeV}$

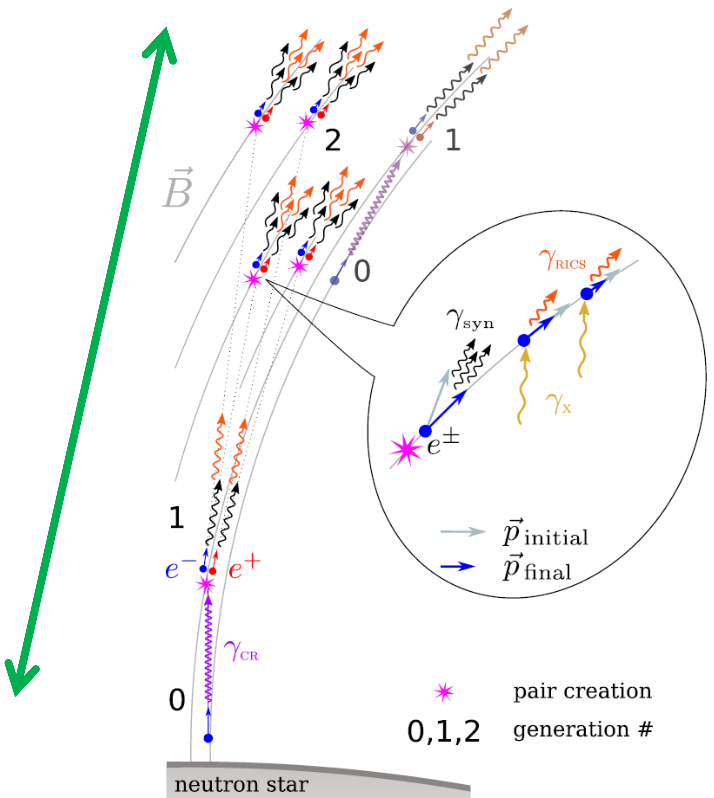
$\to \lambda \sim 1 - 10\ \text{fm.}$

Smaller mfp for a larger energy ~ 1/ω

Implications

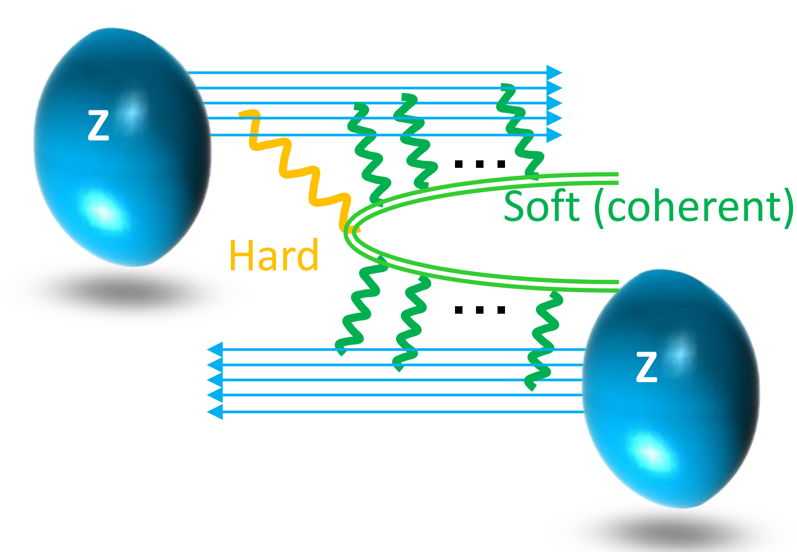
Neutron star magnetosphere

Macroscopic scale $\gtrsim 10 - 100$ km



Timokhin & Harding, ApJ (2019)

Ultraperipheral heavy-ion collisions



✘ Difficult to measure high-energy photon polarizations.

✔ One could analyze the dilepton angle-distribution.

Imaginary part of the refractive index
= Total cross section.

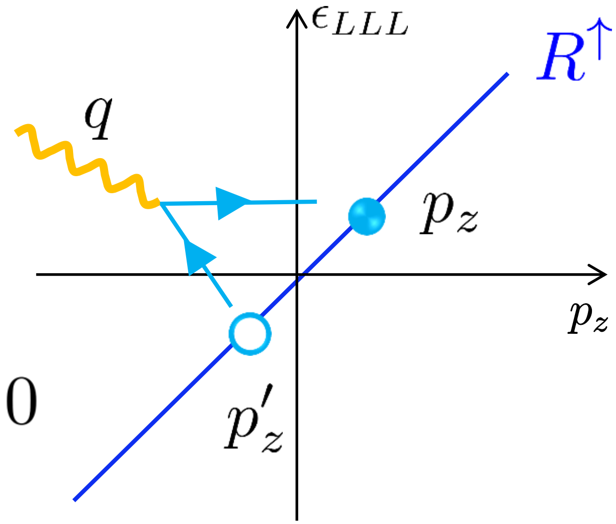
Differential dilepton cross-section may be useful.

