# MAGNETO-HYDRODYNAMICS IN HEAVY-ION COLLISIONS

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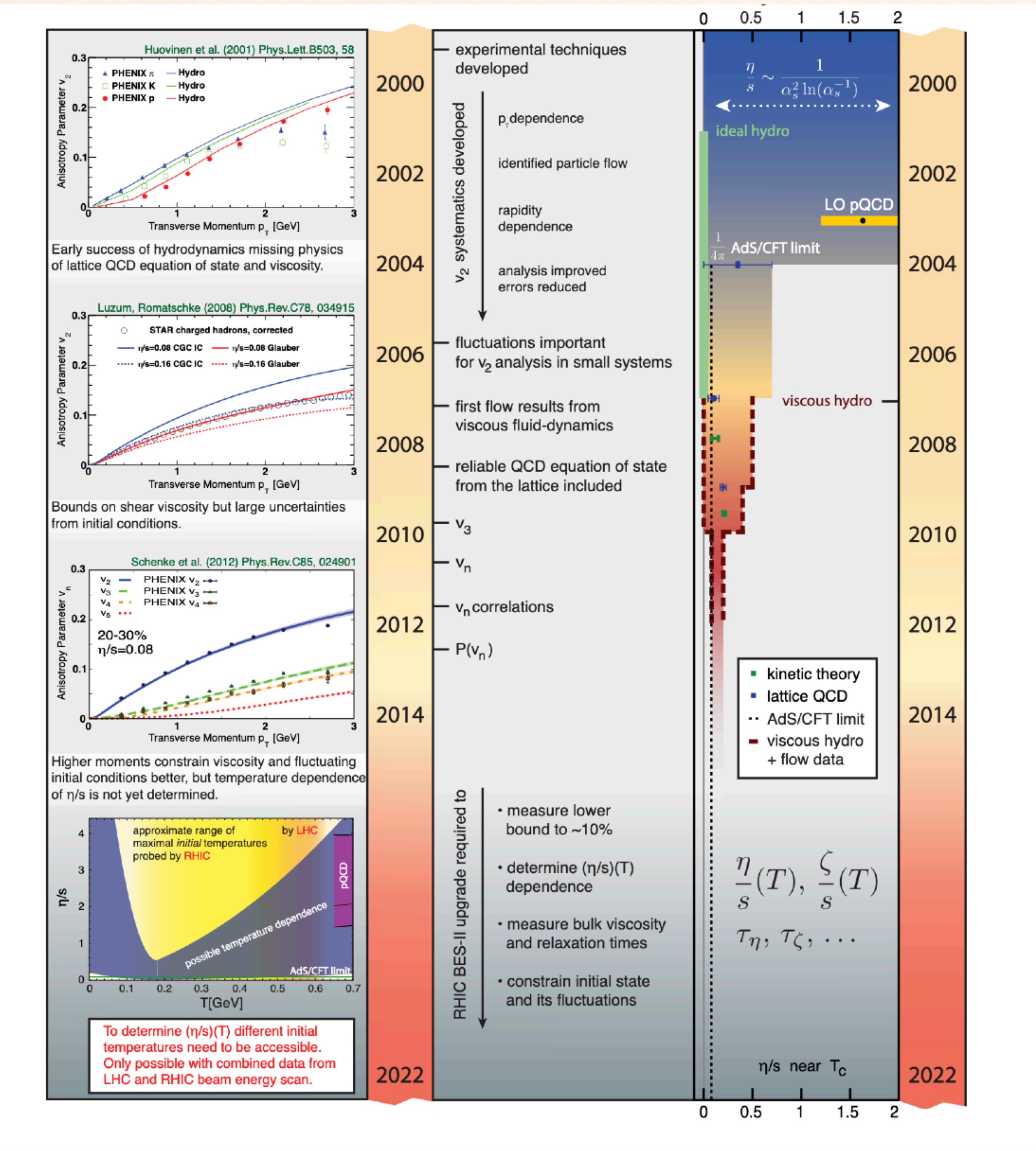
# OUTLINE

- Motivation.
- Relativistic Magneto-hydrodynamics (MHD).
- MHD from the Kinetic theory.
- Results & discussion.

# WHY DO WE CARE ABOUT MAGNETIC FIELD?

## Contemporary high-energy heavy ion collisions : precision measurements




*A Community White Paper on the Future of Relativistic Heavy-Ion Physics in the US*
*https://www.bnl.gov/npp/docs/Bass_RHI_WP_final.pdf*

