the flavour puzzle as a vacuum problem

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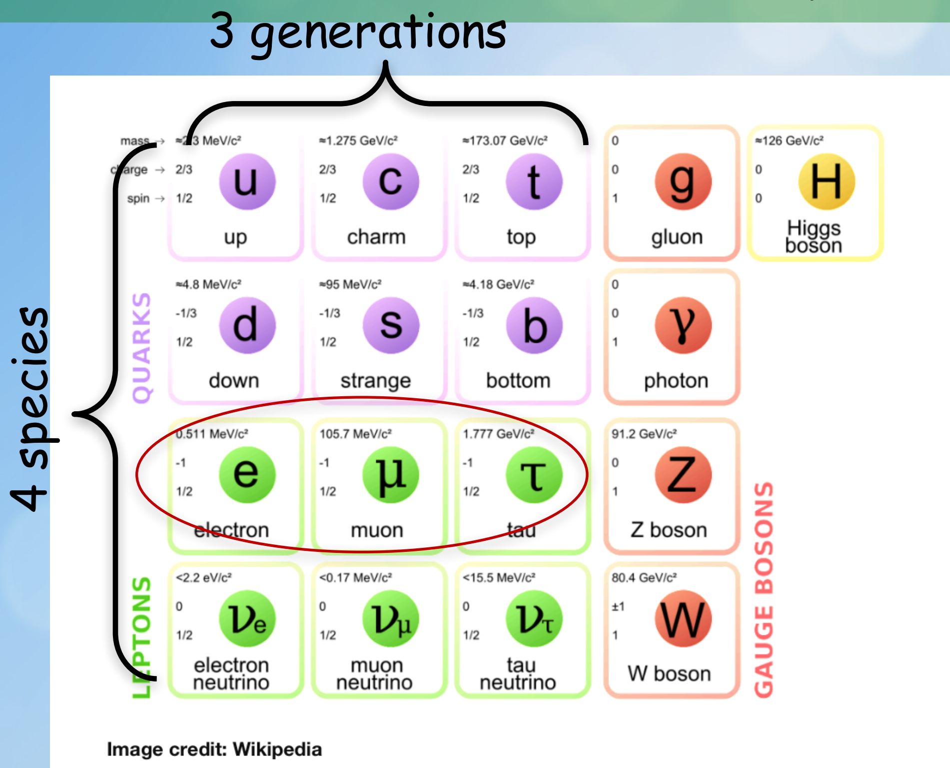
related talks:
Joao Penedo
Ivo de Medeiros
Saul Ramos Sanchez
Andreas Trautner

based on collaboration with Gui-Jun DING, Xiang-Gan LIU

- DF 2003.13448
- DFL 2010.07952
- DFL 2102.06716

The flavour puzzle



- gauge and Higgs boson masses
- all gauge interactions of 3 families of quark and lepton masses (15 x 3 Weyl spinors)

- 3 gauge couplings
- 2 masses (G_F, m_H)

fermion masses and mixing angles require (up to) 22 additional parameters [fermion bilinears]

- 6+6 masses
- 3+3 mixing angles
- 1+3 phases

$$\mathcal{L}_Y = -\overline{\Psi}\,\mathcal{Y}\,\Phi\,\Psi \qquad \mathcal{L}_\nu = -\frac{1}{\Lambda}(\Phi\,\Psi)\mathcal{W}\,(\Phi\,\Psi)$$

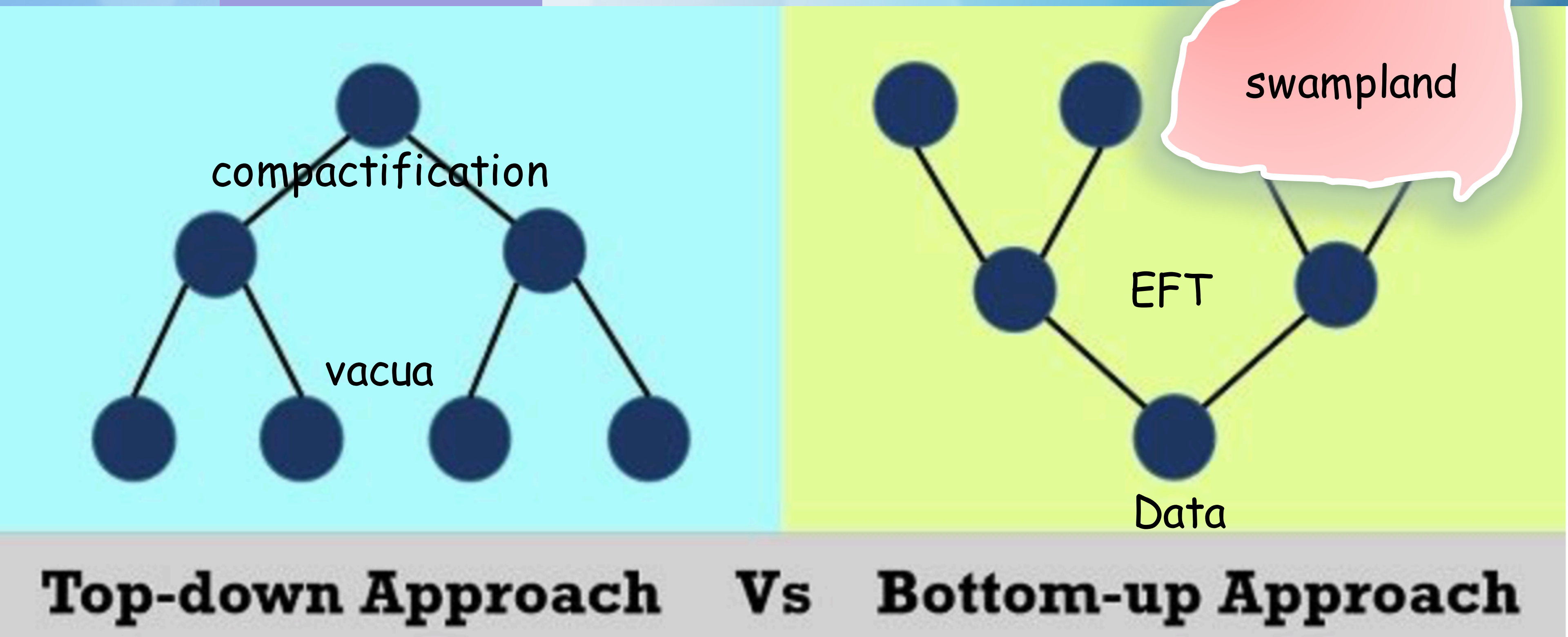
$$m_{ij} = \mathcal{Y}_{ij}\,v$$
$$m_{\nu ij} = \mathcal{W}_{ij} v^2/\Lambda$$