Kinematics in the massless LLL

Linear dispersion relations in the lowest Landau levels

$$\epsilon_p = \pm p_z$$

R handed
L handed (when eB>0)



Perturbative vertex does not mix the R and L.

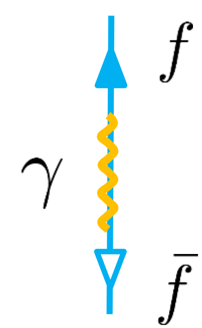
$$q_{\parallel}^2 = \{\pm(p_z + p'_z)\}^2 - (p_z + p'_z)^2 = 0$$

$$\Rightarrow \quad q^2 = -|\boldsymbol{q}_{\perp}|^2$$

Kinematics in the massless limit is satisfied only in the "collinear limit."

⇒ <u>No coupling to the transverse photons</u>.

$$j_\mu A^\mu = 0$$



Canonical quantization in B-field

See a review part in [2010.13492]

$$(i\not{D}_{\rm ext} - m)\psi = 0$$

$$D^\mu_{\rm ext} = \partial^\mu + i|e|A^\mu_{\rm ext}$$

$$\mathrm{rot}\,\boldsymbol{A}_{\rm ext} = \boldsymbol{B}$$

Ritus basis: Eigenspinor in B-field

$$\mathcal{R}_n(x_\perp) = \phi_n \mathcal{P}_+ + \phi_{n-1}\mathcal{P}_-$$

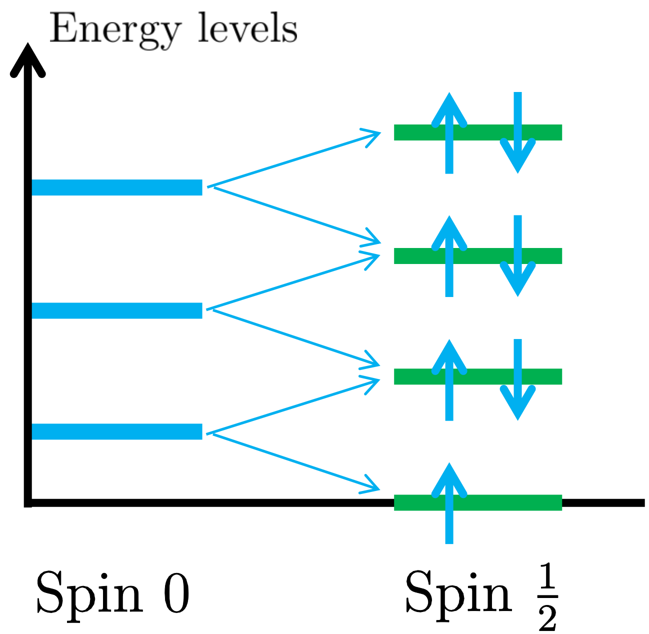
$\phi_n(x_\perp)$: Wave function at the Landau level n

$\mathcal{P}_\pm$: Spin projection operator along B-field

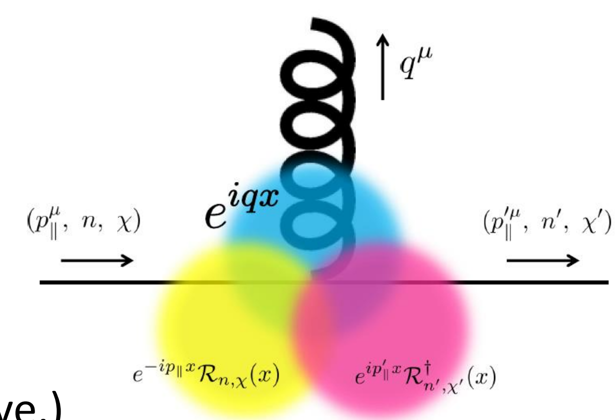
Mode expansion can be performed in this basis.
Fermion propagator gets simplified.

$$\mathcal{R}_n \frac{i}{\not{D}_{\rm ext} - m} \mathcal{R}^\dagger_{n'} = \frac{i}{\not{p}_n - m}\delta_{nn'}$$

$$p^\mu_n = (p^0, \sqrt{2n|eB|}, 0, p^3)$$



Spin up and down states are degenerated except for n = 0.



Price: Convolution of the wave functions at the vertex gets complicated. Fermion wave functions are not orthogonal to photon wave function. (Photon wave function is a plane wave.)

