Quantum correlations in neutrino oscillations

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Summary

1. Entanglement in neutrino oscillations

2. Quantum Correlations & nonlocality in neutrino oscillations

3. Chiral oscillations and lepton/antineutrino entanglement

4. Exact neutrino oscillation formula

Entanglement in neutrino oscillations

Motivations

- Importance of entanglement both at a fundamental level and for quantum information;

- Entanglement in particle physics: entanglement, decoherence, Bell inequalities for the $K^0\bar{K}^0$ (or $B^0\bar{B}^0$) system[*];

- Neutrino mixing and oscillations in Quantum Field Theory[†].

- Necessity for a treatment of entanglement in the context of Quantum Field Theory[‡];

[*] R.A.Bertlmann and B.Hiesmayr, Phys.Rev. A (2001); R.A.Bertlmann, Lect.Not.Phys.(2006); A.Di Domenico et al. Found.Phys.(2012).

[†] M.B and G.Vitiello, Ann. Phys. (1995)

[‡] M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007)

Entanglement in particle mixing[§]

– Flavor mixing (neutrinos)

$$|\nu_e\rangle = \quad \cos\theta\,|\nu_1\rangle \,+\, \sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle \,+\, \cos\theta\,|\nu_2\rangle$$

• Correspondence with two-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2 \equiv |10\rangle, \qquad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2 \equiv |01\rangle,$$

$|\rangle_i$ denotes states in the Hilbert space for neutrinos with mass m_i.

⇒ flavor states are entangled superpositions of the mass eigenstates:

$$|\nu_e\rangle = \quad \cos\theta\,|10\rangle \,+\, \sin\theta\,|01\rangle.$$

[§] M.B., M. Di Mauro and P.Jizba, J. Phys. Conf. Ser. (2007); M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2009).

Entanglement - mathematical definition

- Given a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, a system is entangled, iff

$$\rho_{AB} \neq \sum_k p_k \, \rho_k^{(A)} \otimes \rho_k^{(B)}$$

with $0 \leq p_i \leq 1, \quad \sum_k p_k = 1.$

- For a generic pure state of the form:

$$|\psi\rangle_{AB} \, = \sum_{ij} c_{ij} |i\rangle_A \otimes |j\rangle_B$$

the condition for entanglement reads

$$|\psi\rangle_{AB} \neq |\phi\rangle_A \otimes |\chi\rangle_B$$

Single-particle entanglement*

– A state like $|\psi\rangle_{A,B} = |0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B$ is entangled;

– entanglement among field modes, rather than particles;

– entanglement is a property of composite systems, rather than of many-particle systems;

– entanglement and non-locality are not synonyms;

– single-particle entanglement is as good as two-particle entanglement for applications (quantum cryptography, teleportation, violation of Bell inequalities, etc..).

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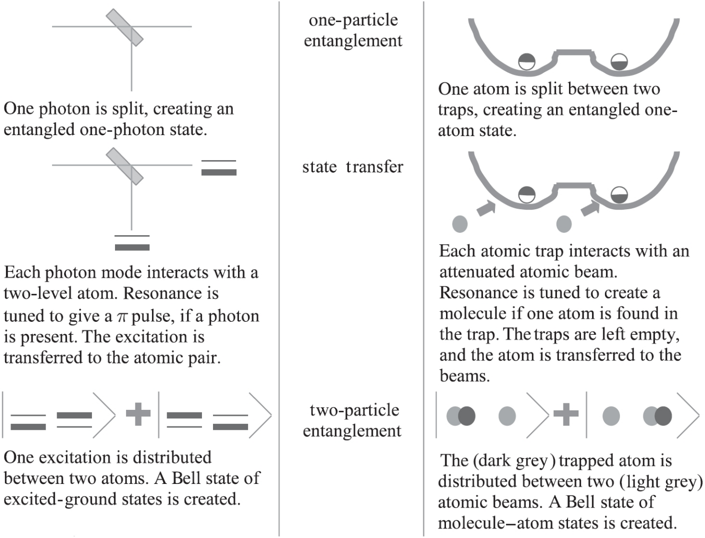
*J.van Enk, Phys. Rev. A (2005), (2006);
M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007);
J.A.Dunningham and V.Vedral, Phys. Rev. Lett. (2007).

Protocols for extraction of single-particle entanglement †



† M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007)

Multipartite entanglement in neutrino mixing[‡]

– Neutrino mixing (three flavors):

$$|\underline{\nu}_f\rangle \quad = \quad U(\tilde{\theta},\delta)\,|\underline{\nu}_m\rangle$$

with $|\underline{\nu}_f\rangle = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle)^T$ and $|\underline{\nu}_m\rangle = (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle)^T$.

– Mixing matrix (PMNS)

$$U(\tilde{\theta},\delta) \quad = \quad \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

where $(\tilde{\theta},\delta) \equiv (\theta_{12},\theta_{13},\theta_{23};\delta)$, $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$.

- Correspondence with three-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2|0\rangle_3 \equiv |100\rangle, \qquad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2|0\rangle_3 \equiv |010\rangle,$$

$$|\nu_3\rangle \equiv |0\rangle_1|0\rangle_2|1\rangle_3 \equiv |001\rangle$$

[‡] M.B., F.Dell'Anno, S.De Siena, M.Di Mauro and F.Illuminati, PRD (2008).

(Flavor) Entanglement in neutrino oscillations[§]

– Two-flavor neutrino states

$$|\underline{\nu}^{(f)}\rangle \;=\; \mathbf{U}(\theta,\delta)\,|\underline{\nu}^{(m)}\rangle$$

where $|\underline{\nu}^{(f)}\rangle \;=\; (|\nu_e\rangle, |\nu_\mu\rangle)^T$ and $|\underline{\nu}^{(m)}\rangle \;=\; (|\nu_1\rangle, |\nu_2\rangle)^T$ and
$\mathbf{U}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

– Flavor states at time t:

$$|\underline{\nu}^{(f)}(t)\rangle \;=\; \mathbf{U}(\theta,\delta)\,\mathbf{U}_0(t)\,\mathbf{U}(\theta,\delta)^{-1}\,|\underline{\nu}^{(f)}\rangle \equiv\; \widetilde{\mathbf{U}}(t)|\underline{\nu}^{(f)}\rangle\,,$$

with $\mathbf{U}_0(t) = \begin{pmatrix} e^{-iE_1 t} & 0 \\ 0 & e^{-iE_2 t} \end{pmatrix}$.

[§] M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2009).

– Transition probability for $\nu_\alpha \to \nu_\beta$

$$P_{\nu_\alpha \to \nu_\beta}(t) \,=\, |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 \,=\, |\widetilde{\mathbf{U}}_{\alpha\beta}(t)|^2 \,.$$

• We now take the flavor states at initial time as our qubits:

$$|\nu_e\rangle \equiv |1\rangle_e |0\rangle_\mu \equiv |10\rangle_f, \quad |\nu_\mu\rangle \equiv |0\rangle_e |1\rangle_\mu \equiv |01\rangle_f,$$

• Starting from $|10\rangle_f$ or $|01\rangle_f$, time evolution generates the (entangled) Bell-like states:

$$|\nu_\alpha(t)\rangle \,=\, \widetilde{\mathbf{U}}_{\alpha e}(t) |1\rangle_e |0\rangle_\mu + \widetilde{\mathbf{U}}_{\alpha\mu}(t) |0\rangle_e |1\rangle_\mu, \quad \alpha = e, \mu.$$

Entanglement measure

- Let $\rho = |\psi\rangle\langle\psi|$ be the density operator for a pure state $|\psi\rangle$

Bipartition of the N-partite system $S = \{S_1, S_2, \ldots, S_N\}$ in two subsystems:

$$S_{A_n} = \{S_{i_1}, S_{i_2}, \ldots, S_{i_n}\}, \quad 1 \leq i_1 < i_2 < \ldots < i_n \leq N; (1 \leq n < N)$$

and

$$S_{B_{N-n}} = \{S_{j_1}, S_{j_2}, \ldots, S_{j_{N-n}}\} \quad , 1 \leq j_1 < j_2 < \ldots < j_{N-n} \leq N; i_q \neq j_p$$

- Reduced density matrix of S_{A_n} after tracing over $S_{B_{N-n}}$:

$$\rho_{A_n} \equiv \rho_{i_1, i_2, \ldots, i_n} = Tr_{B_{N-n}}[\rho] = Tr_{j_1, j_2, \ldots, j_{N-n}}[\rho]$$

• Linear entropy associated to such a bipartition:

$$S_L^{(A_n;B_{N-n})}(\rho) \,=\, \frac{d}{d-1}(1 - Tr_{A_n}[\rho_{A_n}^2])\,,$$

d is the Hilbert-space dimension:
$d = \min\{\dim S_{A_n}\,, \dim S_{B_{N-n}}\} = \min\{2^n, 2^{N-n}\}$.

• Average linear entropy (global entanglement):

$$\langle S_L^{(n:N-n)}(\rho)\rangle \,=\, \left(\begin{array}{c} N \\ n \end{array}\right)^{-1} \sum_{A_n} S_L^{(A_n;B_{N-n})}(\rho)\,,$$

sum over all the possible bi-partitions of the system in two subsystems, respectively with n and $N-n$ elements ($1 \leq n < N$).

Entanglement measures

It is necessary to distinguish the various entanglement measures for pure and mixed states (which may contain classical correlations).

- **Measures for pure states:**
  - von Neumann entropy
  - Geometric Entanglement
- **Measures for mixed states:**
  - Entanglement of Formation and Concurrence
  - Logarithmic negativity
  - Relative Entropy of Entanglement

Entanglement in neutrino oscillations: two-flavors

Consider the density matrix for the electron neutrino state $\rho^{(e)} = |\nu_e(t)\rangle\langle\nu_e(t)|$, and trace over mode $\mu \Rightarrow \rho_e^{(e)}$.