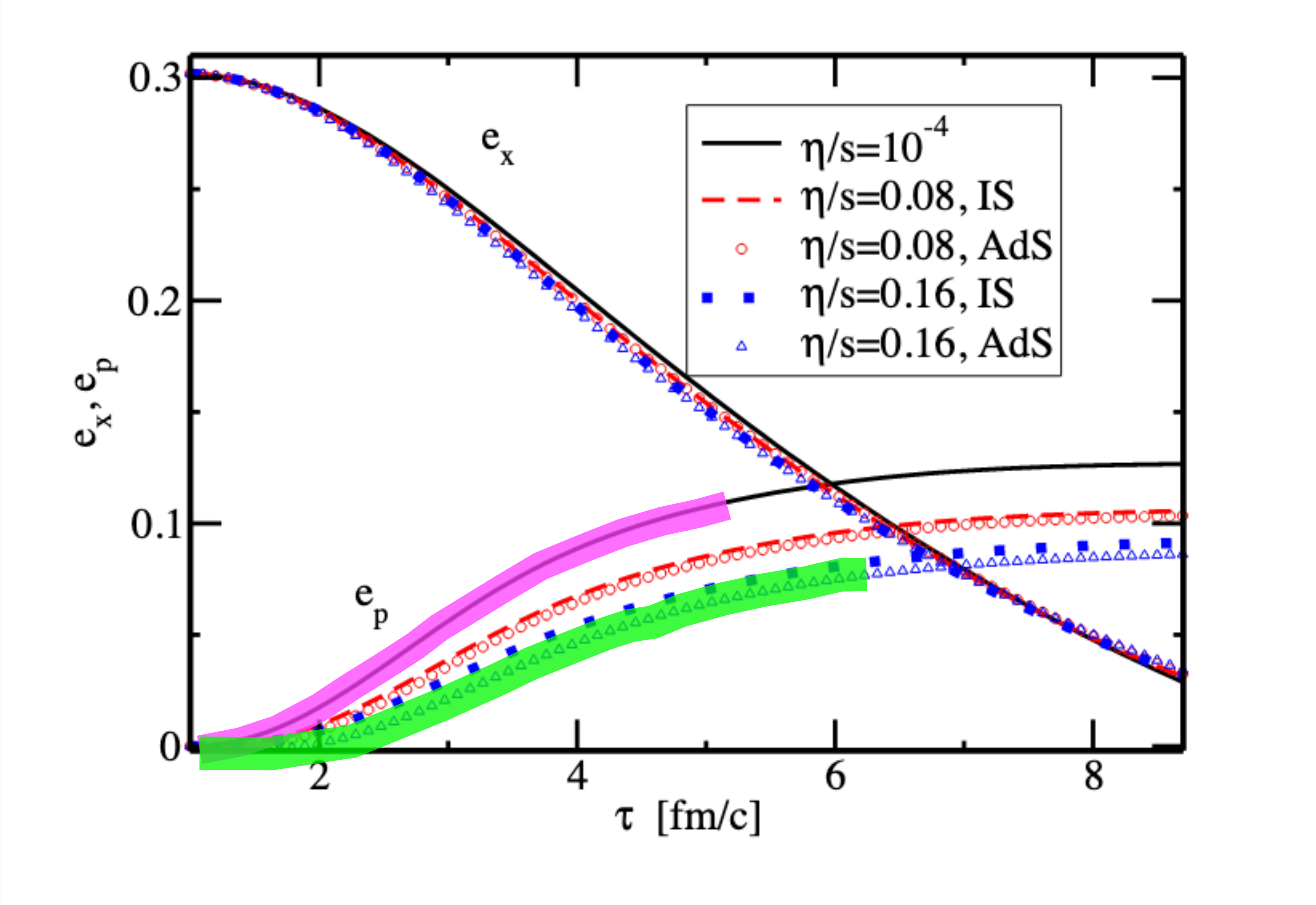
**May not be as precise as claimed**

## Values are obtained from elliptic and higher order flow coefficients.

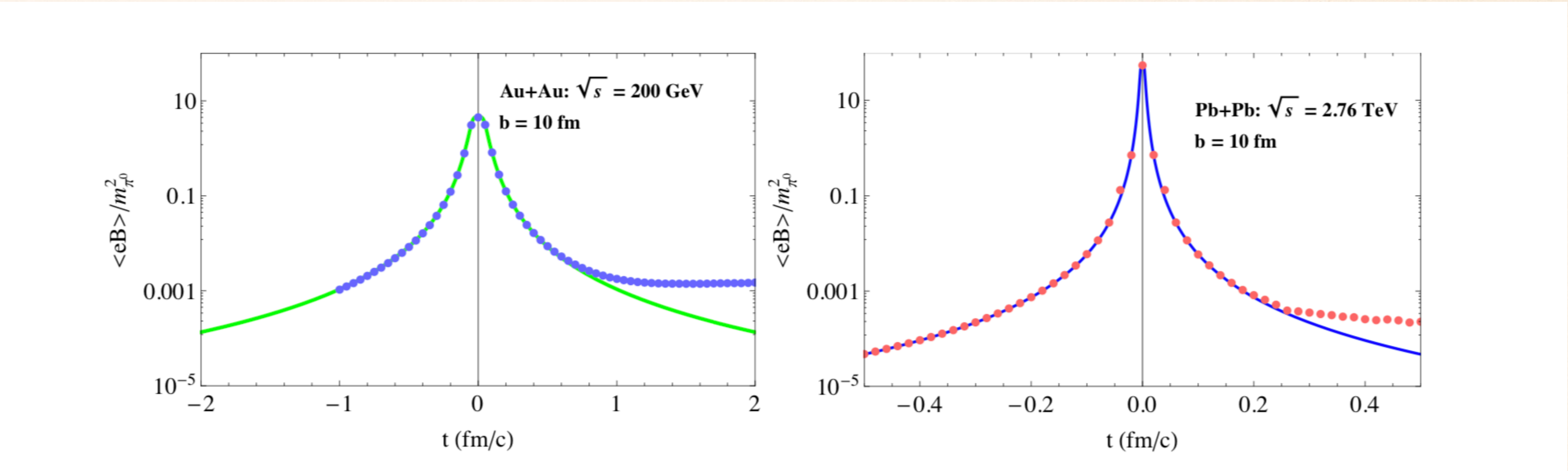
# WHY DO WE CARE ABOUT MAGNETIC FIELD?

Flow development is an early time dynamical phenomena

*M Luzum et al, Phys. Rev. C. 78, 034915 (2008)*




*X G Huang et al, Rep. Prog. Phys. 79, 076302*




Electro-Magnetic fields are strongest during the early stage of heavy ion collisions

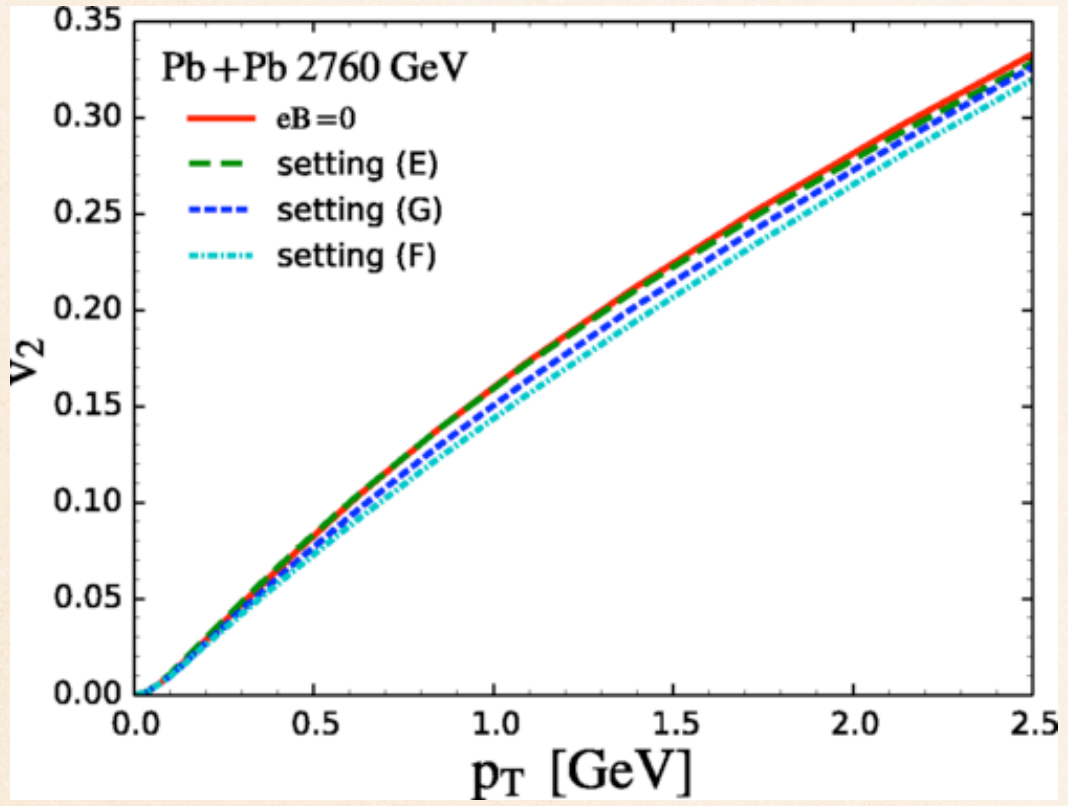
# WE PROBABLY NEED TO CARE ABOUT MAGNETIC FIELD

## Numerical simulations hints that probably we need to care about EM fields

PHYSICAL REVIEW C **93**, 044919 (2016)

**Magnetic-field-induced squeezing effect at energies available at the BNL Relativistic Heavy Ion Collider and at the CERN Large Hadron Collider**

Long-Gang Pang,[1] Gergely Endrődi,[2,3] and Hannah Petersen[1,3,4]



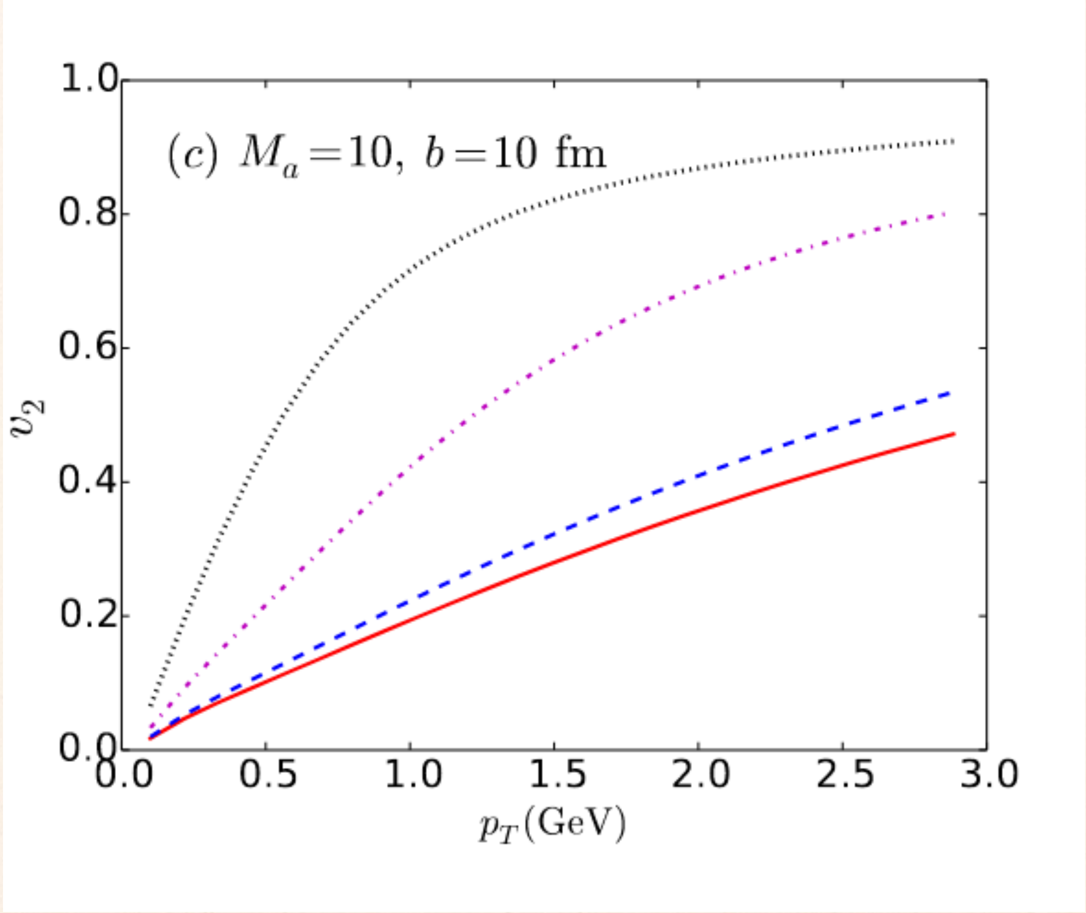

Paramagnetic QGP
parameterise magnetic field

## Charged pion v2

PHYSICAL REVIEW C **96**, 054909 (2017)

**Effect of intense magnetic fields on reduced-magnetohydrodynamics evolution in $\sqrt{s_{NN}}$ = 200 GeV Au + Au collisions**

Victor Roy,[1] Shi Pu,[2] Luciano Rezzolla,[3,4] and Dirk H. Rischke[3,5]