loosely constrained by gauge symmetry

Talks by
Saul Ramos Sanchez
Andreas Trautner

String Theory

Joao Penedo
Ivo de Medeiros
this talk



Baur
Kade
Nilles
Ramos-Sanchez
Trautner
Vaudrevange

Flavour Symmetries

"standard" Flavour Symmetries

$g \in G_{fl}$

$$\Psi \xrightarrow{g} \varrho(g)\,\Psi$$

$$\Psi = \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}$$

$$\varrho(g)^{+}\mathcal{Y}\,\varrho(g) = \mathcal{Y}$$

can be combined with CP

$$\Psi \xrightarrow{CP} X_{CP}\Psi^{*}$$

example: Isospin SU(2) in strong interactions $m_p = m_n$

- flavour symmetries of this type are necessarily broken

e.g. largest flavour symmetry of the quark sector

$\binom{u}{d}\binom{c}{s}\binom{t}{b}$

$u^c \;\; c^c \;\; t^c$

$d^c \;\; s^c \;\; b^c$

$$U(3)^3$$

[up to anomalies]

broken down to $U(1)_B \times U(1)_Y$

no constraint on quark masses/mixing angles

Symmetry Breaking

τ_α

symmetry breaking sector:
set of dimensionless, gauge invariant scalar fields, charged under G_{fl}

[τ_α stands for $\langle\tau_\alpha\rangle/\Lambda_F$ where the scale Λ_F has been set to 1]

reviews:
Ishimori, Kobayashi, Ohki, Shimizu, Okada, Tanimoto , 1003.3552;
King, Luhn, 1301.1340;
King, Merle, Morisi, Shimizu, Tanimoto, 1402.4271;
King, 1701.04413
Hagedorn, 1705.00684;
F.F., Romanino 1912.06028
.
.

<- many free parameters

lowest order Lagrangian parameters

higher dimensional operators

$$m_{ij}(\tau) = m^0_{ij} + m^{1\alpha}_{ij}\ \tau_\alpha + m^{1\bar{\alpha}}_{ij}\ \bar{\tau}_{\bar{\alpha}} + m^{2\alpha\beta}_{ij}\ \tau_\alpha\tau_\beta + ...$$

vacuum alignment in SB sector

SUSY breaking effects
RGE corrections
(Λ_{UV}, m_{SUSY} , $\tan\beta$)

huge number of models: G_{fl} continuous/discrete, global/local,......
no baseline model in bottom-up approach

Example

matter fields

fields		S_4
Higgs & leptons	L	$\mathbf{3'}$
	E_3^c	$\mathbf{1}$
	E_2^c	$\mathbf{1}$
	E_1^c	$\mathbf{1}$
	N_{atm}^c	$\mathbf{1}$
	N_{sol}^c	$\mathbf{1}$
	H_d	$\mathbf{1}$
	H_u	$\mathbf{1}$

Example

matter fields

symmetry breaking sector

	fields	S_4
Higgs & leptons	L	$\mathbf{3'}$
	E_3^c	$\mathbf{1}$
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	N_{atm}^c	$\mathbf{1}$
	N_{sol}^c	$\mathbf{1}$
	H_d	$\mathbf{1}$
	H_u	$\mathbf{1}$
flavon fields	$\phi'_{S,U}$	$\mathbf{3'}$
	$\rho_{S,U}$	$\mathbf{2}$
	$\xi_{S,U}$	$\mathbf{1}$
	ϕ_T	$\mathbf{3}$
	ξ_T	$\mathbf{1}$
	ϕ'_t	$\mathbf{3'}$
	ρ_t	$\mathbf{2}$
	ϕ'_{atm}	$\mathbf{3'}$
	ϕ'_{sol}	$\mathbf{3'}$
	ξ_{atm}	$\mathbf{1}$
	ξ_{sol}	$\mathbf{1}$
driving fields	$X_{3'}$	$\mathbf{3'}$
	X_2	$\mathbf{2}$
	X_1	$\mathbf{1}$
	$X_{1'}$	$\mathbf{1'}$
	Y_3	$\mathbf{3}$
	$Y_{3'}$	$\mathbf{3'}$
	$Z_{3'}$	$\mathbf{3'}$
	$\tilde{Z}_{3'}$	$\mathbf{3'}$
	X_0	$\mathbf{1}$