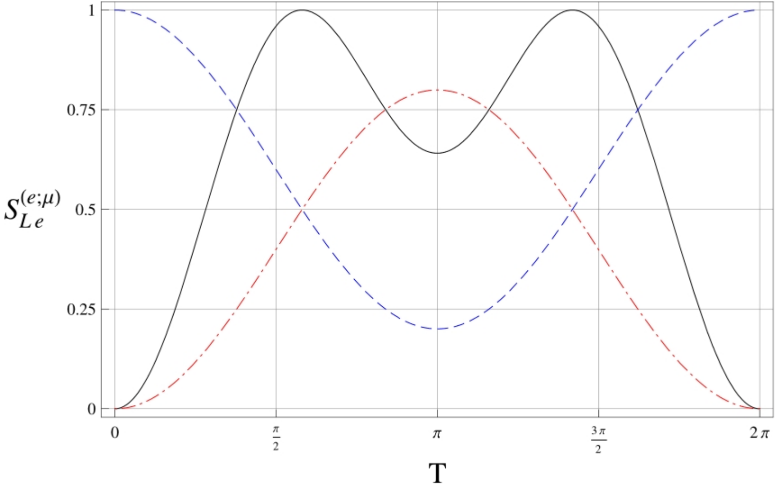
- The associated linear entropy is :

$$S_L^{(e;\mu)}(\rho^{(e)}) = 4\,|\widetilde{\mathbf{U}}_{e\mu}(t)|^2\,|\widetilde{\mathbf{U}}_{ee}(t)|^2 = 4\,P_{\nu_e\to\nu_e}(t)\,P_{\nu_e\to\nu_\mu}(t)$$

The linear entropy for the state $\rho^{(\alpha)}$ is:

$$\begin{aligned} S_{L\alpha}^{(e;\mu)} = S_{L\alpha}^{(\mu;e)} = \langle S_{L\alpha}^{(1:1)}\rangle &= 4\,|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2\,|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2 \\ &= 4|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2\,(1-|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2) \\ &= 4|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2\,(1-|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2)\,. \end{aligned}$$

- Linear entropy given by product of transition probabilities !



Linear entropy $S_{Le}^{(e;\mu)}$ (full) as a function of the scaled time $T = \frac{2Et}{\Delta m_{12}^2}$, with $\sin^2\theta = 0.314$. Transition probabilities $P_{\nu_e \to \nu_e}$ (dashed) and $P_{\nu_e \to \nu_\mu}$ (dot-dashed) are reported for comparison.

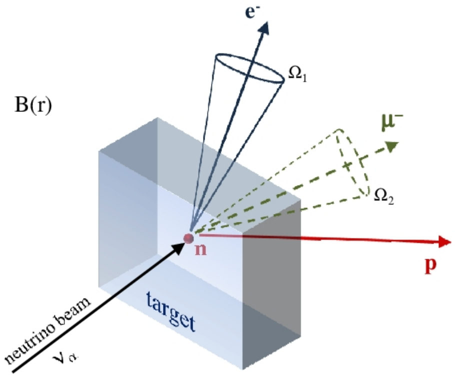
ν–oscillations as a resource for quantum information

• Single-particle entanglement encoded in flavor states $|\underline{\nu}^{(f)}(t)\rangle$ is a real physical resource that can be used, at least in principle, for protocols of quantum information.

– Experimental scheme for the transfer of the flavor entanglement of a neutrino beam into a single-particle system with *spatially separated modes.*

Charged-current interaction between a neutrino ν_α with flavor α and a nucleon N gives a lepton α^- and a baryon X:

$$\nu_\alpha + N \longrightarrow \alpha^- + X\,.$$



Generation of a single-particle entangled lepton state (two flavors):

In the target the charged-current interaction occurs: $\nu_\alpha + n \longrightarrow \alpha^- + p$ with $\alpha = e\,,\mu$. A spatially nonuniform magnetic field $\mathbf{B}(\mathbf{r})$ constraints the momentum of the outgoing lepton within a solid angle Ω_i, and ensures spatial separation between lepton paths. The reaction produces a superposition of electronic and muonic spatially separated states.

• Given the initial Bell-like superposition $|\nu_\alpha(t)\rangle$ the unitary process associated with the weak interaction leads to the superposition

$$|\alpha(t)\rangle \;=\; \Lambda_e|1\rangle_e|0\rangle_\mu + \Lambda_\mu|0\rangle_e|1\rangle_\mu \,,$$

where $|\Lambda_e|^2 + |\Lambda_\mu|^2 = 1$, and $|k\rangle_\alpha$, with $k = 0, 1$, is the lepton qubit.

The coefficients Λ_α are proportional to $\widetilde{\mathbf{U}}_{\alpha\beta}(t)$ and to the cross sections associated with the creation of an electron or a muon.

• Analogy with single-photon system: quantum uncertainty on the "*which path*" of the photon at the output of an unbalanced beam splitter $\Leftrightarrow$ uncertainty on the "*which flavor*" of the produced lepton.

The coefficients Λ_α plays the role of the transmissivity and of the reflectivity of the beam splitter.

Other results

- Generalization to three flavors. Extension to wave packets (decoherence);*

- Flavor entanglement in Quantum Field Theory.†

*M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2015).
†M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2014).

Quantum Correlations & nonlocality in neutrino oscillations

Quantum Correlations

Quantum systems exhibit properties that are beyond our understanding of reality. They show correlations that have no classical counterpart.

Entanglement is the most known of these correlations. But the terminology *quantum correlations* refers to a broader concept:

- **Quantum correlations related to entanglement:**
  - Bell non-locality
  - Entanglement
  - Quantum steering
- **Quantum correlations beyond entanglement:**
  - Quantum discord

Quantum correlations beyond entanglement

The mutual information between two random variables A and B can be expressed in two different ways:

$$I(A:B) = H(A) + H(B) - H(A,B),$$
$$J(A:B) = H(A) - H(A|B).$$

The quantum generalization is given by:

$$I(\rho^{AB}) = S(\rho^{A}) + S(\rho^{B}) - S(\rho^{AB}),$$
$$J(\rho^{AB})_{\{\prod_i^B\}} = S(\rho^{A}) - S(A|\{\prod_i^B\})$$

The **quantum discord** is defined as‡:

$$\delta^{B|A}(\rho^{AB}) = \min_{\{\prod_i^B\}} [I(\rho^{AB}) - J(\rho^{AB})_{\{\prod_i^B\}}],$$

‡H. Ollivier, W.H. Zurek, Phys. Rev. Lett. (2001)
L. Henderson, V. Vedral, J. Phys. (2001)

Quantum Resource Theory

Resource theories are a versatile set of tools developed in quantum information theory[§].

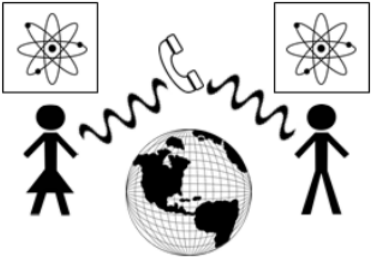
The basic idea of a quantum resource theory is to study quantum information processing under a restricted set of physical operations, called *free* operations.

These allow as to prepare only certain physical states, called *free* states. The other are called **resource** states.

[§] E. Chitambar, G. Gour, Rev. Mod. Phys. (2019)

Entanglement in QRT

Alice and Bob work in their laboratory separated by a large distance. They can communicate only by telephone.



The free operations consist in local operations and classical communication (LOCC). But an entangled state cannot be generated using LOCC $\Rightarrow$ **Entanglement is a (quantum) resource**.

Classification of QRTs

The QRTs can be classified in:

- **Convex resources**:
  - Entanglement
  - Coherence
- **Non-convex resources:**
  - Quantum discord

Quantum correlations in neutrino oscillations

- Recently, quantum correlations have been investigated in the context of high-energy particle physics;

Focus on neutrinos and mesons, which are candidates for quantum information applications beyond photons.

Quantum correlations in neutrino oscillations (partial list):

A.K. Alok et al., *Quantum correlations in terms of neutrino oscillation probabilities* Nuc. Phys. B (2016)

J.A. Formaggio et al., *Violation of the Leggett- Garg Inequality in Neutrino Oscillation* Phys. Rev. Lett. (2016).

J.Naikoo et al. *Leggett-Garg inequality in the context of three flavor neutrino oscillation* Phys. Rev. D (2019)

K. Dixit et al., *Study of coherence and mixedness in meson and neutrino systems* Eur. Phys. J. C (2019)

Quantum correlations in neutrino oscillations

X.-S. Song et al. *Quantifying quantum coherence in experimentally observed neutrino oscillations* Phys. Rev. A (2018)

F. Ming et al. *Quantification of quantumness in neutrino oscillations* Eur. Phys. J. C (2020)

L.-J. Li et al. *Characterizing entanglement and measurement's uncertainty in neutrino oscillations* Eur. Phys. J. C (2021)

P.Kurashvili et al *Coherence and mixedness of neutrino oscillations in a magnetic field* Eur. Phys. J. C (2021)

S.Shafaq and P.Mehta *Enhanced violation of Leggett–Garg inequality in three flavour neutrino oscillations via non-standard interactions* J. Phys. G (2021)

- In Ming et al. (2020), quantumness in neutrino oscillations has been quantified through correlation measures such as Non-local Advantage of Quantum Coherence (NAQC), quantum steering and Bell non-locality.

Non-local Advantage of Quantum Coherence

- A state is said to be coherent provided that there are non-zero non-diagonal elements in its matrix representation.

Coherence is quantified by menas of the l_1-norm of coherence:*

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{i,j}|$$

The upper bound beyond which the effects of non-locality emerge is given by:

$$\sum_{i=x,y,z} C_i^{l_1}(\rho) \leq C_{max}.$$

*T.Baumgratz, M.Cramer and M.B.Plenio, Phys. Rev. Lett. (2014).