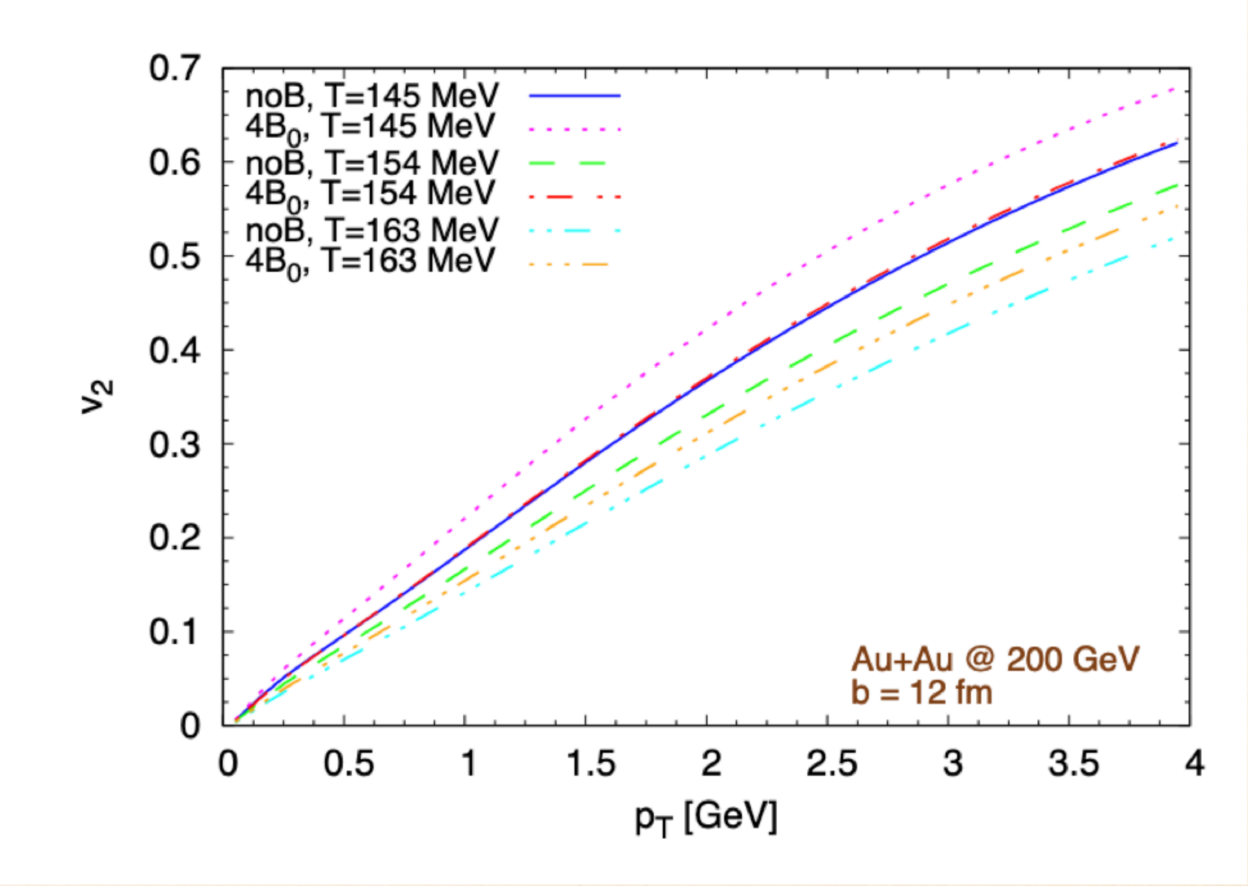

Eur. Phys. J. C (2020) 80:293
https://doi.org/10.1140/epjc/s10052-020-7847-4

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

**Magnetic fields in heavy ion collisions: flow and charge transport**

Gabriele Inghirami[1,2,a], Mark Mace[1,2], Yuji Hirono[3,4], Luca Del Zanna[5,6,7], Dmitri E. Kharzeev[8,9,10], Marcus Bleicher[11,12,13,14]




Ideal-MHD ECHO-QGP

## Some other early simple model based studies also pointed out the same

*K Tuchin J.Phys. G. 39,025010 (2012) ; R. Mohapatra et al Mod. Phys. Lett. A 26, 2477-2486 (2011)*

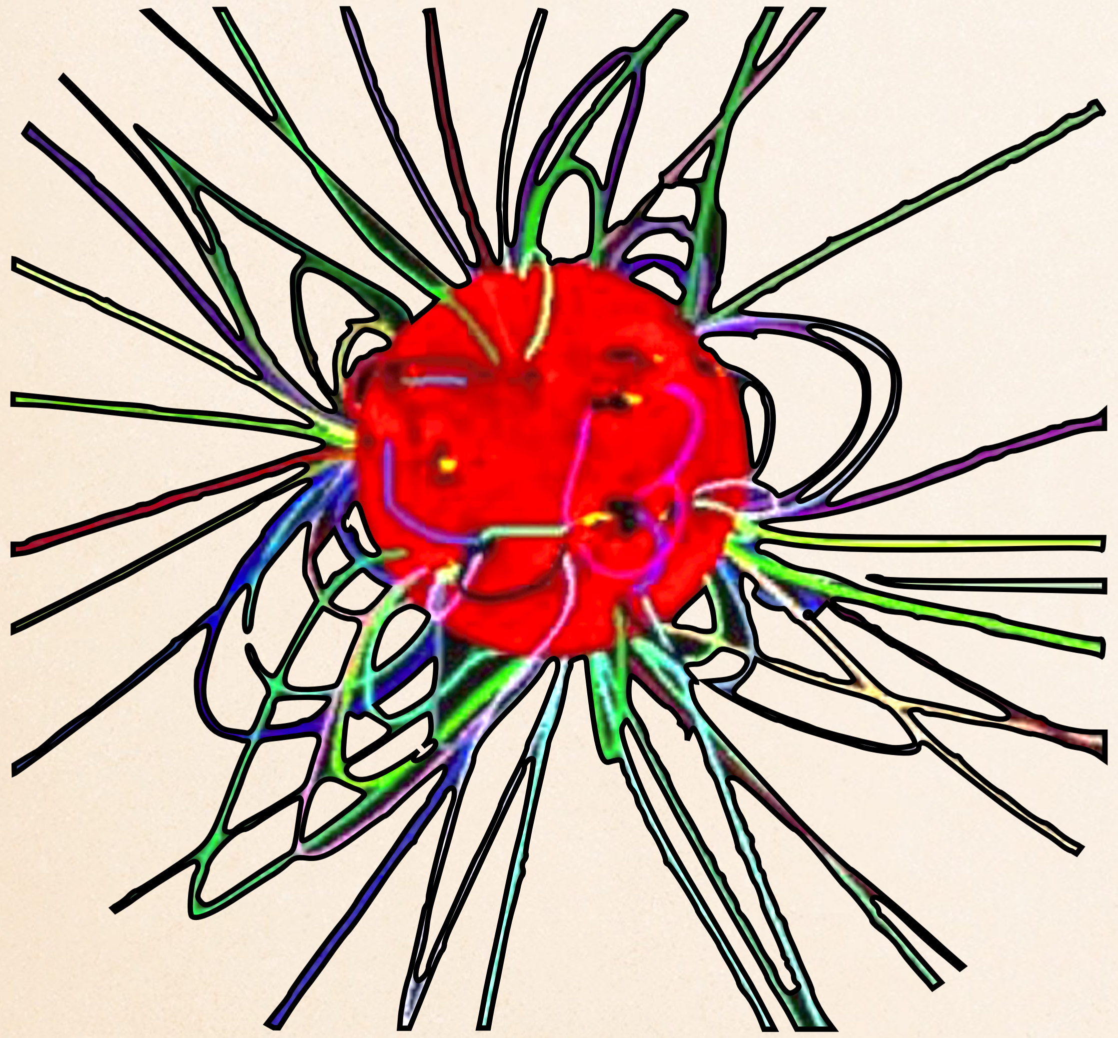
# WE PROBABLY NEED TO CARE ABOUT MAGNETIC FIELD

- Study of Chiral Magnetic Effects
- Charged dependent directed flow
- Polarisation of vector mesons
- Possibility of anisotropic transport coefficients

# INTRO TO MAGNETO-HYDRO

fluid consists of charged particles moving under external electric/magnetic field experience force which changes fluid velocity

fluid constituents are electrically charged hence its motion generates electromagnetic field which changes the external field