Example

matter fields

symmetry breaking sector

	fields	S_4	$U(1)$	$U(1)_x$	$Z_3^{(1)}$	$Z_3^{(2)}$	$Z_3^{(3)}$	$Z_3^{(4)}$	$Z_3^{(5)}$
Higgs & leptons	L	$\mathbf{3'}$	$-x_1+z_1$	1	1	0	0	0	0
	E_3^c	$\mathbf{1}$	$x_1-z_1-z_3$	-1	2	0	2	0	0
	E_2^c	$\mathbf{1}$	$x_1-x_2-z_1-2z_3-z_4+z_5$	-4	2	0	1	2	1
	E_1^c	$\mathbf{1}$	$x_1-2x_2-z_1-3z_3-2z_4+2z_5$	-7	2	0	0	1	2
	N^c_{atm}	$\mathbf{1}$	$-z_1$	0	2	0	0	0	0
	N^c_{sol}	$\mathbf{1}$	$-z_2$	0	0	2	0	0	0
	H_d	$\mathbf{1}$	0	0	0	0	0	0	0
	H_u	$\mathbf{1}$	0	0	0	0	0	0	0
flavon fields	$\phi'_{S,U}$	$\mathbf{3'}$	x_1+x_2	2	0	0	0	0	0
	$\rho_{S,U}$	$\mathbf{2}$	$z_1-z_2+z_3$	0	1	2	1	0	0
	$\xi_{S,U}$	$\mathbf{1}$	z_5	0	0	0	0	0	1
	ϕ_T	$\mathbf{3}$	$-x_2+z_5$	-3	0	0	0	0	1
	ξ_T	$\mathbf{1}$	z_4	0	0	0	0	1	0
	ϕ'_t	$\mathbf{3'}$	z_3	0	0	0	1	0	0
	ρ_t	$\mathbf{2}$	$x_2+z_3+z_4-z_5$	3	0	0	1	1	2
	ϕ'_{atm}	$\mathbf{3'}$	x_1	-1	0	0	0	0	0
	ϕ'_{sol}	$\mathbf{3'}$	$x_1-z_1+z_2$	-1	2	1	0	0	0
	ξ_{atm}	$\mathbf{1}$	$2z_1$	0	2	0	0	0	0
	ξ_{sol}	$\mathbf{1}$	$2z_2$	0	0	2	0	0	0
driving fields	$X_{3'}$	$\mathbf{3'}$	$-2x_1-2x_2$	-4	0	0	0	0	0
	X_2	$\mathbf{2}$	$2x_2-2z_5$	6	0	0	0	0	1
	X_1	$\mathbf{1}$	$-2z_3$	0	0	0	1	0	0
	$X_{1'}$	$\mathbf{1'}$	$x_2-z_3-z_5$	3	0	0	2	0	2
	Y_3	$\mathbf{3}$	$-x_1-x_2-z_1+z_2-z_3$	-2	2	1	2	0	0
	$Y_{3'}$	$\mathbf{3'}$	$-z_3-z_4$	0	0	0	2	2	0
	$Z_{3'}$	$\mathbf{3'}$	$-x_1-z_5$	1	0	0	0	0	2
	$\tilde{Z}_{3'}$	$\mathbf{3'}$	$-x_1-z_3$	1	0	0	2	0	0
	X_0	$\mathbf{1}$	0	0	0	0	0	0	0

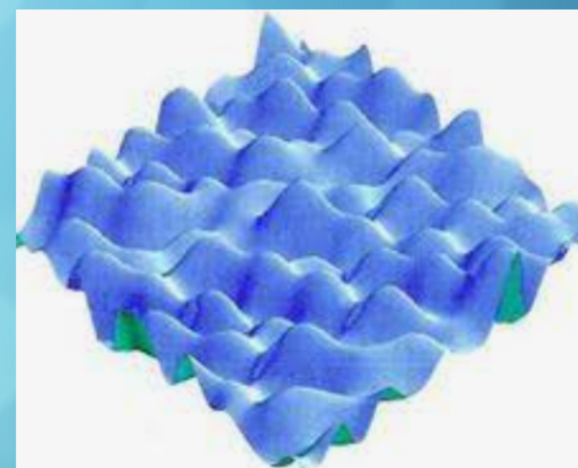
usual path:

choose G_f → assign f to G_f multiplets → look for some SB sector and adjust $\langle\tau_\alpha\rangle$

the crucial sector relegated to the last step

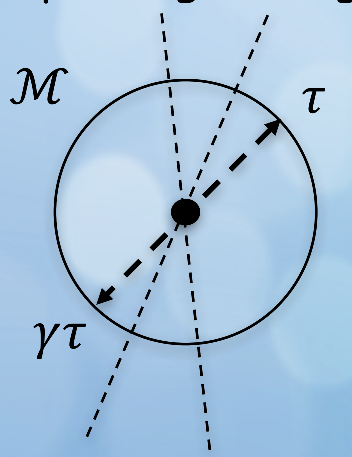
can we reverse the logic?

look for physically and/or mathematically motivated SB sector



$\tau \in \mathcal{M}$ = moduli space parametrizes possible vacua

$\tau \in \mathcal{M}$ = {lines of the plane passing through the origin}



$$\mathcal{L}_{IR}(\mathcal{Y}(\gamma\tau),\varphi') = \mathcal{L}_{IR}(\mathcal{Y}(\tau),\varphi)$$