Resummed polarization tensor

Integrands with strong oscillations

$$\begin{cases} \Gamma_0(\tau,\beta) = \cos(\beta\tau) - \beta\sin(\beta\tau)\cot(\tau) \\ \Gamma_1(\tau,\beta) = (1-\beta^2)\cos(\tau) - \Gamma_0(\tau,\beta) \\ \Gamma_2(\tau,\beta) = 2\frac{\cos(\beta\tau)-\cos(\tau)}{\sin^2(\tau)} - \Gamma_0(\tau,\beta) \quad . \end{cases}$$

$$\phi_{\parallel}(r_{\parallel}^2, B_{\rm r}) = \frac{1}{B_{\rm r}}\left\{1-(1-\beta^2)\ r_{\parallel}^2\right\}$$

$$\phi_{\perp}(r_{\parallel}^2, B_{\rm r}) = -\frac{2r_{\perp}^2}{B_{\rm r}} \cdot \frac{\cos(\beta\tau)-\cos(\tau)}{\sin(\tau)}$$

$$r_{\parallel}^2 = q_{\parallel}^2/4m^2$$

Exponentiated trig-functions generate strongly oscillating behavior with arbitrarily high frequency.

$$\chi_i(r_{\parallel}^2, r_{\perp}^2, B_{\rm r}) = \frac{\alpha B_{\rm r}}{4\pi}\int_{-1}^{1} d\beta \int_0^{\infty} d\tau\ \frac{\Gamma_i(\tau,\beta)}{\sin(\tau)}\ e^{-i(\phi_{\parallel}+\phi_{\perp})\tau}$$

Proper time integrals on the two fermion lines

Vanishing B limit: $\chi_0 \to \Pi_{\rm vac}\ ,\ \ \chi_{1,2} \to 0$

Schwinger, Adler, Shabad, Urrutia, Tsai and Eber, Dittrich and Gies

Decomposing exponential factors

$$\chi_i = \frac{\alpha}{4\pi}\int_{-1}^{1} d\beta \int_0^{\infty} d\tau\, \frac{\Gamma_i(\tau,\beta)}{\sin\tau}\, \mathrm{e}^{-iu\cos(\beta\tau)}\, \mathrm{e}^{i\eta\cot\tau}\, \mathrm{e}^{-i\phi_{\parallel}\tau}$$

Contains arbitrarily higher harmonics

Linear w.r.t. τ in exp.

$$\begin{aligned} &= 1 + c_1\cos(\beta\tau) + c_2\cos^2(\beta\tau) + \cdots c_n\cos^n(\beta\tau) + \cdots \\ &= 1 + d_1\cos(\beta\tau) + d_2\cos(2\beta\tau) + \cdots + d_n\cos(n\beta\tau) + \cdots \end{aligned}$$

1st step: "Partial wave decomposition"

$$e^{-iu\cos(\beta\tau)} = \sum_{n=0}^{\infty}(2-\delta_{n0})I_n(-iu)\, e^{in\beta\tau}$$

Linear w.r.t. τ in exp.

2nd step: Getting Laguerre polynomials

Put $z = \exp(-2i\tau)$ and $x = y = \eta$

Associated Laguerre polynomial

$$\exp\left(-\frac{(x+y)z}{1-z}\right) I_n\left(\frac{2\sqrt{xyz}}{1-z}\right) = (1-z)(xyz)^{\frac{n}{2}} \sum_{\ell=0}^{\infty} \frac{\ell!}{\Gamma(\ell+n+1)} L_\ell^n(x) L_\ell^n(y)\, e^{-2i\ell\tau}$$

Linear w.r.t. τ in exp.

All terms fall in one of three elementary integrals.

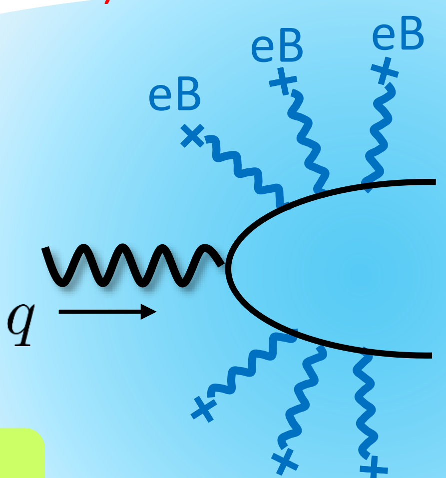
$$
\begin{aligned}
F_\ell^n(r_\parallel^2, B_\mathrm{r}) &= \frac{i}{B_\mathrm{r}} \int_{-1}^{1} d\beta \int_0^\infty d\tau \; e^{-i\left(\phi_\parallel + 2\ell - n\beta + n\right)\tau} \\
G_\ell^n(r_\parallel^2, B_\mathrm{r}) &= \frac{i}{B_\mathrm{r}} \int_{-1}^{1} d\beta \int_0^\infty d\tau \; \beta \, e^{-i\left(\phi_\parallel + 2\ell - n\beta + n\right)\tau} \\
H_\ell^n(r_\parallel^2, B_\mathrm{r}) &= \frac{i}{B_\mathrm{r}} \int_{-1}^{1} d\beta \int_0^\infty d\tau \; \beta^2 \, e^{-i\left(\phi_\parallel + 2\ell - n\beta + n\right)\tau}
\end{aligned}
$$

$$\phi_\parallel(q_\parallel^2, B) = \frac{m^2}{eB} \left\{ 1 - (1-\beta^2)\, \frac{q_\parallel^2}{4m^2} \right\}$$

What are the integers ℓ and n intorudced in the mathematical formulas?