Self consistently solving the Maxwell's equation & Conservation laws

$$\nabla_\mu F^{\mu\nu} = j^\mu \qquad \nabla_\mu \mathscr{F}^{\mu\nu} = 0$$

$$\mathscr{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

# BASICS OF MAGNETO-HYDRO

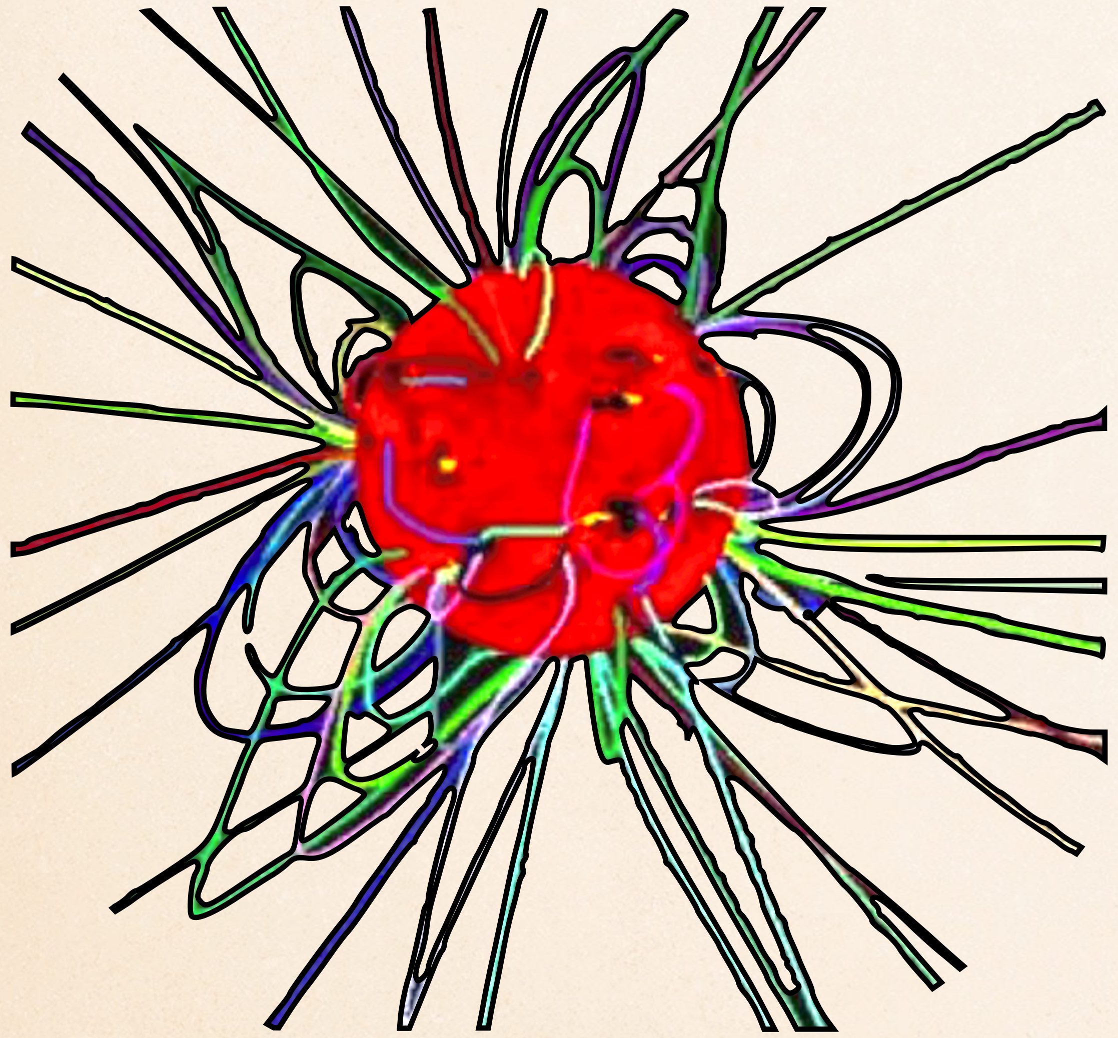
Self consistently solving the Maxwell's equation & Conservation laws

$$\partial_{;\mu} T^{\nu\mu}_{fluid} = F^{\mu\lambda} j_\lambda \qquad \partial_{;\mu} N^{\mu}_{fluid} = 0$$

$$\partial_{;\mu} T^{\nu\mu}_{field} = - F^{\mu\lambda} j_\lambda$$

$$T^{\mu\nu}_{field} = - F^{\mu\lambda} F^{\nu}_{\lambda} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}$$

$$F^{\mu\nu} = E^\mu u^\nu - E^\nu u^\mu + \epsilon^{\mu\nu\alpha\beta} u_\alpha B_\beta$$



For Ideal-MHD: $\partial_\mu T^{\mu\nu}_{fluid} = F^{\nu\lambda} J_\lambda = B^{\nu\lambda} J_\lambda$

$$u_\nu \partial_\mu T^{\mu\nu}_{fluid} = u_\nu B^{\nu\lambda} J_\lambda = 0$$

Fluid energy is conserved in magnetic field, magnetic field does no work.

$$\Delta^\alpha_\nu \partial_\mu T^{\mu\nu}_{fluid} = \Delta^\alpha_\nu B^{\nu\lambda} J_\lambda = B^{\nu\lambda} d_\lambda$$

The magnetic field influence the dynamics of the fluid only by coupling to the dissipative part of the current.

What about dissipative stresses in EM fields?

# MAGNETO-HYDRO FROM KINETIC THEORY: THE BOLTZMANN EQUATION

$$\frac{df}{d\mathcal{T}} = \frac{dt}{d\mathcal{T}}\frac{\partial f}{\partial t} + \frac{d\mathbf{x}}{d\mathcal{T}}\frac{\partial f}{\partial \mathbf{x}} + \frac{dP^\alpha}{d\mathcal{T}}\frac{\partial f}{\partial P^\alpha} = \mathscr{C}[f]$$

$f(X^\mu, P^\mu) \rightarrow$ single particle distribution $\qquad \mathscr{C}[f] \rightarrow$ collision term

$$f(X^\mu, P^\mu) = f_0 + \delta f$$

$f_0 \rightarrow$ equilibrium distribution function $\qquad \delta f \rightarrow$ off-equilibrium correction

Without an external force the Boltzmann equation along with $f_0 = 1/[exp(\beta p^\mu u_\mu - \beta\mu) + r]$

$$DT = D\mu = 0$$

$$\nabla^\alpha T = TDu^\alpha$$

$$\beta u^\alpha = \Omega^{\alpha\beta}x_\beta + \Omega^\alpha$$

# THE BOLTZMANN EQUATION WITH EM FIELDS IN RTA

$$\frac{\partial f}{\partial x^\mu} + \mathscr{F}^\alpha \frac{\partial f}{\partial P^\alpha} = -\frac{u \cdot p}{\tau_c}\delta f$$

*J Anderson* *Physica 74, 466-488 (1974),*

$\mathscr{F}^\nu := qF^{\nu\beta}p_\beta \rightarrow$ Four force $\quad\quad \delta f = f - f_0$

Presence of EM force introduces two new dimensionless parameter along with the Knudsen number.

$$\chi \sim qB\tau_c/T \qquad \xi \sim qE\tau_c/T$$

Presence of magnetic fields allows the existence of non-dissipative transport coefficients.

# THE DISSIPATIVE STRESSES FROM THE KINETIC THEORY

Order by order expansion in dimensionless small numbers

$$f = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\tau_c}{u\,.\,p}\right)^n \left(p^\mu \partial_\mu + qF^{\mu\nu} p_\nu \frac{\partial}{\partial p^\mu}\right)^n f_0 \,.$$

$$\delta f = \delta f_{(1)} + \delta f_{(2)} + \ldots$$

$$\Pi_{(n)} = -\frac{\Delta_{\mu\nu}}{3} \int dp p^\mu p^\nu \left(\delta f^{(n)} + \delta \bar{f}^{(n)}\right) \rightarrow \text{Bulk} \quad \text{pressure}$$

$$V^\mu_{(n)} = \Delta^\mu_\alpha \int dp p^\alpha \left(\delta f^{(n)} - \delta \bar{f}^{(n)}\right) \rightarrow \text{Diffusion} \quad \text{current}$$

$$\pi^{\mu\nu}_{(n)} = \Delta^{\mu\nu}_{\alpha\beta} \int dp p^\alpha p^\beta \left(\delta \tilde{f}^{(n)} + \delta \tilde{\bar{f}}^{(n)}\right) \rightarrow \text{Shear} \quad \text{tensor}$$