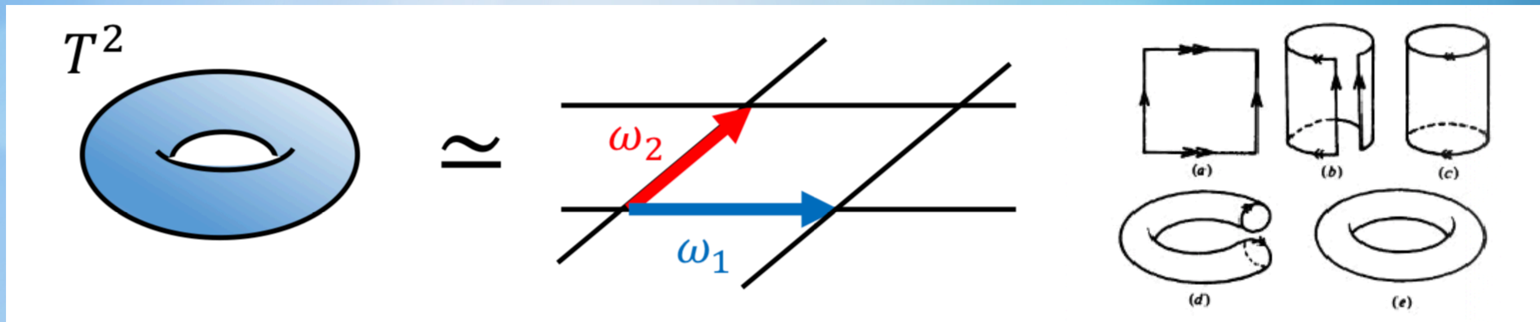
is a gauge symmetry

$$\begin{cases} \tau \to \gamma\tau \\ \varphi' = \xi(\gamma)\,\varphi \end{cases}$$

flavour symmetry G_f action on matter fields ↔ derived from $\mathcal{M}$ now

a less trivial example

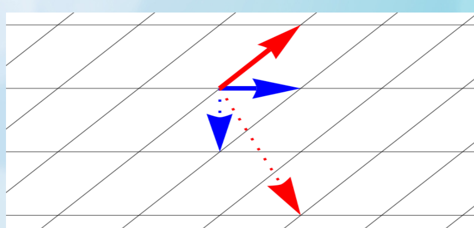
$\tau \in \mathcal{M}$ = {classes of conformally equivalent metrics on the torus}



tori parametrized by

$$\mathcal{M} = \left\{ \tau = \frac{\omega_2}{\omega_1} \quad Im(\tau) > 0 \right\}$$

lattice left invariant by modular transformations:

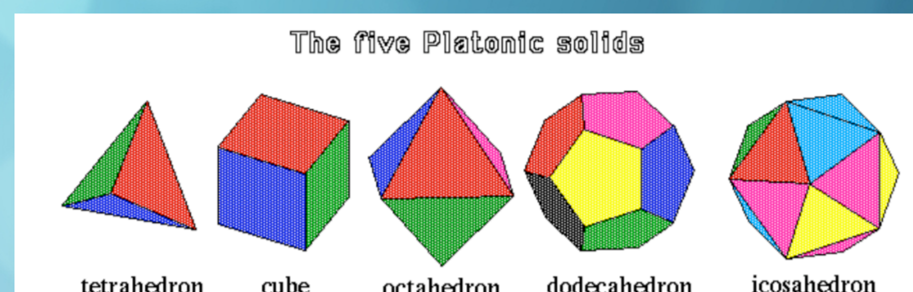


$$\tau \rightarrow \gamma\tau \equiv \frac{a\tau + b}{c\tau + d} \in SL(2, Z)$$

$$\mathcal{M} = \left\{ \tau = \frac{\omega_2}{\omega_1} \quad Im(\tau) > 0 \right\}$$

$$G_f = SL(2, Z)$$

$$\varphi' = \xi(\gamma)\, \varphi$$



$$\xi(\gamma) = (c\,\tau + d)^{k_\varphi}\, \rho_\varphi(\gamma)$$

choose the space $\mathcal{M}$ and guess the flavor symmetry G_f

- interesting candidates for $\mathcal{M}$: Hermitian Symmetric Spaces [HSS]

- HSS are Kähler

  - [Ding, F., Liu, 2010.07952]

- non-compact HSS as moduli space in sugra and compactified strings
- related to automorphic forms, building blocks of Yukawa couplings

$$\mathcal{M} = G/K$$

G: connected Lie group

K: compact subgroup

[classification: Calabi, Visentini 1960, Wolf 1964]

Lie algebra

$$\mathcal{G} = \mathcal{V} \oplus \mathcal{A} \qquad [\mathcal{V},\mathcal{V}] \subset \mathcal{V} \qquad [\mathcal{V},\mathcal{A}] \subset \mathcal{A} \qquad [\mathcal{A},\mathcal{A}] \subset \mathcal{V}$$

automorphism $\quad \mathcal{V} + \mathcal{A} \leftrightarrow \mathcal{V} - \mathcal{A}$

- flavour symmetry: a discrete subgroup Γ of G

example $G = Sp(2g,\mathbb{R}) \quad K = U(g)$ $\qquad \mathcal{M} = G/K$ noncompact

$$\mathcal{M}=\{\tau \in GL(g,\mathbb{C}) | \tau^t = \tau, Im(\tau) > 0\}$$

Siegel upper half plane complex dimension $g(g+1)/2$

action of G on τ

$$\tau \rightarrow \gamma\tau = (A\tau + B)(C\tau + D)^{-1}$$

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \gamma^t J \gamma = J \qquad J = \begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix}$$

candidate flavour group

$\Gamma = Sp(2g,\mathbb{Z})$ Siegel modular group

action on matter fields

- **automorphy factor** $j(\gamma,\tau)$ $\quad j(\gamma_1\gamma_2,\tau)=j(\gamma_1,\gamma_2\,\tau)\,j(\gamma_2,\tau)$

  cocycle condition

- **unitary representation** $\rho^I(\gamma)$ of (a finite copy of) Γ

  finite copy of Γ: Γ/G_d
  where $G_d\subset\Gamma$ normal subgroup of finite index