Quantification of quantumness in neutrino oscillations

– The criterion for NAQC can be also written as[†]:

$$N^{l_1}(\rho_{AB}) = \frac{1}{2}\sum_{i,j,b} p(\rho^b_{\Pi_{j\neq i}}) C^{\sigma_i}_{l_1}(\rho_{B|\Pi_{j\neq i}}) > \sqrt{6}.$$

– The Bell non-locality can be detected by the violation of CHSH inequality:

$$B(\rho_{AB}) = |\langle B_{CHSH}\rangle| \leq 2.$$

– The criterion of quantum steering for a bipartite system is:

$$F_n(\rho_{AB}, \varsigma) = \frac{1}{\sqrt{n}}\left|\sum_{i=1}^{n} Tr(\rho_{AB} A_i \otimes B_i)\right| \leq 1.$$

[†]F. Ming, X-K. Song, D. Wang, Eur. Phys. J. C (2020)

– The NAQC quantifier is defined as:

$$\mathcal{N}^{l_1}(\rho_{AB}) = \max\left\{0, \frac{N^{l_1}(\rho_{AB}) - \sqrt{6}}{N^{l_1}_{max}(\rho_{AB}) - \sqrt{6}}\right\}.$$

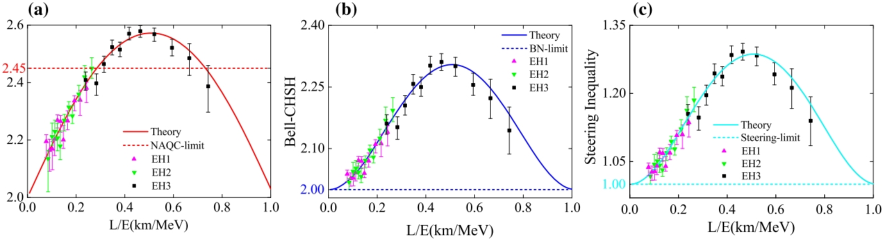
– The Bell nonlocality quantifier can be written as:

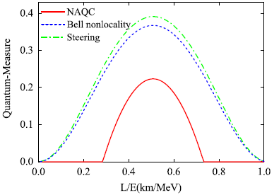
$$\mathcal{B}(\rho_{AB}) = \max\left\{0, \frac{B(\rho_{AB}) - 2}{B_{max}(\rho_{AB}) - 2}\right\},$$

– The steering quantifier is:

$$\mathcal{F}_3(\rho_{AB}) = \max\left\{0, \frac{F_3(\rho_{AB}) - 1}{F_3^{max}(\rho_{AB}) - 1}\right\},$$

Quantumness in neutrino oscillations (Daya Bay) *

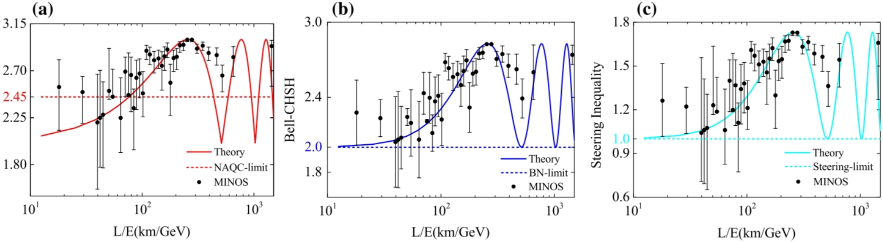


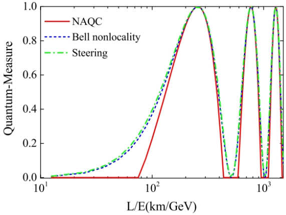


- Daya Bay: $\sin^2 2\theta_{13} = 0.084 \pm 0.005$ and $\Delta m^2_{ee} = 2.42^{+0.10}_{-0.11} \times 10^{-3} eV^2$
- NAQC is a stronger nonclassical correlation than Bell non-locality and quantum steering.

*F. Ming, X-K. Song, D. Wang, Eur. Phys. J., (2020)

Quantumness in neutrino oscillations (MINOS) †





- MINOS: $\sin^2 2\theta_{23} = 0.95^{+0.035}_{-0.036}$ and $\Delta m^2_{32} = 2.32^{+0.12}_{-0.08} \times 10^{-3} eV^2$.
- The NAQC is a stronger nonclassical correlation than Bell non-locality and quantum steering.

† F. Ming, X-K. Song, D. Wang, Eur. Phys. J., (2020)

NAQC & Bell nonlocality in the wave packet approach

• We have extended the studies on quantumness of neutrino oscillations through NAQC using the wave packet approach.[‡]

Neutrino with definite flavor:

$$|\nu_\alpha(x,t)\rangle = \sum_j U^*_{\alpha j}\psi_j(x,t)|\nu_j\rangle$$

where:

$$\psi_j(x,t) = \frac{1}{\sqrt{2\pi}}\int dp\ \psi_j(p)e^{ipx-iE_j(p)t}$$

with:

$$\psi_j(p) = \left(2\pi {\sigma_p^P}^2\right)^{-\frac{1}{4}}\exp - \frac{(p-p_j)^2}{4{\sigma_p^P}^2}$$

[‡]M.B., S. De Siena and C. Matrella, Eur. Phys. J. C (2021)

Wave packet description of neutrino oscillations

Assume the condition $\sigma_p^P \ll E_j^2(p_j)/m_j$. Then we have:

$$E_j(p) \simeq E_j + v_j(p - p_j)$$

Integrating on p, one gets the wave packet in coordinate space:

$$\psi_j(x,t) = \left(2\pi{\sigma_x^P}^2\right)^{-\frac{1}{4}} \exp\left[-iE_j t + ip_j x - \frac{(x - v_j t)^2}{4{\sigma_x^P}^2}\right]$$

Write density matrix operator $\rho_\alpha(x,t) = |\nu_\alpha(x,t)\rangle\langle\nu_\alpha(x,t)|$. After time integration, one gets the oscillation formula in space

$$P_{\alpha\beta}(L) = \sum_{j,k} U^*_{\alpha j} U_{\alpha k} U^*_{\beta j} U_{\beta k} \exp\left[-2\pi i \frac{L}{L^{osc}_{jk}} - \left(\frac{L}{L^{coh}_{jk}}\right)^2 - 2\pi^2(1-\xi)^2\left(\frac{\sigma_x}{L^{osc}_{jk}}\right)^2\right]$$

Wave packet description of neutrino oscillations[§]

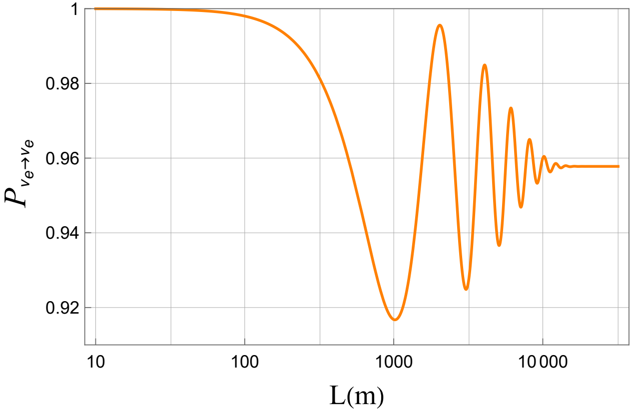
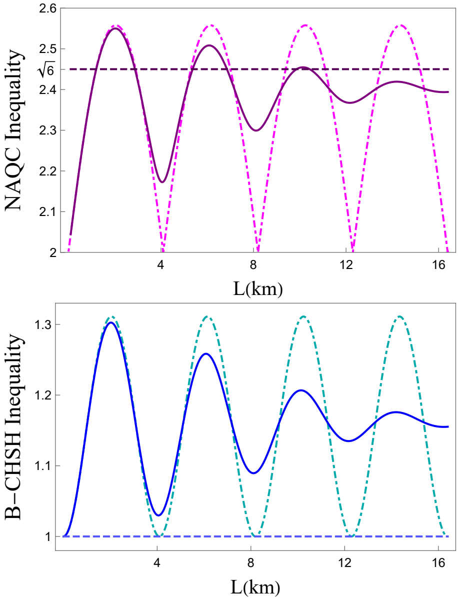


Figure 1: Survival transition for an electronic neutrino in the wave packet approach. $E = 2\,MeV$, $\xi = 0$, $\sin^2 2\theta_{13} = 0.084 \pm 0.005$ and $\Delta m^2_{ee} = 2.42^{+0.10}_{-0.11} \times 10^{-3} eV^2$ and $\sigma_x = 3.3 \times 10^{-6} m$.

[§] C. Giunti, C.W. Kim, *Fundamentals of Neutrino Physics and Astrophysics*, ,Oxford University Press (2007)

NAQC in the wave packet approach (Daya Bay)



NAQC in the wave packet approach (Daya Bay)

The amount of coherence increases with the wave packet width σ_x.

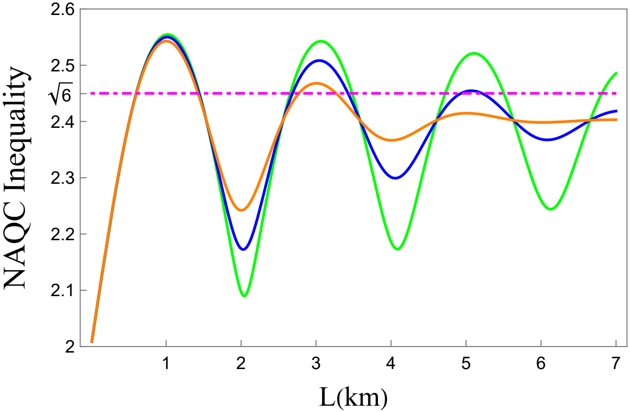
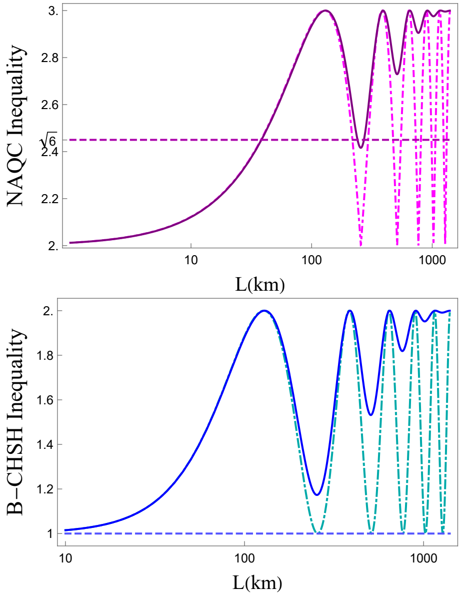


Figure 2: NAQC inequality as a function of the distance for three different wave packet widths σ_x: $\sigma_x = 5 \times 10^{-6}$ (green line), $\sigma_x = 2.5 \times 10^{-6} m$ (blue line) and $\sigma_x = 1.7 \times 10^{-6} m$ (orange line). The value of the energy is $E = 2MeV$.