# THE DISSIPATIVE STRESSES IN RESISTIVE-MHD

| | First-order | Second-order |
|---|---|---|
| Bulk: | $\Pi_{(1)} = -\tau_c \beta_\Pi \theta$ | $\frac{\Pi}{\tau_c} = -\dot{\Pi} - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu} - \tau_{\Pi V} V\cdot\dot{u} - \lambda_{\Pi V} V\cdot\nabla\alpha - l_{\Pi V}\partial\cdot V - \beta_\Pi\theta$ <br> $-qB\lambda_{\Pi VB} b^{\mu\beta} V_\beta V_\mu + \tau_c \tau_{\Pi VB}\dot{u}_\alpha qBb^{\alpha\beta}V_\beta - q\delta_{\Pi VB}\nabla_\mu\left(\tau_c Bb^{\mu\beta}V_\beta\right) - q^2\tau_c\chi_{\Pi EE}E^\mu E_\mu$ |
| Diffusion: | $V^\mu_{(1)} = \tau_c \beta_V \left(\nabla^\mu \alpha + \beta q E^\mu\right)$ | $\frac{V^\mu}{\tau_c} = -\dot{V}^{\langle\mu\rangle} - V_\nu\omega^{\nu\mu} + \lambda_{VV}V^\nu\sigma^\mu_\nu - \delta_{VV}V^\mu\theta + \lambda_{V\Pi}\Pi\nabla^\mu\alpha - \lambda_{V\pi}\pi^{\mu\nu}\nabla_\nu\alpha - \tau_{V\pi}\pi^\mu_\nu\dot{u}^\nu$ <br> + shear-diffusion + diffusion+bulk + ... <br> $-qB\delta_{VB}b^{\mu\gamma}V_\gamma + \chi_{VE}qE^\mu + q\Delta^\mu_\alpha\chi_{VE}D\left(\tau_c E^\alpha\right) - q\tau_c\rho_{VE}E^\mu\theta - q\tau_{VVB}\Delta^\mu_\gamma D\left(\tau_c Bb^{\gamma\nu}V_\nu\right)$ |
| Shear: | $\pi^{\mu\nu}_{(1)} = 2\tau_c\beta_\pi\sigma^{\mu\nu}$ | $\frac{\pi^{\mu\nu}}{\tau_c} = -\dot{\pi}^{\mu\nu} + 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} +$ shear-shear <br> + shear-bulk + shear- diffusion + ... <br> $+\delta_{\pi B}\Delta^{\mu\nu}_{\eta\beta}qBb^{\gamma\eta}g^{\beta\rho}\pi_{\gamma\rho} - \tau_c qB\tau_{\pi VB}\dot{u}^{\langle\mu}b^{\nu\rangle\sigma}V_\sigma$ |

*A.K. Panda et al*

*Phys.Rev.D 104 (2021) 5, 054004,*
*JHEP 03 (2021) 216.*

# THE TRANSPORT COEFFICIENTS IN THE MASSLESS LIMIT

Massless Boltzmann gas

: Bulk :

| Transport Coefficients | CE | Denicol et al. |
|---|---|---|
| $\tau_{\Pi V}$ | 0 | 0 |
| $\chi_{\Pi EE}$ | $\beta^2 P/36$ | – |
| $\lambda_{\Pi V}$ | $1/(3\beta)$ | 0 |
| $l_{\Pi V}$ | 0 | 0 |
| $\lambda_{\Pi VB}$ | $3/(\beta P)$ | – |

: Diffusion :

| Transport Coefficients | CE | Denicol et al. |
|---|---|---|
| $\lambda_{VV}$ | $2/5$ | $3/5$ |
| $\delta_{VV}$ | $22/3$ | 1 |
| $\delta_{VB}$ | $2\beta$ | $5\beta/12$ |
| $\rho_{VE}$ | $P\beta^2/18$ | – |
| $\chi_{VE}$ | $P\beta^2/12$ | $P\beta^2/12$ |

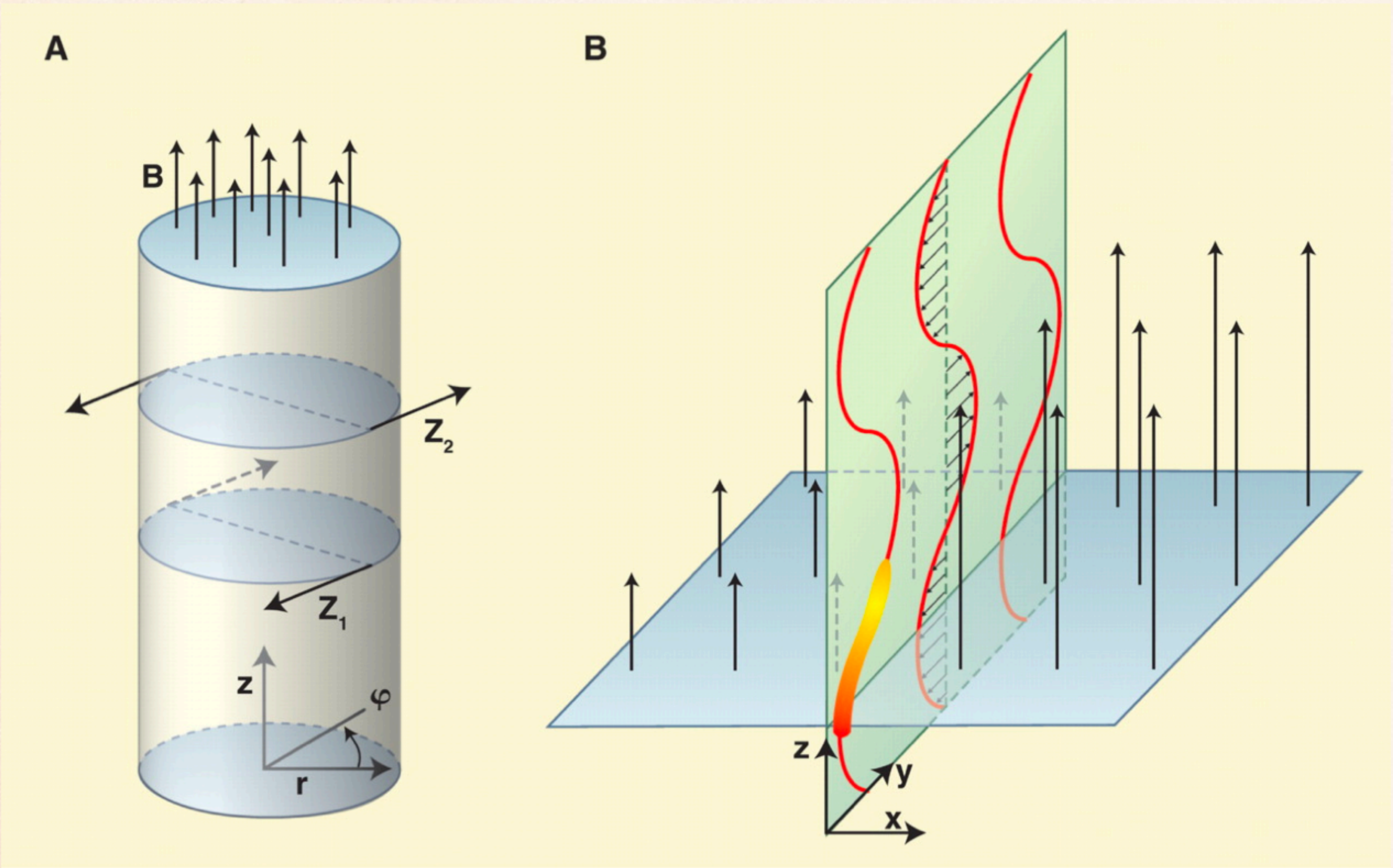
: Shear :

| Transport Coefficients | CE | Denicol et al. |
|---|---|---|
| $\tau_{\pi V}$ | $12/(5\beta)$ | 0 |
| $l_{\pi V}$ | $12/(5\beta)$ | 0 |
| $\lambda_{\pi V}$ | $11/5\beta$ | 0 |
| $\lambda_{\pi VB}$ | $24/5(\beta P)^{-1}$ | – |
| $\chi_{\pi EE}$ | $2\beta^2 P/15$ | – |