Integers specify the fermion spectrum encoded in the photon spectrum.
(Remember the lesson in introduction)

$$\mathrm{Im}\,\Pi^{\mu\nu} =$$



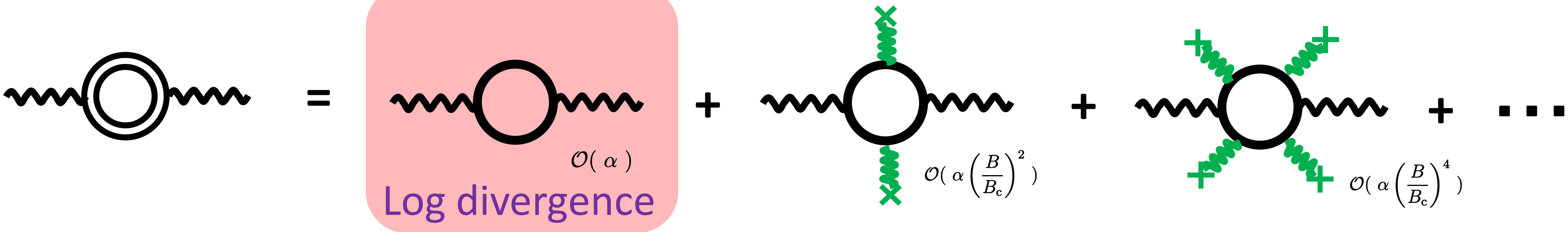
Invariant mass:

$$q^2 \geq (\epsilon_f + \epsilon_{\bar{f}})^2$$

Fermion-antifermion spectrum

Square of the decay amplitude
(Optical theorem)

Renormalization

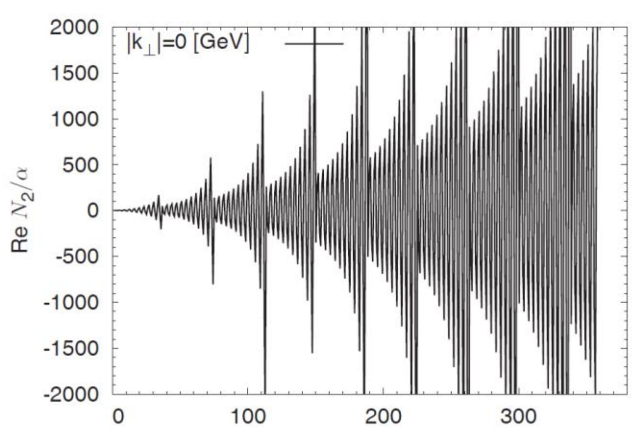


$$
\begin{aligned}
\Pi_{\rm ren}(q^2) &= \Pi(q_\parallel^2, q_\perp^2) - \Pi_{\rm vac}(q^2=0) \\
&= \underbrace{\Pi(q_\parallel^2, q_\perp^2) - \Pi(0, q_\perp^2)}_{\text{Term-by-term subtraction}} + \{\, \underbrace{\Pi(0, q_\perp^2) - \Pi_{\rm vac}(q^2=0)}_{\text{Finite}} \,\}
\end{aligned}
$$

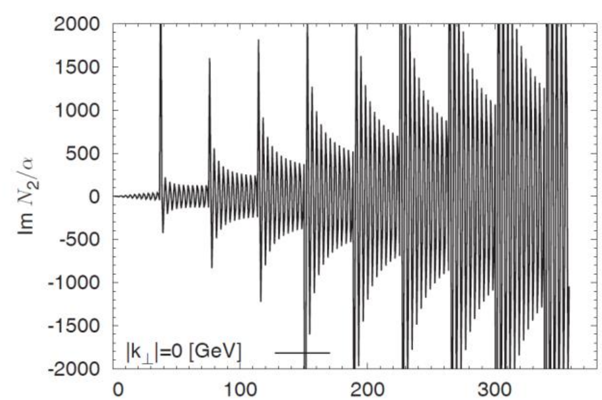
$\Pi(0, q_\perp^2)$ can be evaluated both by directly integrating the proper-time integrals and decomposing into the seires of Landau levels.

Ishikawa, Kimura, Shigaki, Tsuji (2013)

Re



Im



Taken from Ishikawa, et al. (2013)