NAQC in the wave packet approach (MINOS)



Results and perspectives

- Our treatment based on wave packets leads to a better agreement with experimental data in the case of MINOS.*

- NAQC has a different long-distance behaviour for the two experiments, due to the different values of the mixing angle.

- Existence of a "critical" angle for which NAQC exceeds the bound.

*M.B., S. De Siena and C. Matrella, Eur. Phys. J. C (2021)

Complete Complementarity Relations

To better understand the above results, we resort to CCR.

- N.Bohr (1928): complementarity principle
- W.K.Wootters and W.H.Zurek, *Complementarity in the double-slit experiment: quantum nonseparability and a quantitative statement of Bohr's principle*, Phys. Rev. D (1979)
- M.Jakob and J.A.Bergou, *Quantitative complementarity relations in bipartite systems: entanglement as a physical reality*, Opt. Comm. (2010)

- For a pure state, $|\Psi_{A,B}\rangle = \sum_{i,j=0}^{d_A-1,d_B-1} a_{ij}\,|i,j\rangle_{A,B}$ *:

$$P_{hs}(\rho_A) + C_{hs}(\rho_A) + C_{hs}^{nl}(\rho_{A|B}) = \frac{d_A - 1}{d_A}$$

with

– Predictability

$$P_{hs}(\rho_A) \equiv \sum_{i=0}^{d_A-1} (\rho_{ii}^A)^2 - \frac{1}{d_A}\,,$$

– Quantum coherence (visibility)

$$C_{hs}(\rho_A) \equiv \sum_{i\neq k}^{d_A-1} |\rho_{ik}^A|^2$$

– Non-local quantum coherence (entanglement)

$$C_{hs}^{nl}(\rho_{A|B}) = \sum_{i\neq k, j\neq l} |\rho_{ij,kl}|^2 - 2\sum_{i\neq k, j<l} \Re(\rho_{ij,kj}\rho_{il,kl}^*)$$

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* M.L.W.Basso and J.Maziero, J. Phys. A (2020)

CCR for oscillating neutrinos

Neutrino state:

$$|\nu_e(t)\rangle = a_{ee}\,|10\rangle + a_{e\mu}\,|01\rangle$$

with $|10\rangle = |\nu_e\rangle$, $|01\rangle = |\nu_\mu\rangle$.

The density matrix is:

$$\rho_{e\mu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |a_{e\mu}|^2 & a_{ee}a_{e\mu}^* & 0 \\ 0 & a_{e\mu}a_{10}^* & |a_{ee}|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The state of subsystems e and μ are:

$$\rho_e = \begin{pmatrix} |a_{ee}|^2 & 0 \\ 0 & |a_{e\mu}|^2 \end{pmatrix}; \qquad \rho_\mu = \begin{pmatrix} |a_{e\mu}|^2 & 0 \\ 0 & |a_{ee}|^2 \end{pmatrix}$$

We find:[†]

$$P_{hs}(\rho_e) = P_{ee}^2 + P_{e\mu}^2 - \frac{1}{2}$$

$$C_{hs}(\rho_e) = 0$$

$$C_{hs}^{nl}(\rho_{e\mu}) = 2P_{ee}P_{e\mu}$$

where $|a_{ee}|^2 = P_{ee}$, $|a_{e\mu}|^2 = P_{e\mu}$ and $P_{ee} + P_{e\mu} = 1$.

• When wave-packets are considered, an integration over time is performed, and the state $\rho_{e\mu}$ is mixed.

[†]M.B, S. De Siena, C. Matrella, in preparation

- For mixed states, we have to consider the following relation[‡]:

$$\log_2 d_A = I_{A:B}(\rho_{AB}) + S_{A|B}(\rho_{AB}) + P_{vn}(\rho_A) + C_{re}(\rho_A),$$

  - $C_{re}(\rho_A) = S_{vn}(\rho_{A\,diag}) - S_{vn}(\rho_A)$ relative entropy of coherence;
  - $P_{vn}(\rho_A) \equiv \ln d_A - S_{vn}(\rho_{A\,diag})$ predictability;
  - $I_{A:B}(\rho_{AB})$ mutual information of A and B
  - $S_{A|B}(\rho_{AB}) = S_{vn}(\rho_{AB}) - S_{vn}(\rho_B)$ indicates how much it is convenient knowing about the subsystem A with respect the whole system.

- Work in progress. Interesting preliminary results.

[‡] M.L.W.Basso and J.Maziero, (2021)

Chiral oscillations and lepton/antineutrino entanglement

Chiral oscillations

- Taking into account (bi)spinorial nature of neutrinos and chiral nature of weak interaction, one naturally gets chiral oscillations [*]
- They occur even with one flavor; interplay with flavor oscillations in the non-relativistic region[†]
- For neutrinos from CνB, chiral oscillations reduce detection by a factor of 2.[‡]
- Application: lepton-antineutrino entanglement and chiral oscillations in pion decay.[§]

[*] A. E. Bernardini and S. De Leo, Phys. Rev. D (2005)
[†] V.A.Bittencourt, A.Bernardini and M.B., Eur. Phys. J. C (2021)
[‡] S.-F. Ge and P.Pasquini, Phys. Lett. B (2020)
[§] V.A.Bittencourt, A.Bernardini and M.B., Universe (2021)

Chiral oscillations

Chiral representation of the Dirac matrices

$$\hat{\alpha}_i = \begin{bmatrix} \hat{\sigma}_i & 0 \\ 0 & -\hat{\sigma}_i \end{bmatrix}, \qquad \hat{\beta} = \begin{bmatrix} 0 & \hat{I}_2 \\ \hat{I}_2 & 0 \end{bmatrix},$$

Any bispinor $|\xi\rangle$ can be written in this representation as

$$|\xi\rangle = \begin{bmatrix} |\xi_R\rangle \\ |\xi_L\rangle \end{bmatrix},$$

The Dirac equation $\hat{H}_D\,|\xi\rangle = i|\dot{\xi}\rangle$ can then be written as

$$\hat{\mathbf{p}} \cdot \hat{\sigma}\,|\xi_R\rangle + m\,|\xi_L\rangle = i\partial_t\,|\xi_R\rangle\,,$$
$$-\hat{\mathbf{p}} \cdot \hat{\sigma}\,|\xi_L\rangle + m\,|\xi_R\rangle = i\partial_t\,|\xi_L\rangle\,,$$

- Evolution under the free Dirac Hamiltonian $\hat{H}_D$ induces left-right chiral oscillations.

Take initial state $|\psi(0)\rangle = [0,\ 0,\ 0,\ 1]^T$ which has negative helicity and negative chirality: $\hat{\gamma}_5 |\psi(0)\rangle = -|\psi(0)\rangle$.

The time evolved state $|\psi_m(t)\rangle = e^{-i\hat{H}_D t} |\psi(0)\rangle$ is given by

$$|\psi_m(t)\rangle = \sqrt{\frac{E_{p,m}+m}{4E_{p,m}}}\left[\left(1+\frac{p}{E_{p,m}+m}\right)e^{-iE_{p,m}t}|u_-(p,m)\rangle - \left(1-\frac{p}{E_{p,m}+m}\right)e^{iE_{p,m}t}|v_-(-p,m)\rangle\right],$$

with (for one-dimensional propagation along the $\mathbf{e}_z$ direction)

$$|u_\pm(p,m)\rangle = \sqrt{\frac{E_{p,m}+m}{4E_{p,m}}}\begin{bmatrix}\left(1\pm\frac{p}{E_{p,m}+m}\right)|\pm\rangle \\ \left(1\mp\frac{p}{E_{p,m}+m}\right)|\pm\rangle\end{bmatrix},$$

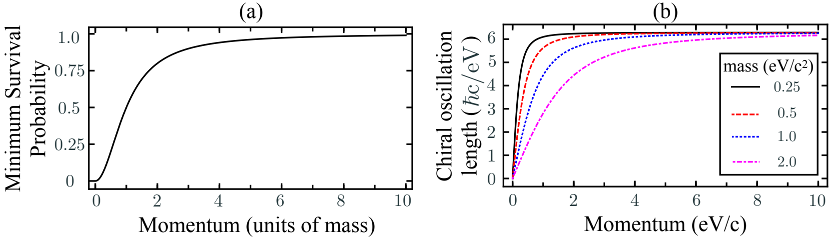
$$|v_\pm(p,m)\rangle = \sqrt{\frac{E_{p,m}+m}{4E_{p,m}}}\begin{bmatrix}\left(1\pm\frac{p}{E_{p,m}+m}\right)|\pm\rangle \\ -\left(1\mp\frac{p}{E_{p,m}+m}\right)|\pm\rangle\end{bmatrix},$$

- Survival probability of initial left-handed state

$$\mathcal{P}(t) = |\langle\psi_m(0)|\psi_m(t)\rangle|^2 = 1 - \frac{m^2}{E_{p,m}^2}\sin^2(E_{p,m}t),$$

Average value of the chiral operator $\langle\hat{\gamma}_5\rangle(t)$

$$\langle\hat{\gamma}_5\rangle(t) = \langle\psi_m(t)|\hat{\gamma}_5|\psi_m(t)\rangle = -1 + \frac{2m^2}{E_{p,m}^2}\sin^2(E_{p,m}t).$$



Chiral and flavor oscillations

- State of a neutrino of flavor α at a given t:

$$|\nu_\alpha(t)\rangle = \sum_i U_{\alpha,i}\,|\psi_{m_i}(t)\rangle \otimes |\nu_i\rangle\,,$$

where $|\psi_{m_i}(t)\rangle$ are bispinors.