- nonlinear realization of Γ

$$\varphi^{(I)}\to j(\gamma,\tau)^{k_I}\,\rho^I(\gamma)\,\varphi^{(I)} \qquad \gamma\in\Gamma$$

weight (k_I) — unitary representation of Γ/G_d ($\rho^I(\gamma)$)

back to example $G = Sp(2g, \mathbb{R})$ $K = U(g)$

$\Gamma = Sp(2g, \mathbb{Z})$ Siegel modular group

- automorphy factor

$$j(\gamma, \tau) = \det(C\tau + D) \qquad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

- finite copies of $Sp(2g, \mathbb{Z})$

$$\Gamma / G_d(n) = \Gamma_{g,n}$$ finite Siegel modular group

genus level

$G_d(n) = \{\gamma \in \Gamma | \, \gamma = \mathbb{1}_{2g} \, mod \, n\}$ principal congruence subgroup

$$\varphi^{(I)} \to \det(C\tau + D)^{k_I} \rho^I(\gamma) \, \varphi^{(I)}$$

$\mathcal{N}$=1 SUSY invariant theories

- Yukawa interactions in $\mathcal{N}$=1 global SUSY [extension to $\mathcal{N}$=1 SUGRA straightforward]

$$S = \int d^4x d^2\theta\, w(\tau,\varphi) + h.c + \int d^4x d^2\theta d^2\bar{\theta}\, K(\tau,\varphi,\bar{\tau},\bar{\varphi})$$

superpotential =
Yukawa interactions

Kahler potential =
kinetic terms

- a minimal Kahler potential

$$K = -h\log Z(\tau,\bar{\tau}) + \sum_I Z(\tau,\bar{\tau})^{k_I} \left|\varphi^{(I)}\right|^2$$

for general
G, K, Γ

$$Z(\tau,\bar{\tau}) = [j^+(\gamma,\tau_0) j(\gamma,\tau_0)]^{-1}$$

$$\mathcal{M} \ni \tau = \gamma\cdot\tau_0 \qquad \gamma\in G$$

$$K\cdot\tau_0 = \tau_0$$

in our previous example:

$$K = -h\log\det(-i\tau + i\tau^+) + \sum_I [\det(-i\tau + i\tau^+)]^{k_I} \left|\varphi^{(I)}\right|^2$$

$h > 0$

$$w(\tau,\varphi) = \sum_n Y_{I_1 \dots I_n}(\tau)\varphi^{(I_1)} \dots \varphi^{(I_n)}$$

field-dependent Yukawa couplings

invariance of $w(\Phi)$ guaranteed by an holomorphic $Y_{I_1 \dots I_n}(\tau)$ such that

$$Y_{I_1 \dots I_n}(\gamma\tau) = j(\gamma,\tau)^{k_Y(n)}\rho(\gamma)\, Y_{I_1 \dots I_n}(\tau) \qquad \gamma \in \Gamma$$

1. $k_Y(n) + k_{I_1} + \cdots + k_{I_n} = 0$
2. $\rho \times \rho^{I_1} \times \dots \times \rho^{I_n} \supset 1$

modular forms of Γ/G_d and weight k_Y

form a linear space $\mathcal{M}_k(\Gamma/G_d))$ of finite dimension

special case of automorphic forms for G, K, Γ

- $\Psi(\gamma g) = \Psi(g)$
- $\Psi(gk) = j(k,\tau_0)^{-1}\Psi(g)$
- $\Psi(g)$ eigenfunction of G-Casimir operators
- suitable growth conditions

$$Y(\tau) = j(g,\tau_0)\Psi(g)$$

CP invariance in $Sp(2g,\mathbb{Z})$ [Ding, F., Liu 2102.06716]

CP belongs to Out(Γ)

$$\mathcal{CP}\,\gamma\,\mathcal{CP}^{-1} = u(\gamma) \qquad \gamma \in \Gamma = Sp(2g,\mathbb{Z})$$

$$\tau \to \tau_{CP} = -\tau^*$$ up to In(Γ) [moduli]

$$\begin{cases} \varphi^{(I)} \to \det(C\tau + D)^{\,k_I}\,\rho^I(\gamma)\,\varphi^{(I)} \\ \varphi^{(I)} \xrightarrow{CP} X_I\,\bar{\varphi}^{(I)}(x_P) \end{cases}$$

[matter fields]

$$\begin{cases} Y(\gamma\tau) = \det(C\tau + D)^{\,k_Y}\rho_Y(\gamma)\,Y(\tau) \\ Y^a(\tau) \xrightarrow{CP} Y^a(-\tau^*) = \lambda_b^a\,X_Y\,Y^{b*}(\tau) \end{cases}$$

[modular forms]

$$X_I\,\rho^{I*}(\gamma)\,X_I^{-1} = \chi(\gamma)^{g\,k_I}\,\rho_Y(u(\gamma))$$

[Nilles, Ramos-Sanchez and Vaudrevange 2001.01736]
Baur, Nilles, Trautner and Vaudrevange, 1901.03251,
H. Ohki, S. Uemura and R. Watanabe, 2003.04174]

CP violation as a property of the vacuum

[Novichkov, Penedo, Petcov and Titov 1905.11970]

[Baur, Kade, Nilles, Ramos-Sanchez, Vaudrevange 2012.09586]

a model of lepton masses at genus $g = 2$

$$\mathcal{M} = \left\{\tau = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix} \middle| \ \det(Im(\tau)) > 0, tr\big(Im(\tau)\big) > 0 \right\}$$