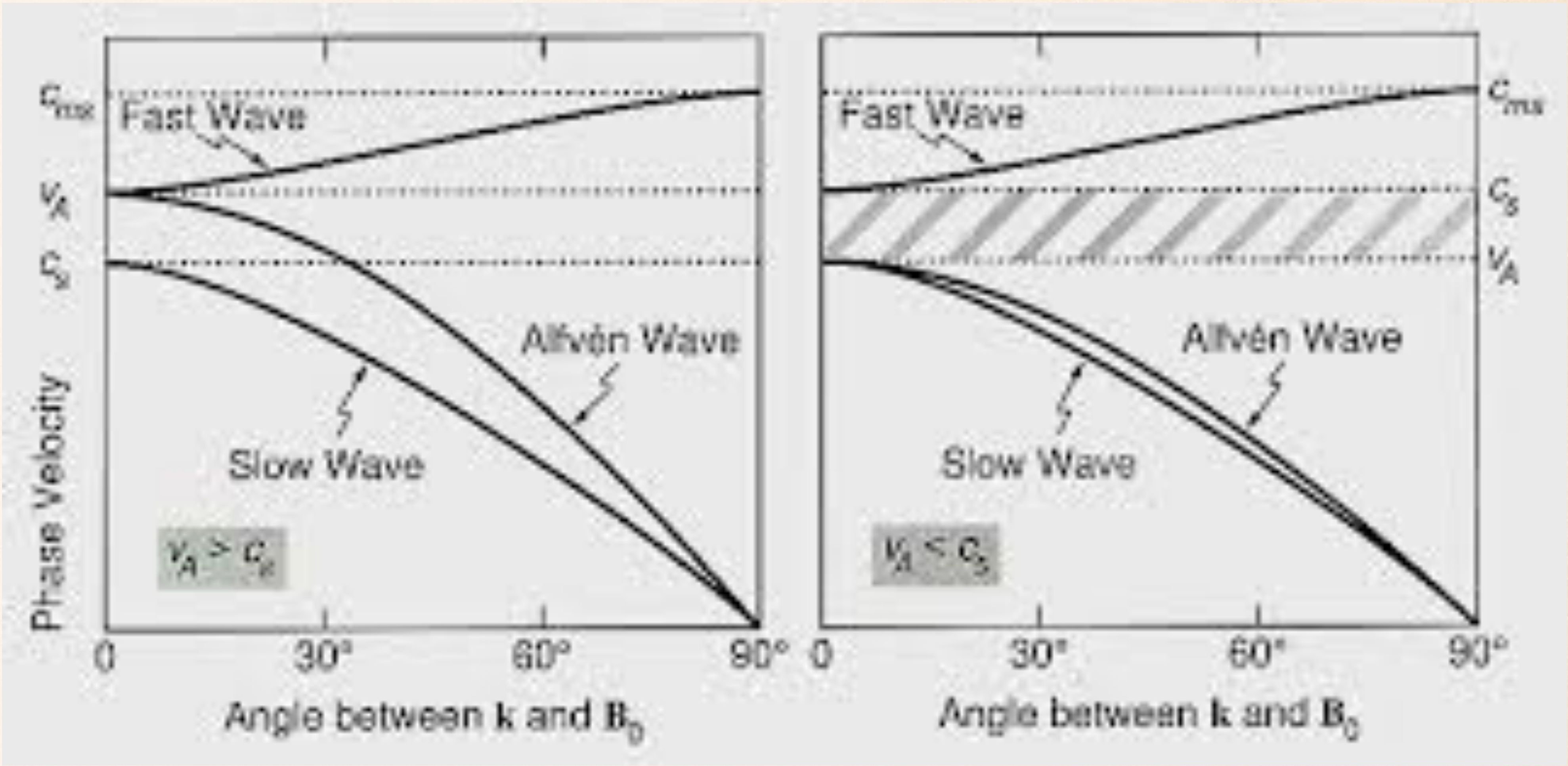
CE : *A.K. Panda et al Phys.Rev.D 104 (2021) 5, 054004,* *G. Denicol et al Phys.Rev.D 99 (2019) 056017,*

# THE PROPAGATION OF PERTURBATIONS IN MHD

Propagation of the perturbation through charged fluid
Three basic linear MHD waves




$$v_A = \frac{B}{\sqrt{\mu\rho}}$$

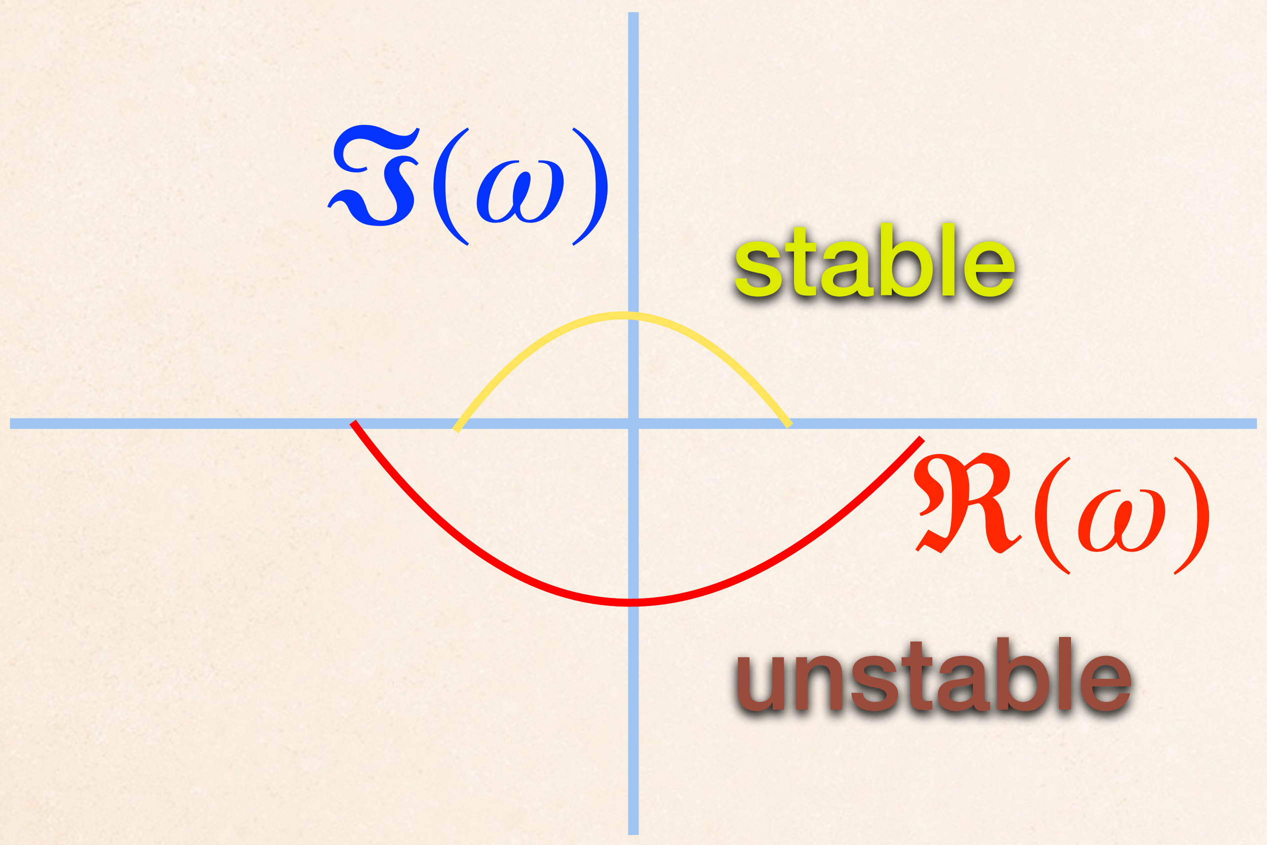



"An Alfven wave is a low frequency wave of ions and magnetic fields in plasma,
the ion mass provide the inertia and the magnetic field line tension provide the restoring force"

"In the absence of magnetic field the fast magnetosonic wave reduces to normal speed of sound
the slow magnetosonic and the Alfven waves become zero. "

*R Biswas et al*
*JHEP 10 (2020) 171*

# THE STABILITY AND CAUSALITY



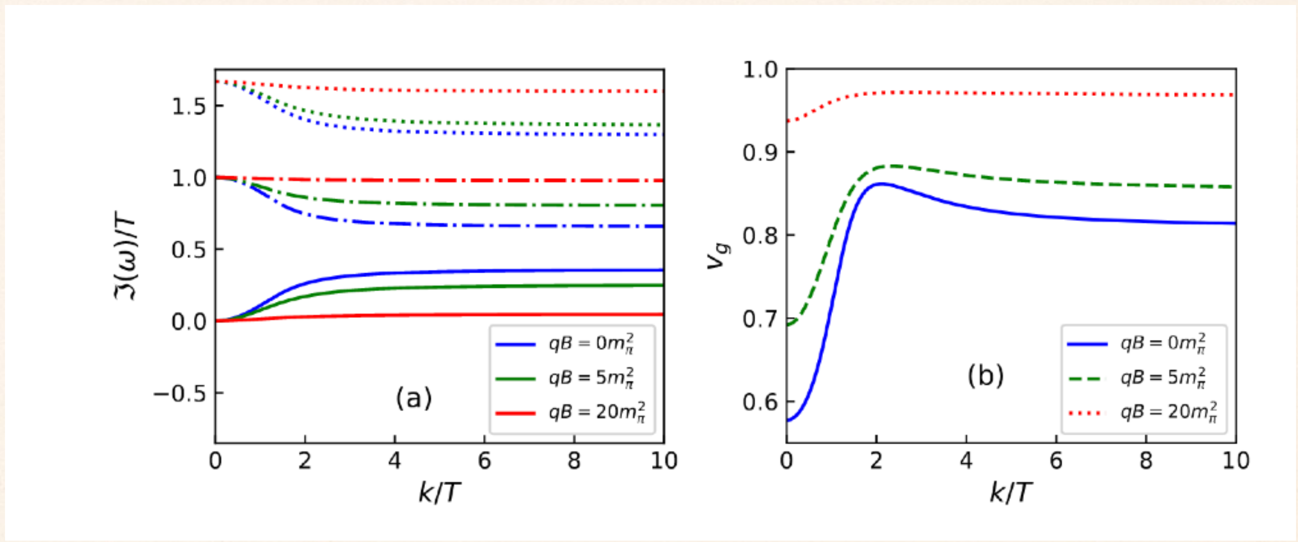

Apply perturbation and consider Fourier modes $\sim e^{i(\omega t - kx)}$

Linearize the MHD equations

Find the dispersion relations $\omega = \omega(k)$

$$v_g = \frac{\partial \mathfrak{R}(\omega)}{\partial k}$$

$$lim_{k\to\infty} v_g < 1 \implies \text{causal}$$




Under small perturbation and in linear regime IS like 2nd order MHD theory found to be Causal and Stable