- The state at $t = 0$ reads

$$|\nu_\alpha(0)\rangle = |\psi(0)\rangle \otimes \sum_i U_{\alpha,i}\,|\nu_i\rangle = |\psi(0)\rangle \otimes |\nu_\alpha\rangle\,,$$

where $|\psi(0)\rangle$ is a left handed bispinor.

- Survival probability:

$$\mathcal{P}_{\alpha\to\alpha} = |\,\langle\nu_\alpha(0)|\nu_\alpha(t)\rangle\,|^2 = |\sum_i |U_{\alpha,i}|^2\,\langle\psi(0)|\psi_{m_i}(t)\rangle\,|^2.$$

Two flavor mixing:

$$|\nu_e(t)\rangle = \left[\cos^2\theta\,|\psi_{m_1}(t)\rangle + \sin^2\theta\,|\psi_{m_2}(t)\rangle\right] \otimes |\nu_e\rangle$$
$$+ \sin\theta\cos\theta\,\left[|\psi_{m_1}(t)\rangle - |\psi_{m_2}(t)\rangle\right] \otimes |\nu_\mu\rangle\,,$$

- The survival probability can be decomposed as

$$\mathcal{P}_{e\to e}(t) = \mathcal{P}^S_{e\to e}(t) + \mathcal{A}_e(t) + \mathcal{B}_e(t).$$

$\mathcal{P}^S_{e\to e}(t)$ is the standard flavor oscillation formula

$$\mathcal{P}^S_{e\to e}(t) = 1 - \sin^2 2\theta \sin^2\left(\frac{E_{p,m_2} - E_{p,m_1}}{2} t\right)$$

and

$$\mathcal{A}_e(t) = -\left[\frac{m_1}{E_{p,m_1}}\cos^2\theta\sin\left(E_{p,m_1}t\right) + \frac{m_2}{E_{p,m_2}}\sin^2\theta\sin\left(E_{p,m_2}t\right)\right]^2,$$

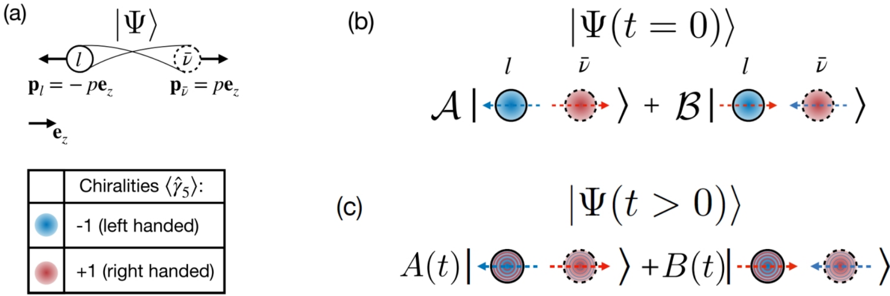
$$\mathcal{B}_e(t) = \frac{1}{2}\sin^2 2\theta\sin(E_{p,m_1}t)\sin(E_{p,m_2}t)\left(\frac{p^2 + m_1 m_2}{E_{p,m_1}E_{p,m_2}} - 1\right),$$

are correction terms due to the bispinorial structure.

- Agreement with the QFT formula (see below).

Lepton-Antineutrino Entanglement and Chiral Oscillations*

- As an application of chiral oscillations, we consider induced spin correlations in pion decay products ($\pi \rightarrow l + \bar{\nu}$)



*V.A.Bittencourt, A.Bernardini and M.B., Universe (2021)

- For the creation process, we assume the following superposition

$$|\Phi\rangle = \frac{|v_\uparrow(p, m_{\bar{\nu}})\rangle \otimes |u_\downarrow(-p, m_l)\rangle - |v_\downarrow(p, m_{\bar{\nu}})\rangle \otimes |u_\uparrow(-p, m_l)\rangle}{\sqrt{2}}$$

$$|u_\uparrow(p,m)\rangle = N_{p,m}\begin{bmatrix} f_+(p,m)|\uparrow\rangle \\ f_-(p,m)|\uparrow\rangle \end{bmatrix}, \qquad |u_\downarrow(p,m)\rangle = N_{p,m}\begin{bmatrix} f_-(p,m)|\downarrow\rangle \\ f_+(p,m)|\downarrow\rangle \end{bmatrix},$$

$$|v_\uparrow(p,m)\rangle = N_{p,m}\begin{bmatrix} f_+(p,m)|\uparrow\rangle \\ -f_-(p,m)|\uparrow\rangle \end{bmatrix}, \qquad |v_\downarrow(p,m)\rangle = N_{p,m}\begin{bmatrix} f_-(p,m)|\downarrow\rangle \\ -f_+(p,m)|\downarrow\rangle \end{bmatrix},$$

with

$$N_{p,m} = \sqrt{\frac{E_{p,m}+m}{4E_{p,m}}}, \quad f_\pm(p,m) = 1 \pm \frac{p}{E_{p,m}+m},$$

- At $t = 0$ the state is given by

$$|\Psi(0)\rangle = \frac{\hat{\Pi}_R^{(\bar{\nu})} \otimes \hat{\Pi}_L^{(l)} |\Phi\rangle}{\langle \Phi | \hat{\Pi}_R^{(\bar{\nu})} \otimes \hat{\Pi}_L^{(l)} |\Phi\rangle}.$$

Chirality projectors $\hat{\Pi}_{R(L)}^{(A)} = (\hat{I}^{(A)} + (-)\hat{\gamma}_5^{(A)})/2$, with $\hat{\gamma}_5 = \mathrm{diag}\left[\hat{I}_2, -\hat{I}_2\right]$ and $A = \bar{\nu}$, l, such that

$$|\Psi(0)\rangle = \mathcal{A}(p, m_l, m_{\bar{\nu}}) |\bar{\nu}_{\uparrow}(0)\rangle \otimes |l_{\downarrow}(0)\rangle - \mathcal{B}(p, m_l, m_{\bar{\nu}}) |\bar{\nu}_{\downarrow}(0)\rangle \otimes |l_{\uparrow}(0)\rangle.$$

The chirality projected states at $t = 0$ are

$$|\bar{\nu}_{\uparrow(\downarrow)}(0)\rangle = \begin{bmatrix} |\uparrow(\downarrow)\rangle \\ 0 \end{bmatrix}, \quad |l_{\uparrow(\downarrow)}(0)\rangle = \begin{bmatrix} 0 \\ |\uparrow(\downarrow)\rangle \end{bmatrix},$$

and the coefficients of the superposition are given by

$$\mathcal{A}(p, m_l, m_{\bar{\nu}}) = N_{p,m_l} N_{p,m_{\bar{\nu}}} f_+(p, m_{\bar{\nu}}) f_-(p, m_l) \left[\frac{1}{2} - \frac{p^2}{2E_{p,m_l} E_{p,m_{\bar{\nu}}}}\right]^{-\frac{1}{2}},$$

$$\mathcal{B}(p, m_l, m_{\bar{\nu}}) = N_{p,m_l} N_{p,m_{\bar{\nu}}} f_-(p, m_{\bar{\nu}}) f_+(p, m_l) \left[\frac{1}{2} - \frac{p^2}{2E_{p,m_l} E_{p,m_{\bar{\nu}}}}\right]^{-\frac{1}{2}}.$$

Spin entanglement at $t = 0$

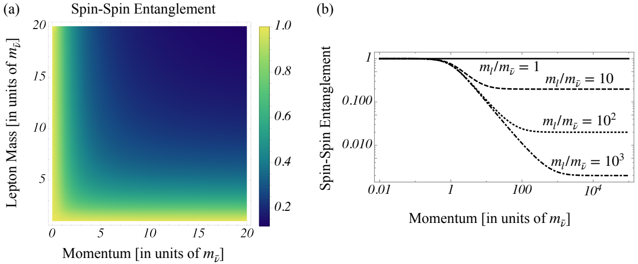
- The state of the lepton-antineutrino pair is then described in the composite Hilbert space $\mathcal{H}_{C_{\bar{\nu}}} \otimes \mathcal{H}_{S_{\bar{\nu}}} \otimes \mathcal{H}_{C_l} \otimes \mathcal{H}_{S_l}$.

- It is a 4-qubit entangled state.

- We can write $|\Psi(0)\rangle = |+_{C_{\bar{\nu}}}\rangle \otimes |-_{C_l}\rangle \otimes |\Psi_{S_{\bar{\nu}},S_l}\rangle$, with $|\pm_A\rangle$ denoting the positive (negative) chirality of $A = C_{\bar{\nu},l}$, and

$$|\Psi_{S_{\bar{\nu}},S_l}\rangle = \mathcal{A}(p, m_l, m_{\bar{\nu}})|\uparrow_{S_{\bar{\nu}}}\rangle \otimes |\downarrow_{S_l}\rangle - \mathcal{B}(p, m_l, m_{\bar{\nu}})|\downarrow_{S_{\bar{\nu}}}\rangle \otimes |\uparrow_{S_l}\rangle$$

is the joint spin state at $t = 0$.