$\mathcal{M}$ restricted to $\tau_1 = \tau_2$

$\Gamma_{2,2} \to S_4 \times Z_2$ CP

	E_D^c	E_3^c	L	H_u	H_d
k_φ	-3	-1	-1	0	0
$S_4 \times Z_2$	2	1	$3'$	1	1

τ optimized at

$$\tau = \begin{pmatrix} \tau_1 & \tau_1/2 \\ \tau_1/2 & \tau_1 \end{pmatrix}$$

5 real parameters + 1 complex τ 12 observables → 5 predictions

$$\sin^2\theta_{12} = 0.3036\,, \quad \sin^2\theta_{13} = 0.02215\,, \quad \sin^2\theta_{23} = 0.5291\,, \quad \delta_{CP} = 1.41\pi\,,$$
$$\alpha_{21} = 0.17\pi\,, \quad \alpha_{31} = 1.13\pi\,, \quad m_e/m_\mu = 0.00480, \quad m_\mu/m_\tau = 0.05801\,,$$
$$m_1 = 10.08 \text{ meV}\,, \quad m_2 = 13.26 \text{ meV}\,, \quad m_3 = 51.26 \text{ meV}\,,$$
$$m_\beta = 13.40 \text{ meV}\,, \quad m_{\beta\beta} = 11.26 \text{ meV}\,.$$

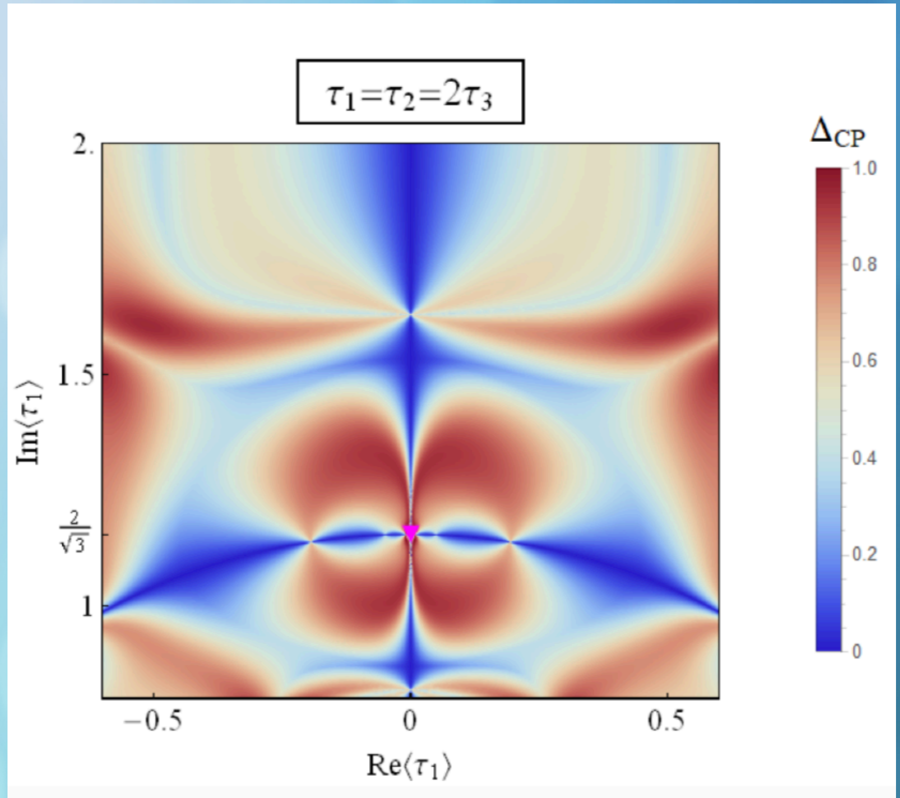
$\tau_1 = \tau_2 = 2\tau_3 =$
$-0.0283 + i\,1.1761$

best fit point of τ
close to

$$\hat{\tau} = \frac{i}{\sqrt{3}}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$2/\sqrt{3} = 1.1547$

enhanced symmetry point:
CP & $S_3 \times Z_2$



CP violation
masses/mixing pattern

$\leftrightarrow$ small departure from $\hat{\tau}$

Orbifolds from Sp(4,Z) and their modular symmetries

Nilles, Ramos-Sanchez, Trautner and Vaudrevange, 2105.08078

Conclusions

- flavour puzzle as a vacuum problem
- Flavour symmetry and representations from Moduli Space
- Flavour symmetry is a (discrete) gauge symmetry ⟺ redundancy of description in moduli space
- HSS=G/H as symmetry breaking space $\mathcal{M}$
- Γ flavour (gauge) symmetry is a discrete group of G
- Yukawa couplings $\mathcal{Y}(\tau)$ are automorphic forms
- Open questions:
  - model building
  - vacuum selection
  - nonminimal Kahler potential

[Chen, Ramon-Sanchez, Ratz 1909.06910]

THANK YOU!

back-up slides

$\mathcal{N}$=1 sugra invariant theories

This setup can be easily extended to the case of $N = 1$ local supersymmetry where Kahler potential and superpotential are not independent functions since the theory depends on the combination

$$\mathcal{G}(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) + \log w(\Phi) + \log w(\bar{\Phi}) \quad . \tag{20}$$

The modular invariance of the theory can be realized in two ways. Either $K(\Phi, \bar{\Phi})$ and $w(\Phi)$ are separately modular invariant or the transformation of $K(\Phi, \bar{\Phi})$ under the modular group is compensated by that of $w(\Phi)$. An example of this second possibility is given by the Kahler potential of eq. (14), with the superpotential $w(\Phi)$ transforming as

$$w(\Phi) \to e^{i\alpha(\gamma)}(c\tau + d)^{-h} w(\Phi) \tag{21}$$

In the expansion (17) the Yukawa couplings $Y_{I_1 \ldots I_n}(\tau)$ should now transform as

$$Y_{I_1 \ldots I_n}(\gamma\tau) = e^{i\alpha(\gamma)}(c\tau + d)^{k_Y(n)} \rho(\gamma) \; Y_{I_1 \ldots I_n}(\tau) \quad , \tag{22}$$

with $k_Y(n) = k_{I_1} + \ldots. + k_{I_n} - h$ and the representation ρ subject to the requirement 2. When we have $k_{I_1} + \ldots. + k_{I_n} = h$, we get $k_Y(n) = 0$ and the functions $Y_{I_1 \ldots I_n}(\tau)$ are τ-independent constants. This occurs for supermultiplets belonging to the untwisted sector in the orbifold compactification of the heterotic string.