*R Biswas et al JHEP 10 (2020) 171*

# ANISOTROPIC TRANSPORT COEFFICIENTS IN EM FIELDS

electric current density ∝ electric field

$$j_i = \sigma_{ij} E_j$$

For isotropic case $\sigma_{ij} \sim \delta_{ij}$

EoM $m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - m\tau^{-1}\mathbf{v}$

Considering steady state and solving for velocity

$$\sigma_{ij} = \sigma^{\parallel} h_i h_j + \sigma^{\perp}\left(\delta_{ij} - h_i h_j\right) + \sigma^{\times}\varepsilon_{ijk} h_k$$

$$\sigma^{\parallel} = \sigma_0 \equiv ne^2\tau/m, \quad \sigma^{\perp} = \sigma_0\left(1+\varphi^2\right)^{-1}, \quad \sigma^{\times} = \sigma_0\varphi\left(1+\varphi^2\right)^{-1}$$

$$\varphi = eB\tau/m$$

*Hernandez et al JHEP 1705 (2017) 001*
*Gabriel Denicol et al. Phys.Rev.D 98 (2018) 076009*
*Kiril Tuchin J. Phys. G: Nucl. Part. Phys. 39 025010*
*Arpan das et al Phys.Rev.D 102 (2020) 1, 014030*
*Ashutosh dash et al Phys.Rev.D 102 (2020) 1, 016016*
*Payal Mohanty et al. Eur.Phys.J.A 55 (2019) 35*
*Manu Kurian et al. Phys.Rev.D 101 (2020) 9, 094024*
*A.K. Panda et al Phys.Rev.D 104 (2021) 5, 054004,*

$$R_{\mu\nu}(\varphi) = \sum_{m=-1}^{1} P^{(m)}_{\mu\kappa}(\exp[\varphi H])_{\kappa\nu} = \sum_{m=-1}^{1} \exp[im\varphi] P^{(m)}_{\mu\nu}$$

$$P^{(0)}_{\mu\nu} = h_\mu h_\nu \equiv P^{\parallel}_{\mu\nu}, \quad P^{(1)}_{\mu\nu} + P^{(-1)}_{\mu\nu} = \delta_{\mu\nu} - h_\mu h_\nu \equiv P^{\perp}_{\mu\nu}$$

*S. Hess Tensor for physics,*

# CONCLUSION AND DISCUSSION

- Important to use RMHD in heavy-ion collisions.
- Need to take into account anisotropic nature of transport coefficient while extracting them.
- Resistive RMHD code for heavy-ion collisions.

# BACKUP SLIDES

| | |
|---|---|
| $\tau_{\Pi V}$ | $-\beta\frac{\partial}{\partial\beta}\left[\frac{5}{3\beta_V}\left(J_{42}^{(2)-}-\frac{n_f}{\epsilon+P}J_{42}^{(1)-}\right)\right] - \left[\frac{2}{3\beta_V}\left(\frac{J_{31}^{(0)-}}{h}-J_{31}^{(1)-}\right)-\beta\frac{\partial}{\partial\beta}\left(\frac{5}{3h\beta_V}J_{42}^{(1)-}-\frac{5}{3\beta_V}J_{42}^{(2)-}\right)\right]$ |
| $\lambda_{\Pi V}$ | $\left(\frac{\partial}{\partial\alpha}+h^{-1}\frac{\partial}{\partial\beta}\right)\left[\frac{5}{3\beta_V}\left(J_{42}^{(2)-}-\frac{n_f}{\epsilon+P}J_{42}^{(1)-}\right)\right] - \frac{1}{3\beta\beta_V}\left(5J_{42}^{(3)+}-\frac{5}{h}J_{42}^{(2)-}+2J_{31}^{(2)+}-\frac{2}{h}J_{31}^{(1)-}\right) - \left(\frac{\partial}{\partial\alpha}+h^{-1}\frac{\partial}{\partial\beta}\right)\left[\frac{5}{3h\beta_V}J_{42}^{(1)-}-\frac{5}{3\beta_V}J_{42}^{(2)-}\right]$ |
| $l_{\Pi V}$ | $\frac{5}{3\beta_V}\left(J_{42}^{(2)-}-\frac{n_f}{\epsilon+P}J_{42}^{(1)-}\right)-\frac{1}{\beta\beta_V}\left[\frac{5\beta}{3h}J_{42}^{(1)-}-\frac{5}{3}\beta J_{42}^{(2)-}\right]$ |
| $\lambda_{\Pi VB}$ | $\frac{1}{\beta_V}\left(\frac{\partial}{\partial\alpha}+h^{-1}\frac{\partial}{\partial\beta}\right)\left[\frac{5J_{42}^{(1)+}\beta}{3(\epsilon+P)}\right]+\frac{1}{\beta_V}\left[\frac{-1}{3(\epsilon+p)}\left(5J_{42}^{(2)-}+2J_{31}^{(1)-}\right)\right]$ |
| $\chi_{\Pi EE}$ | $\frac{-\beta}{3}\left(5J_{42}^{(3)+}+2J_{31}^{(2)+}-\frac{5}{h}J_{42}^{(2)+}-\frac{2}{h}J_{31}^{(1)+}\right)$ |

TABLE II. Transport coefficients appearing in bulk-stress equation Eq. (28).

| | |
|---|---|
| $\lambda_{VV}$ | $-\left(1+\frac{2}{\beta_V}\left(\frac{n_f}{\epsilon+P}J_{42}^{(2)+}-J_{42}^{(3)+}\right)-\frac{1}{\beta_V}\left\{\left(J_{31}^{(2)+}+4J_{42}^{(3)+}\right)-\frac{1}{h}\left(2J_{42}^{(2)+}+J_{31}^{(1)+}\right)\right\}\right)$ |
| $\delta_{VV}$ | $\frac{4}{3}+\frac{5}{3\beta_V}\left(\frac{n_f J_{42}^{(2)+}}{\epsilon+P}-J_{42}^{(3)+}\right)+\frac{1}{\beta\beta_V}\left[\frac{\beta}{h}\left(\frac{4}{3}J_{31}^{(1)+}+\frac{5}{3}J_{42}^{(2)+}\right)\right] - \frac{1}{\beta\beta_V}\left[\beta\left(\frac{7}{3}J_{31}^{(2)+}+\frac{10}{3}J_{42}^{(3)+}\right)-\left(J_{20}^{(1)+}+J_{21}^{(1)+}\right)\mathcal{X}-\left(J_{20}^{(2)+}+J_{21}^{(2)+}\right)\mathcal{Y}\right]$ |
| $\delta_{VB}$ | $\left(\frac{n_f J_{21}^{(1)-}}{\epsilon+P}-J_{21}^{(2)-}\right)/\beta_V+\frac{1}{\beta_V}\left(\frac{J_{21}^{(1)-}}{h}-J_{21}^{(2)-}\right)$ |
| $\chi_{VE}$ | $\beta\beta_V$ |
| $\rho_{VE}$ | $-\left(\frac{n_f}{D_{20}}\left[\left(J_{20}^{(0)+}\frac{\partial\chi_{VE}}{\partial\alpha}+J_{10}^{(0)+}\frac{\partial\chi_{VE}}{\partial\beta}\right)h-\left(J_{30}^{(0)+}\frac{\partial\chi_{VE}}{\partial\alpha}+J_{20}^{(0)+}\frac{\partial\chi_{VE}}{\partial\beta}\right)\right]\right)$ |

TABLE III. Transport coefficients appearing in Diffusion evolution equation Eq. (31)

| | |
|---|---|
| $\tau_{\pi V}$ | $\beta\frac{\partial}{\partial\beta}\left[\frac{2}{\beta_V}\left(J_{42}^{(2)-}-\frac{n_f}{\epsilon+P}J_{42}^{(1)-}\right)\right]-\frac{2}{\beta_V}\left[J_{31}^{(1)-}-\frac{J_{31}^{(0)-}}{h}\right]-\beta\frac{\partial}{\partial\beta}\frac{1}{\chi_{VE}}\left[-\beta J_{42}^{(2)-}+2J_{42}^{(1)-}\left(\frac{\beta}{h}\right)\right]$ |
| $\lambda_{\pi V}$ | $\left(\frac{\partial}{\partial\alpha}+h^{-1}\frac{\partial}{\partial\beta}\right)\left[\frac{2}{\beta_V}\left(J_{42}^{(2)-}-\frac{n_f}{\epsilon+P}J_{42}^{(1)-}\right)\right]+\frac{2}{h\beta\beta_V}\left(J_{31}^{(1)-}+J_{42}^{(2)-}\right)-\frac{2}{\beta\beta_V}\left(J_{31}^{(2)-}+J_{42}^{(3)-}\right)-\left(\frac{\partial}{\partial\alpha}+h^{-1}\frac{\partial}{\partial\beta}\right)\frac{1}{\chi_{VE}}\left[-\beta J_{42}^{(2)-}+2J_{42}^{(1)-}\left(\frac{\beta}{h}\right)\right]$ |
| $l_{\pi V}$ | $-\frac{2}{\beta_V}\left(J_{42}^{(2)-}-\frac{n_f}{\epsilon+P}J_{42}^{(1)-}\right)+\frac{1}{\chi_{VE}}\left[-\beta J_{42}^{(2)-}+2J_{42}^{(1)-}\left(\frac{\beta}{h}\right)\right]$ |
| $\lambda_{\pi VB}$ | $\frac{1}{\beta_V}\left(\frac{\partial}{\partial\alpha}+h^{-1}\frac{\partial}{\partial\beta}\right)\left[2\beta J_{42}^{(1)+}/(\epsilon+P)\right]+\frac{1}{\beta\beta_V}\left[-\frac{2\beta}{(\epsilon+P)}\left(J_{31}^{(1)-}+J_{42}^{(2)-}\right)\right]$ |
| $\chi_{\pi EE}$ | $2\beta\left(\frac{\left(J_{31}^{(1)-}+J_{42}^{(2)-}\right)}{h}-\left(J_{42}^{(3)+}+J_{31}^{(2)+}\right)\right)$ |