Spin entanglement at $t = 0$

- Reduced density matrix

$$\begin{aligned}\rho_{S_{\bar{\nu}},S_l}(0) &= \mathrm{Tr}_{C_l,C_{\bar{\nu}}}\Big[|\Psi(0)\rangle\langle\Psi(0)|\Big] = |\Psi_{S_{\bar{\nu}},S_l}\rangle\langle\Psi_{S_{\bar{\nu}},S_l}| \\ &= \mathcal{A}^2(p,m_l,m_{\bar{\nu}})|\uparrow_{\bar{\nu}}\downarrow_l\rangle\langle\uparrow_{\bar{\nu}}\downarrow_l| + \mathcal{B}^2(p,m_l,m_{\bar{\nu}})|\downarrow_{\bar{\nu}}\uparrow_l\rangle\langle\downarrow_{\bar{\nu}}\uparrow_l| \\ &- \mathcal{A}(p,m_l,m_{\bar{\nu}})\mathcal{B}(p,m_l,m_{\bar{\nu}})\Big[|\uparrow_{\bar{\nu}}\downarrow_l\rangle\langle\downarrow_{\bar{\nu}}\uparrow_l| + |\downarrow_{\bar{\nu}}\uparrow_l\rangle\langle\uparrow_{\bar{\nu}}\downarrow_l|\Big].\end{aligned}$$

- Partial transposition yields $\rho^T_{S_{\bar{\nu}},S_l}$ from which we obtain the spin-spin negativity for the state at $t = 0$

$$\mathcal{N}_{S_{\bar{\nu}},S_l}(0) \equiv \mathcal{N}[\rho_{S_{\bar{\nu}},S_l}(0)] = 2|\mathcal{A}(p,m_l,m_{\bar{\nu}})\mathcal{B}(p,m_l,m_{\bar{\nu}})|.$$

Spin entanglement at $t \neq 0$

- For the joint lepton-antineutrino state we get

$$|\Psi(t)\rangle = \mathcal{A}(p, m_l, m_{\bar{\nu}})|\bar{\nu}_\uparrow(t)\rangle \otimes |l_\downarrow(t)\rangle - \mathcal{B}(p, m_l, m_{\bar{\nu}})|\bar{\nu}_\downarrow(t)\rangle \otimes |l_\uparrow(t)\rangle,$$

– antineutrino components:

$$|\bar{\nu}_\uparrow(t)\rangle = \mathcal{N}_{p,m_{\bar{\nu}}}\left[e^{-iE_{p,m_{\bar{\nu}}}t} f_+(p, m_{\bar{\nu}})|u_\uparrow(p, m_{\bar{\nu}})\rangle + e^{iE_{p,m_{\bar{\nu}}}t} f_-(p, m_{\bar{\nu}})|v_\uparrow(-p, m_{\bar{\nu}})\rangle\right],$$

$$|\bar{\nu}_\downarrow(t)\rangle = \mathcal{N}_{p,m_{\bar{\nu}}}\left[e^{-iE_{p,m_{\bar{\nu}}}t} f_-(p, m_{\bar{\nu}})|u_\downarrow(p, m_{\bar{\nu}})\rangle + e^{iE_{p,m_{\bar{\nu}}}t} f_+(p, m_{\bar{\nu}})|v_\downarrow(-p, m_{\bar{\nu}})\rangle\right],$$

– lepton components:

$$|l_\uparrow(t)\rangle = \mathcal{N}_{p,m_l}\left[e^{-iE_{p,m_l}t} f_+(p, m_l)|u_\uparrow(-p, m_l)\rangle - e^{iE_{p,m_l}t} f_-(p, m_l)|v_\uparrow(p, m_l)\rangle\right],$$

$$|l_\downarrow(t)\rangle = \mathcal{N}_{p,m_l}\left[e^{-iE_{p,m_l}t} f_-(p, m_l)|u_\downarrow(-p, m_l)\rangle - e^{iE_{p,m_l}t} f_+(p, m_l)|v_\downarrow(p, m_l)\rangle\right].$$

Spin entanglement at $t \neq 0$

- The reduced matrix $\rho_{S_{\bar{\nu}},S_l}(t) = \mathrm{Tr}_{\mathrm{Chirality}}\left[|\Psi(t)\rangle\langle\Psi(t)|\right]$ describes a mixed state with entanglement affected by chiral oscillations.
- Entanglement between the spins at time t

$$\mathcal{N}_{S_{\bar{\nu}},S_l}(t) \equiv \mathcal{N}[\rho_{S_{\bar{\nu}},S_l}(t)] = ||\rho^{T}_{S_{\bar{\nu}},S_l}(t)|| - 1 = \mathcal{N}_{S_{\bar{\nu}},S_l}(0)\Gamma(t)$$

with

$$\Gamma(t) = \prod_{j=\bar{\nu},l}\left[1 - \frac{p^2}{m_j^2}\left(\langle\hat{\gamma}_5\rangle_j(t) - 1\right)^2\right]^{\frac{1}{2}}.$$

The average chiralities are given by $\langle\hat{\gamma}_5\rangle_A(t) = \mathrm{Tr}_A[\rho_A(t)]$ with $A = \bar{\nu}, l$:

$$\begin{aligned}
\langle\hat{\gamma}_5\rangle_{\bar{\nu}}(t) &= 1 - \frac{m_{\bar{\nu}}^2}{E_{p,m_{\bar{\nu}}}^2}\left[1 - \cos\left(2E_{p,m_{\bar{\nu}}}t\right)\right], \\
\langle\hat{\gamma}_5\rangle_l(t) &= -1 + \frac{m_l^2}{E_{p,m_l}^2}\left[1 - \cos\left(2E_{p,m_l}t\right)\right].
\end{aligned}$$

Spin entanglement at $t \neq 0$

- Degree of mixedeness of the spin density matrix:

$$\mathrm{Tr}[\rho^2_{S_{\bar{\nu}},S_l}(t)] = 1 - \frac{\mathcal{N}^2_{S_{\bar{\nu}},S_l}(0)(1-|\Gamma(t)|^2)}{2},$$

quantifies entanglement in the bipartition $(S_{\bar{\nu}}, S_l); (C_{\bar{\nu}}, C_l)$, i.e. between spins and chiralities.

- $\mathrm{Tr}[\rho^2_{S_{\bar{\nu}},S_l}(t)] < 1 \Rightarrow$ entanglement initially encoded only in the spins redistributes into spin-chirality entanglement.

- Entanglement encoded in the bipartition $(C_{\bar{\nu}}, S_{\bar{\nu}}); (C_l, S_l)$ is conserved:

$$\begin{aligned}\mathrm{Tr}[\rho^2_{\bar{\nu}}(t)] = \mathrm{Tr}[\rho^2_l(t)] &= \mathcal{A}^4(p, m_l, m_{\bar{\nu}}) + \mathcal{B}^4(p, m_l, m_{\bar{\nu}}) \\ &= 1 - \frac{\mathcal{N}^2_{S_{\bar{\nu}},S_l}(0)}{2} < 1.\end{aligned}$$

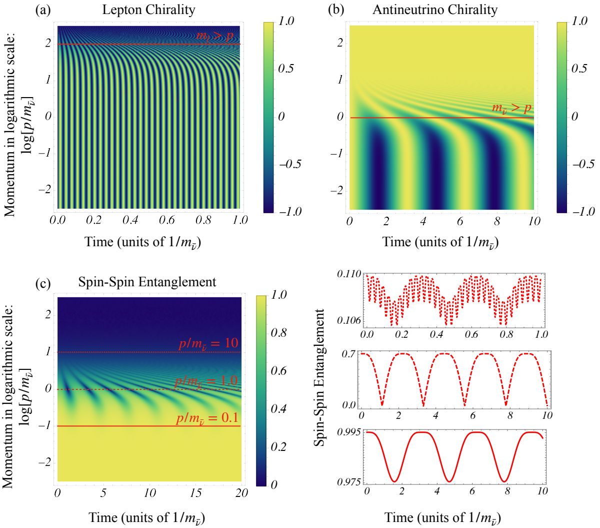


Figure 3: (a) Average lepton chirality, (b) average antineutrino chirality and (c) spin-spin entanglement as a function of the momentum and of time.

Chiral oscillations and Bell spin correlations

- The quantity

$$B[\rho(t)] = |\langle \hat{S}_{\bar{\nu},1} \otimes \hat{S}_{l,1} \rangle + \langle \hat{S}_{\bar{\nu},1} \otimes \hat{S}_{l,2} \rangle + \langle \hat{S}_{\bar{\nu},2} \otimes \hat{S}_{l,1} \rangle - \langle \hat{S}_{\bar{\nu},2} \otimes \hat{S}_{l,2} \rangle|,$$

is the Bell observable used to investigate non-local correlations[†].

For pure states, $B[\rho] > 2$ indicates that the correlations shared between the spins are non-local and that the state is entangled.

[†] N.Brunner et al., Rev. Mod. Phys. (2014)

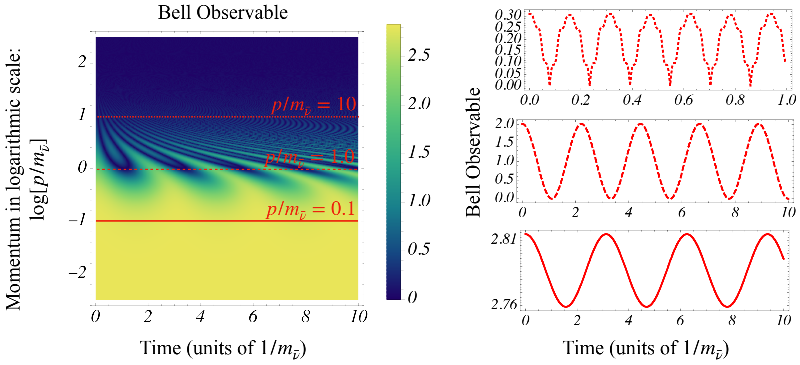


Figure 4: Bell observable as a function of the momentum (in units of the antineutrino mass and in log scale) and of time for $m_l/m_{\bar{\nu}} = 10^2$.