choose the space $\mathcal{M}$ and guess the flavor symmetry G_f

interesting candidates for $\mathcal{M}$: Hermitian Symmetric Spaces [HSS]

- [Ding, F., Liu, 2010.07952]

- HSS are Kähler
- non-compact HSS as moduli space in sugra and compactified strings
- related to automorphic forms, building blocks of Yukawa couplings

$$\mathcal{M} = G/K$$

G: connected Lie group

K: compact subgroup

$\mathcal{M}$: Riemannian metric; integrable complex structure

Lie algebra

$$\mathcal{G} = \mathcal{V} \oplus \mathcal{A} \qquad [\mathcal{V},\mathcal{V}] \subset \mathcal{V} \qquad [\mathcal{V},\mathcal{A}] \subset \mathcal{A} \qquad [\mathcal{A},\mathcal{A}] \subset \mathcal{V}$$

automorphism $\quad \mathcal{V}+\mathcal{A} \leftrightarrow \mathcal{V}-\mathcal{A}$

compact $\quad B(\mathcal{A},\mathcal{A}) < 0$

noncompact $\quad B(\mathcal{A},\mathcal{A}) > 0$

euclidean $\quad B(\mathcal{A},\mathcal{A}) = 0$

$B(\mathcal{V},\mathcal{V}) < 0$

$B(X,Y)$ Killing form

[classification: Calabi, Visentini 1960, Wolf 1964]

a model of lepton masses at genus $g = 2$

$$\mathcal{M} = \left\{ \tau = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix} \middle| \ \det(Im(\tau)) > 0, tr\big(Im(\tau)\big) > 0 \right\}$$

$G = Sp(4,\mathbb{R})$ $K = U(2)$ $\mathcal{M}$ restricted to $\tau_1 = \tau_2$ $\Gamma_{2,2} \to S_4 \times Z_2$ CP

	E_D^c	E_3^c	L	H_u	H_d
k_φ	-3	-1	-1	0	0
$S_4 \times Z_2$	2	1	$3'$	1	1

$$w_e = \alpha(E_D^c L Y^{(4)}_{\mathbf{3}'a})_{\mathbf{1}} H_d + \beta(E_D^c L Y^{(4)}_{\mathbf{3}'b})_{\mathbf{1}} H_d + \gamma(E_3^c L Y_{\mathbf{3}'})_{\mathbf{1}} H_d$$

$$w_\nu = \frac{g_1}{\Lambda}(LLY_{\mathbf{3}'})_{\mathbf{1}} H_u H_u + \frac{g_2}{\Lambda}(LLY_{\mathbf{1}})_{\mathbf{1}} H_u H_u \, .$$

5 real parameters

τ optimized at $\tau = \begin{pmatrix} \tau_1 & \tau_1/2 \\ \tau_1/2 & \tau_1 \end{pmatrix}$

$$\tau_1 = -0.02827 + 1.17613i \, , \quad (\tau_1 = \tau_2 = 2\tau_3)$$
$$\beta/\alpha = -1.02608 \, , \quad \gamma/\alpha = 0.01695 \, , \quad g_2/g_1 = 1.42981 \, ,$$
$$\alpha v_d = 38.52395 \text{ MeV} \, , \quad g_1^2 v_u^2/\Lambda = 6.77168 \text{ meV} \, .$$

12 observables → 5 predictions

an example at genus $g = 2$

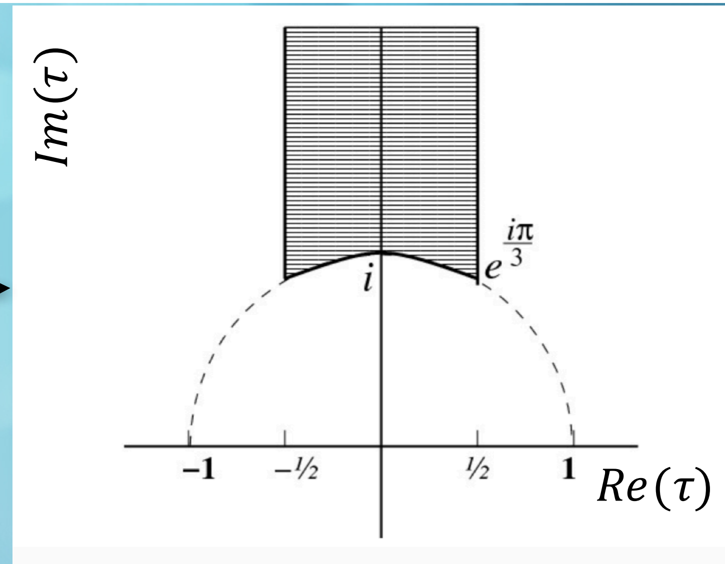
$$\mathcal{M} = \left\{ \tau = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix} \middle| \ \det(Im(\tau)) > 0, tr\big(Im(\tau)\big) > 0 \right\}$$

fundamental domain $\mathcal{M}/\Gamma$: physically inequivalent vacua

$$\mathcal{F}_2 = \left\{ \tau \in \mathcal{H}_2 \ \middle| \ \begin{cases} |\mathrm{Re}(\tau_1)| \leq 1/2, \quad |\mathrm{Re}(\tau_3)| \leq 1/2. \quad |\mathrm{Re}(\tau_2)| \leq 1/2, \\ \mathrm{Im}(\tau_2) \geq \mathrm{Im}(\tau_1) \geq 2\mathrm{Im}(\tau_3) \geq 0 \\ |\tau_1| \geq 1, \quad |\tau_2| \geq 1, \quad |\tau_1 + \tau_2 - 2\tau_3 \pm 1| \geq 1 \\ |\det(\tau + \mathcal{E}_i)| \geq 1 \end{cases} \right\}$$