TABLE IV. Transport coefficients appearing in shear-stress evolution equation Eq. (34)

# BACKUP SLIDES
# RELATIVISTIC "J" INTEGRALS

The $n$-th moments integral for the distribution function is defined as:

$$I^{(m)\pm}_{\mu_1\mu_2\cdots\mu_n} = \int \frac{dp}{(u\cdot p)^m} p_{\mu_1} p_{\mu_2} \cdots p_{\mu_n} \left(f_0 \pm \bar{f}_0\right), \tag{A.1}$$

which can be docomposed as:

$$\begin{aligned} I^{(m)\pm}_{\mu_1\mu_2\ldots\mu_n} &= I^{(m)\pm}_{n0} u_{\mu_1}\cdots u_{\mu_n} + I^{(m)\pm}_{n1}\left(\Delta_{\mu_1\mu_2} u_{\mu_3}\cdots u_{\mu_n} + \text{perm.}\right) + \cdots \\ &\cdots + I^{(m)\pm}_{nq}\left(\Delta_{\mu_1\mu_2}\Delta_{\mu_3\mu_4}\cdots\Delta_{\mu_{n-1}\mu_n} + \text{perm.}\right). \end{aligned} \tag{A.2}$$

where $n \geq 2q$.

Similarly the auxiliary moments integral

$$J^{(m)\pm}_{\mu_1\mu_2\cdots\mu_n} = \int \frac{dp}{(u\cdot p)^m} p_{\mu_1} p_{\mu_2} \cdots p_{\mu_n} \left(f_0\tilde{f}_0 \pm \bar{f}_0\tilde{\bar{f}}_0\right), \tag{A.3}$$

can be decomposed as:

$$J^{(m)\pm}_{\mu_1\mu_2\ldots\mu_n} = J^{(m)\pm}_{n0} u_{\mu_1}\cdots u_{\mu_n} + J^{(m)\pm}_{n1}\left(\Delta_{\mu_1\mu_2} u_{\mu_3}\cdots u_{\mu_n} + \text{perm.}\right) + \cdots \tag{A.4}$$

$$\cdots + J^{(m)\pm}_{nq}\left(\Delta_{\mu_1\mu_2}\Delta_{\mu_3\mu_4}\cdots\Delta_{\mu_{n-1}\mu_n} + \text{perm.}\right). \tag{A.5}$$

$$I^{(m)\pm}_{nq} = \frac{1}{(2q+1)!!}\int dp (u\cdot p)^{n-2q-m}(\Delta_{\alpha\beta}p^\alpha p^\beta)^q \left(f_0 \pm \bar{f}_0\right), \tag{A.6}$$

and

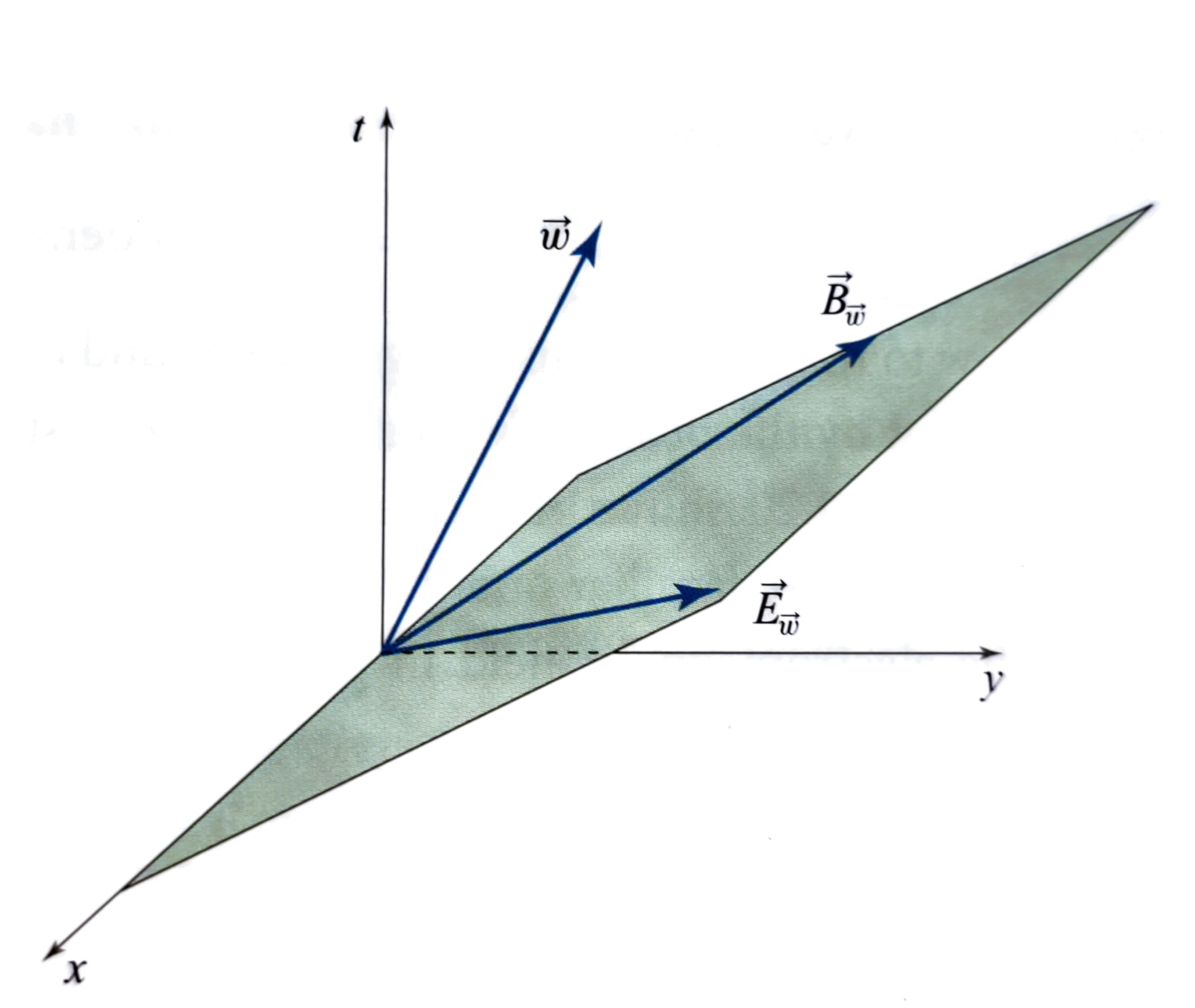
$$J^{(m)\pm}_{nq} = \frac{1}{(2q+1)!!}\int dp (u\cdot p)^{n-2q-m}(\Delta_{\alpha\beta}p^\alpha p^\beta)^q \left(f_0\tilde{f}_0 \pm \bar{f}_0\tilde{\bar{f}}_0\right). \tag{A.7}$$

One can write the $J$ in terms of $I$ as:

$$J^{(0)\pm}_{nq} = \frac{1}{\beta}\left[-I^{(0)\pm}_{n-1,q-1} + (n-2q) I^{(0)\pm}_{n-1,q}\right]. \tag{A.8}$$

The general expression of $D_{nq}$ used in eq. (2.41) and eq. (2.42) is given by: $D_{nq} = J^{(0)+}_{n+1,q} J^{(0)+}_{n-1,q} - J^{(0)-}_{nq} J^{(0)-}_{nq}$.

# MAGNETIC FOUR VECTOR




- Electric and magnetic field are elements of $F^{\mu\nu}$
- We need to define a rest frame where electric and magnetic

  fields are measured
- In the rest frame $E^0 = 0; \overrightarrow{E} = E^i$ same is true for magnetic field
- The four velocity in the rest frame is $u^0 = 1; \overrightarrow{u} = \overrightarrow{0}$

One can define in the rest frame the following four vectors

$$E_w^\alpha = F^{\alpha\beta}u_\beta \qquad B_w^\beta = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}u_\alpha$$

They are tensor equations and hence holds in all reference frames

$$F^{\alpha\beta} = u^\alpha E^\beta - E^\alpha u^\beta + \epsilon^{\alpha\beta}_{\gamma\delta}u^\gamma B^\delta$$