Results & Perspectives

- We find that chiral oscillations do affect spin-spin correlations for the entangled lepton–antineutrino couple.

- Resonance of oscillation amplitude at neutrino mass: possibility of extracting fundamental information via quantum correlations

- We plan to study Leggett-Garg inequalities for the reduced system involving only leptonic d.o.f.

- Inclusion of flavor oscillations.

Exact neutrino oscillation formula

Mixing of neutrino fields

– Mixing relations for two Dirac fields

$$\nu_e(x) = \cos\theta\,\nu_1(x) + \sin\theta\,\nu_2(x)$$

$$\nu_\mu(x) = -\sin\theta\,\nu_1(x) + \cos\theta\,\nu_2(x)$$

ν_1, ν_2 are fields with definite masses.

– Mixing transformations connect the two quadratic forms:

$$\mathcal{L} = \bar{\nu}_1\,(i\,\not\partial - m_1)\,\nu_1 + \bar{\nu}_2\,(i\,\not\partial - m_2)\,\nu_2$$

and

$$\mathcal{L} = \bar{\nu}_e\,(i\,\not\partial - m_e)\,\nu_e + \bar{\nu}_\mu\,(i\,\not\partial - m_\mu)\,\nu_\mu - m_{e\mu}\,(\bar{\nu}_e\,\nu_\mu + \bar{\nu}_\mu\,\nu_e)$$

with

$$m_e = m_1\cos^2\theta + m_2\sin^2\theta,\; m_\mu = m_1\sin^2\theta + m_2\cos^2\theta,\; m_{e\mu} = (m_2 - m_1)\sin\theta\cos\theta.$$

– ν_i are free Dirac field operators:

$$\nu_i(x) = \sum_{\mathbf{k},r} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} \left[u^r_{\mathbf{k},i}(t)\, \alpha^r_{\mathbf{k},i} \;+\; v^r_{-\mathbf{k},i}(t)\, \beta^{r\dagger}_{-\mathbf{k},i} \right], \quad i = 1, 2.$$

– Anticommutation relations:

$$\{\nu_i^\alpha(x), \nu_j^{\beta\dagger}(y)\}_{t=t'} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}\delta_{ij}\,; \quad \{\alpha^r_{\mathbf{k},i}, \alpha^{s\dagger}_{\mathbf{q},j}\} = \{\beta^r_{\mathbf{k},i}, \beta^{s\dagger}_{\mathbf{q},j}\} = \delta^3(\mathbf{k}-\mathbf{q})\delta_{rs}\delta_{ij}$$

– Orthonormality and completeness relations:

$$u^r_{\mathbf{k},i}(t) \;=\; e^{-i\omega_{k,i}t}\, u^r_{\mathbf{k},i} \quad ; \quad v^r_{\mathbf{k},i}(t) \;=\; e^{i\omega_{k,i}t}\, v^r_{\mathbf{k},i} \quad ; \quad \omega_{k,i} = \sqrt{k^2 + m_i^2}$$

$$u^{r\dagger}_{\mathbf{k},i} u^s_{\mathbf{k},i} = v^{r\dagger}_{\mathbf{k},i} v^s_{\mathbf{k},i} = \delta_{rs} \quad , \quad u^{r\dagger}_{\mathbf{k},i} v^s_{-\mathbf{k},i} = 0 \quad , \quad \sum_r (u^{r\alpha*}_{\mathbf{k},i} u^{r\beta}_{\mathbf{k},i} + v^{r\alpha*}_{-\mathbf{k},i} v^{r\beta}_{-\mathbf{k},i}) = \delta_{\alpha\beta}\ .$$

– Fock space for ν_1, ν_2:

$$\mathcal{H}_{1,2} = \left\{ \alpha^\dagger_{1,2}\ ,\ \beta^\dagger_{1,2}\ ,\ |0\rangle_{1,2} \right\}.$$

– Vacuum state $|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2$.

Neutrino mixing in QFT

• Mixing relations for two Dirac fields

$$\nu_e(x) = \cos\theta\, \nu_1(x) + \sin\theta\, \nu_2(x)$$

$$\nu_\mu(x) = -\sin\theta\, \nu_1(x) + \cos\theta\, \nu_2(x)$$

can be written as[‡]

$$\nu_e^\alpha(x) = G_\theta^{-1}(t)\, \nu_1^\alpha(x)\, G_\theta(t)$$

$$\nu_\mu^\alpha(x) = G_\theta^{-1}(t)\, \nu_2^\alpha(x)\, G_\theta(t)$$

– Mixing generator:

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]$$

For ν_e, we get $\frac{d^2}{d\theta^2}\,\nu_e^\alpha = -\nu_e^\alpha$ with i.c. $\nu_e^\alpha\big|_{\theta=0} = \nu_1^\alpha$, $\frac{d}{d\theta}\nu_e^\alpha\big|_{\theta=0} = \nu_2^\alpha$.

[‡]M.B. and G.Vitiello, Annals Phys. (1995)

• The vacuum $|0\rangle_{1,2}$ is not invariant under the action of $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)\;|0\rangle_{1,2}$$

• Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$\lim_{V\to\infty}\ {}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty}\ e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\ \ln\left(1-\sin^2\theta\,|V_\mathbf{k}|^2\right)^2} = 0$$

with

$$|V_\mathbf{k}|^2 \;\;\equiv\;\; \sum_{r,s} |\ v^{r\dagger}_{-\mathbf{k},1} u^s_{\mathbf{k},2}\ |^2 \neq 0 \quad for \quad m_1 \neq m_2$$

.

Quantum Field Theory vs. Quantum Mechanics

- Quantum Mechanics:

- finite $\sharp$ of degrees of freedom.

- unitary equivalence of the representations of the canonical commutation relations (von Neumann theorem).

- Quantum Field Theory:

- infinite $\sharp$ of degrees of freedom.

- ∞ many unitarily inequivalent representations of the field algebra $\Leftrightarrow$ many vacua .

- The mapping between interacting and free fields is "weak", i.e. representation dependent (LSZ formalism)*. Example: theories with spontaneous symmetry breaking.

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*F.Strocchi, *Elements of Quantum Mechanics of Infinite Systems* (W. Sc., 1985).

- The "flavor vacuum" $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state[†]:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k},r} \Big[(1-\sin^2\theta\,|V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\,\cos\theta\,|V_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})$$

$$+\,\epsilon^r \sin^2\theta\,|V_{\mathbf{k}}||U_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\,|V_{\mathbf{k}}|^2\,\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\,|0\rangle_{1,2}$$

- Condensation density:

$$_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; _{e,\mu}\langle 0(t)|\beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; \sin^2\theta\,|V_{\mathbf{k}}|^2$$

vanishing for $m_1 = m_2$ and/or $\theta = 0$ (in both cases no mixing).

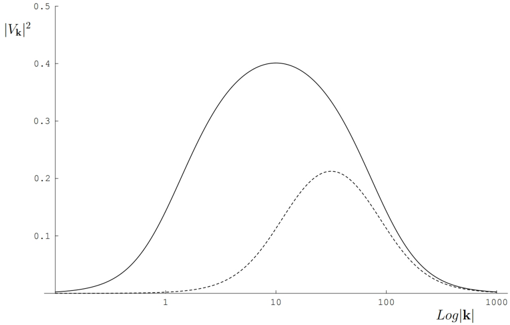
– Condensate structure as in systems with SSB (e.g. superconductors)

– Exotic condensates: mixed pairs

– Note that $|0\rangle_{e\,\mu} \neq |a\rangle_1 \otimes |b\rangle_2 \Rightarrow$ entanglement.

[†] A. Perelomov, *Generalized Coherent States,* (Springer V., 1986)

Condensation density for mixed fermions



Solid line: $m_1 = 1$, $m_2 = 100$; Dashed line: $m_1 = 10$, $m_2 = 100$.

- $V_{\mathbf{k}} = 0$ when $m_1 = m_2$ and/or $\theta = 0$.
- Max. at $k = \sqrt{m_1 m_2}$ with $V_{max} \to \frac{1}{2}$ for $\frac{(m_2 - m_1)^2}{m_1 m_2} \to \infty$.
- $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2 - m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$.

• Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\,\alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},2}\right)$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\,\alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},1}\right)$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\,\beta^r_{-\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right)$$

$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\,\beta^r_{-\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},1} + \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\right)$$

• Mixing transformation = Rotation + Bogoliubov transformation .

– Bogoliubov coefficients:

$$U_{\mathbf{k}}(t) \;=\; u^{r\dagger}_{\mathbf{k},2} u^r_{\mathbf{k},1}\, e^{i(\omega_{k,2}-\omega_{k,1})t} \quad ; \qquad V_{\mathbf{k}}(t) = \epsilon^r\, u^{r\dagger}_{\mathbf{k},1} v^r_{-\mathbf{k},2}\, e^{i(\omega_{k,2}+\omega_{k,1})t}$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$$

Decomposition of mixing generator *

Mixing generator function of m_1, m_2, and θ. Try to disentangle the mass dependence from the one by the mixing angle.

Let us define:

$$R(\theta) \equiv \exp\left\{\theta \sum_{\mathbf{k},r} \left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{\mathbf{k},1}\beta^{r}_{\mathbf{k},2}\right)e^{i\psi_k} - \left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \beta^{r\dagger}_{\mathbf{k},2}\beta^{r}_{\mathbf{k},1}\right)e^{-i\psi_k}\right]\right\},$$

$$B_i(\Theta_i) \equiv \exp\left\{\sum_{\mathbf{k},r} \Theta_{\mathbf{k},i}\ \epsilon^r \left[\alpha^{r}_{\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}e^{-i\phi_{k,i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}e^{i\phi_{k,i}}\right]\right\}, \qquad i = 1,2$$

Since $[B_1, B_2] = 0$ we put

$$B(\Theta_1, \Theta_2) \equiv B_1(\Theta_1)\, B_2(\Theta_2)$$

*M.B., M.V.Gargiulo and G.Vitiello, Phys. Lett. B (2017)

• We find:

$$G_\theta \; = \; B(\Theta_1, \Theta_2) \;\; R(\theta) \;\; B^{-1}(\Theta_1, \Theta_2)$$

which is realized when the $\Theta_{\mathbf{k},i}$ are chosen as:

$$U_{\mathbf{k}} = e^{-i\psi_k} \;\; \cos(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})\,; \qquad V_{\mathbf{k}} = e^{\frac{(\phi_{k,1}+\phi_{k,2})}{2}} \;\; \sin(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})$$

The $B_i(\Theta_{\mathbf{k},i})$, $i = 1, 2$ are Bogoliubov transformations implementing a mass shift, and $R(\theta)$ is a rotation.