$\{\mathcal{E}_i\}$ includes the following 15 matrices:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \pm1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \pm1 \end{pmatrix}, \begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix},$$
$$\begin{pmatrix} \pm1 & 0 \\ 0 & \mp1 \end{pmatrix}, \begin{pmatrix} 0 & \pm1 \\ \pm1 & 0 \end{pmatrix}, \begin{pmatrix} \pm1 & \pm1 \\ \pm1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \pm1 \\ \pm1 & \pm1 \end{pmatrix}$$



[cfr. g=1]

An automorphic form for G_d is a smooth complex function $\Psi(\mathsf{g})$ that

1. is invariant under the action of the discrete group G_d:

$$\Psi(\gamma \mathsf{g}) = \Psi(\mathsf{g}), \qquad \gamma \in G_d \quad , \tag{2.19}$$

2. is K-finite: $\Psi(\mathsf{g}\ \mathsf{k})$, with k varying in K, span a finite dimensional vector space [37]. In all cases of interest discussed in this paper, such a condition is realized through the relation:

$$\Psi(\mathsf{g}\ \mathsf{k}) = j(\mathsf{k}, \tau_0)^{-1}\ \Psi(\mathsf{g})\,, \qquad \mathsf{k} \in K\,, \qquad \mathsf{k}\tau_0 = \tau_0 \quad , \tag{2.20}$$

which defines the transformation property of $\Psi(\mathsf{g})$ under K. In all such cases the space obtained by $\Psi(\mathsf{g}\ \mathsf{k})$, varying k in K, is one-dimensional.

3. $\Psi(\mathsf{g})$ is required to be an eigenfunction of the algebra $\mathcal{D}$ of invariant differential operators on G, that is an eigenfunction of all the Casimir operators of G.

4. The definition is completed by suitable growth conditions [36, 37].

$$Y(\tau) = j(\mathsf{g}, \tau_0)\ \Psi(\mathsf{g})$$

$$Y(\gamma\tau) = j(\gamma, \tau)\ Y(\tau)$$

$$\mathcal{M} = \mathcal{M}_c \times \mathcal{M}_{nc} \times \mathcal{M}_e$$

classification of irreducible HHS [Calabi, Visentini 1960, Wolf 1964]

$$\mathcal{M}_{irr} = G/K$$

G → simply connected Lie group

K → maximal compact subgroup

for noncompact

Type	Group G	Compact subgroup K	$\dim_{\mathbb{C}} G/K$
$\mathbf{I}_{m,n}$	$U(m,n)$	$U(m) \times U(n)$	mn
$\mathbf{II}_m$	$SO^*(2m)$	$U(m)$	$\frac{1}{2}m(m-1)$
$\mathbf{III}_m$	$Sp(2m)$	$U(m)$	$\frac{1}{2}m(m+1)$
$\mathbf{IV}_m$	$SO(m,2)$	$SO(m) \times SO(2)$	m
$\mathbf{V}$	$E_{6,-14}$	$SO(10) \times SO(2)$	16
$\mathbf{VI}$	$E_{7,-25}$	$E_6 \times U(1)$	27

compact case reproduced through $(\mathcal{V}, \mathcal{A}) \rightarrow (\mathcal{V}, i\,\mathcal{A})$

the finite Siegel modular groups $\Gamma_{2,n}$ are too big: $|\Gamma_{2,2}| = 720, |\Gamma_{2,3}| = 51840, \ldots$

choose an invariant subspace of $G/K = Sp(4,\mathbb{R})/U(2)$

$$G/K \to \Omega = \{\tau \in G/K |\, H\tau = \tau\}$$

$$H \subset \Gamma$$

here the flavour group can be restrict to the normalizer $N(H)$

$$\Gamma \to N(H)=\{\gamma \in \Gamma | \gamma^{-1} H \gamma = H\}$$

for instance $\tau_1 = \tau_2$

$$\Omega=\left\{\tau = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_1 \end{pmatrix} \middle| \tau \in G/K \right\} \qquad H=Z_2 \times Z_2$$

working here $\Gamma_{2,2}$
is projected into the smaller group $S_4 \times Z_2$

Parameters	Best fit value and 1σ error
m_e/m_μ	0.0048 ± 0.0002
m_μ/m_τ	0.0565 ± 0.0045
$\Delta m_{21}^2/10^{-5}\text{eV}^2$	$7.42^{+0.21}_{-0.20}$
$\Delta m_{31}^2/10^{-3}\text{eV}^2$	$2.517^{+0.026}_{-0.028}$
δ_{CP}/π	$1.0944^{+0.1500}_{-0.1333}$
$\sin^2\theta_{12}$	$0.304^{+0.012}_{-0.012}$
$\sin^2\theta_{13}$	$0.02219^{+0.00062}_{-0.00063}$
$\sin^2\theta_{23}$	$0.573^{+0.016}_{-0.020}$

Table 2: The best fit values and the 1σ ranges of the charged lepton mass ratios and the lepton mixing parameters. The charged lepton mass ratios averaged over $\tan\beta$ [5] are taken from ref. [82], and we adopt the values of the lepton mixing parameters from NuFIT v5.0 with Super-Kamiokanda atmospheric data for normal ordering [83].