– Their action on the vacuum is given by:

$$|\widetilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1, \Theta_2)|0\rangle_{1,2} \;\; = \;\; \prod_{\mathbf{k},r,i} \left[\cos\Theta_{\mathbf{k},i} + \epsilon^r \sin\Theta_{\mathbf{k},i} \alpha^{r\dagger}_{\mathbf{k},i} \beta^{r\dagger}_{-\mathbf{k},i} \right] |0\rangle_{1,2}$$

$$R^{-1}(\theta)|0\rangle_{1,2} \;\; = \;\; |0\rangle_{1,2} \; .$$

Bogoliubov vs Pontecorvo

Bogoliubov and Pontecorvo do not commute!

[ , ] ≠ 0

As a result, flavor vacuum gets a non-trivial term:

$$|0\rangle_{e,\mu} \equiv G_\theta^{-1}|0\rangle_{1,2} \,=\, |0\rangle_{1,2} \,+\, \left[B(m_1,m_2)\,,\; R^{-1}(\theta)\right]\; |\widetilde{0}\rangle_{1,2}$$

- Non-diagonal Bogoliubov transformation

$$|0\rangle_{e,\mu} \cong \left[\mathbb{1} + \theta\, a \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\; \tilde{V}_\mathbf{k} \sum_r \epsilon^r \left(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\right)\right]|0\rangle_{1,2}\,,$$

with $a \equiv \frac{(m_2-m_1)^2}{m_1 m_2}$.

Currents and charges for mixed fermions *

– Lagrangian in the mass basis:

$$\mathcal{L} \,=\, \bar{\nu}_m \, (i \not\partial - M_d) \, \nu_m$$

where $\nu_m^T = (\nu_1, \nu_2)$ and $M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$.

- $\mathcal{L}$ invariant under global $U(1)$ with conserved charge Q= total charge.
  – Consider now the $SU(2)$ transformation:

$$\nu'_m \,=\, e^{i\alpha_j \tau_j} \, \nu_m \qquad ; \qquad j \,=\, 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

* M. B., P. Jizba and G. Vitiello, Phys. Lett. B (2001)

The associated currents are:

$$\delta\mathcal{L} \;=\; i\alpha_j\,\bar{\nu}_m\,[\tau_j, M_d]\,\nu_m \;=\; -\alpha_j\,\partial_\mu J^\mu_{m,j}$$

$$J^\mu_{m,j} \;=\; \bar{\nu}_m\,\gamma^\mu\,\tau_j\,\nu_m$$

– The charges $Q_{m,j}(t) \equiv \int d^3\mathbf{x}\,J^0_{m,j}(x)$, satisfy the $su(2)$ algebra:

$$[Q_{m,j}(t), Q_{m,k}(t)] \;=\; i\,\epsilon_{jkl}\,Q_{m,l}(t)\,.$$

– Casimir operator proportional to the total charge: $C_m = \frac{1}{2}Q$.

- $Q_{m,3}$ is conserved $\Rightarrow$ charge conserved separately for ν_1 and ν_2:

$$Q_1 \;=\; \frac{1}{2}Q + Q_{m,3} \;=\; \int d^3\mathbf{x}\,\nu_1^\dagger(x)\,\nu_1(x)$$

$$Q_2 \;=\; \frac{1}{2}Q - Q_{m,3} \;=\; \int d^3\mathbf{x}\,\nu_2^\dagger(x)\,\nu_2(x).$$

These are the flavor charges in the absence of mixing.

The currents in the flavor basis

– Lagrangian in the flavor basis:

$$\mathcal{L} = \bar{\nu}_f \left(i \not\partial - M\right) \nu_f$$

where $\nu_f^T = (\nu_e, \nu_\mu)$ and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$.

– Consider the $SU(2)$ transformation:

$$\nu'_f = e^{i\alpha_j \tau_j} \nu_f \qquad ; \qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

– The charges $Q_{f,j} \equiv \int d^3\mathbf{x}\, J^0_{f,j}$ satisfy the $su(2)$ algebra:

$$[Q_{f,j}(t), Q_{f,k}(t)] = i\,\epsilon_{jkl}\, Q_{f,l}(t).$$

– Casimir operator proportional to the total charge $C_f = C_m = \frac{1}{2}Q$.

- $Q_{f,3}$ is not conserved $\Rightarrow$ exchange of charge between ν_e and ν_μ.

Define the flavor charges as:

$$Q_e(t) \equiv \frac{1}{2}Q + Q_{f,3}(t) = \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x)$$

$$Q_\mu(t) \equiv \frac{1}{2}Q - Q_{f,3}(t) = \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)$$

where $Q_e(t) + Q_\mu(t) = Q$.

– We have:

$$Q_e(t) = \cos^2\theta\, Q_1 + \sin^2\theta\, Q_2 + \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right]$$

$$Q_\mu(t) = \sin^2\theta\, Q_1 + \cos^2\theta\, Q_2 - \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right]$$

In conclusion:

– In presence of mixing, neutrino flavor charges are defined as

$$Q_e(t) \equiv \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x) \quad ; \qquad Q_\mu(t) \equiv \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)$$

– They are not conserved charges ⇒ flavor oscillations.

– They are still (approximately) conserved in the vertex ⇒ define flavor neutrinos as their eigenstates

- Problem: find the eigenstates of the above charges.

- Flavor charge operators are diagonal in the flavor ladder operators:

$$:: Q_\sigma(t) :: \;\; \equiv \;\; \int d^3\mathbf{x} \; :: \nu_\sigma^\dagger(x)\, \nu_\sigma(x) ::$$

$$= \;\; \sum_r \int d^3\mathbf{k} \, \left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t) \alpha^r_{\mathbf{k},\sigma}(t) \, - \, \beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \beta^r_{-\mathbf{k},\sigma}(t) \right) \, , \quad \sigma = e, \mu.$$

Here $:: \ldots ::$ denotes normal ordering w.r.t. flavor vacuum:

$$:: A :: \equiv \; A \, - \, {}_{e,\mu}\langle 0|A|0\rangle_{e,\mu}$$

- Define flavor neutrino states with definite momentum and helicity:

$$|\nu^r_{\mathbf{k},\sigma}\rangle \; \equiv \; \alpha^{r\dagger}_{\mathbf{k},\sigma}(0) \, |0\rangle_{e,\mu}$$

– Such states are eigenstates of the flavor charges (at t=0):

$$:: Q_\sigma :: |\nu^r_{\mathbf{k},\sigma}\rangle \; = \; |\nu^r_{\mathbf{k},\sigma}\rangle$$

Neutrino oscillation formula (QFT)

– We have, for an electron neutrino state:

$$\mathcal{Q}_{\mathbf{k},\sigma}(t) \equiv \langle \nu^r_{\mathbf{k},e}| :: Q_\sigma(t) :: |\nu^r_{\mathbf{k},e}\rangle$$

$$= \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2$$

• Neutrino oscillation formula (exact result)*:

$$\mathcal{Q}_{\mathbf{k},e}(t) = 1 - |U_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}\, t\right) - |V_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}\, t\right)$$

$$\mathcal{Q}_{\mathbf{k},\mu}(t) = |U_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}\, t\right) + |V_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}\, t\right)$$

- For $k \gg \sqrt{m_1 m_2}$, $|U_\mathbf{k}|^2 \to 1$ and $|V_\mathbf{k}|^2 \to 0$.

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* M.B., P.Henning and G.Vitiello, Phys. Lett. **B** (1999).

Lepton charge violation for Pontecorvo states[†]

– Pontecorvo states:

$$|\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos\theta\,|\nu^r_{\mathbf{k},1}\rangle \;+\; \sin\theta\,|\nu^r_{\mathbf{k},2}\rangle$$

$$|\nu^r_{\mathbf{k},\mu}\rangle_P \;=\; -\sin\theta\,|\nu^r_{\mathbf{k},1}\rangle \;+\; \cos\theta\,|\nu^r_{\mathbf{k},2}\rangle\,,$$

are *not* eigenstates of the flavor charges.

⇒ *violation of lepton charge conservation* in the production/detection vertices, at tree level:

$${}_P\langle\nu^r_{\mathbf{k},e}|:Q_e(0):|\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos^4\theta+\sin^4\theta+2|U_{\mathbf{k}}|\,\sin^2\theta\cos^2\theta \;<\; 1,$$

for any $\theta \neq 0$, $\mathbf{k} \neq 0$ and for $m_1 \neq m_2$.

[†]M. B., A. Capolupo, F. Terranova and G. Vitiello, Phys. Rev. **D** (2005)
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Other results

- Rigorous mathematical treatment for any number of flavors [*]
- Three flavor fermion mixing: CP violation[†];
- QFT spacetime dependent neutrino oscillation formula[‡];
- Boson mixing[§]; Majorana neutrinos[¶];
- Flavor vacuum and cosmological constant[‖]
- Flavor vacuum induced by condensation of D-particles.[**]
- Geometric phase for mixed particles[††].

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- Dynamical generation of fermion mixing*.
- Flavor-energy uncertainty relations for mixed states†.
- Poincaré invariance for flavor neutrinos‡.
- Violation of equivalence principle§.

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