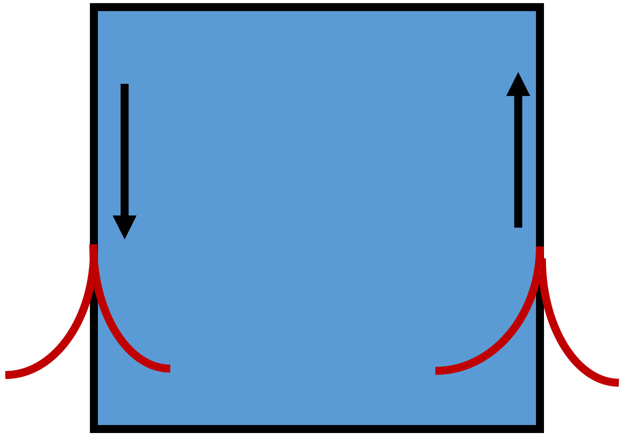
Generalized Hall current from index theorem for gapless edge fermions

Srimoyee Sen,
Iowa State University

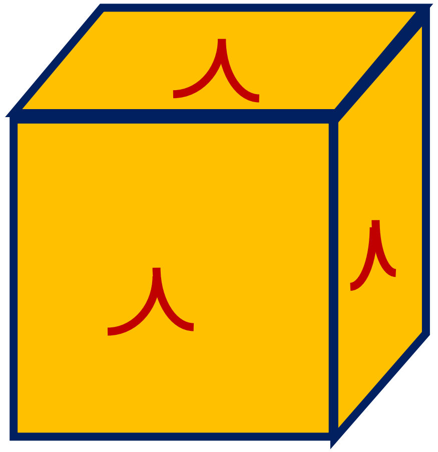
Paper in preparation with
David Kaplan,
University of Washington

Discrete 2021, Nov 29, 2021

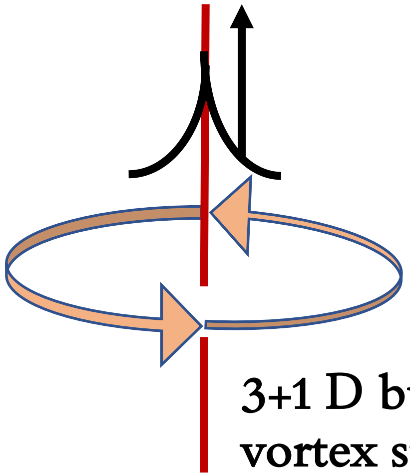
Fermion edge states



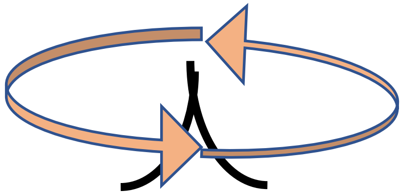
2+1 D bulk, 1+1 edge



3+1 D bulk, 2+1 surface states

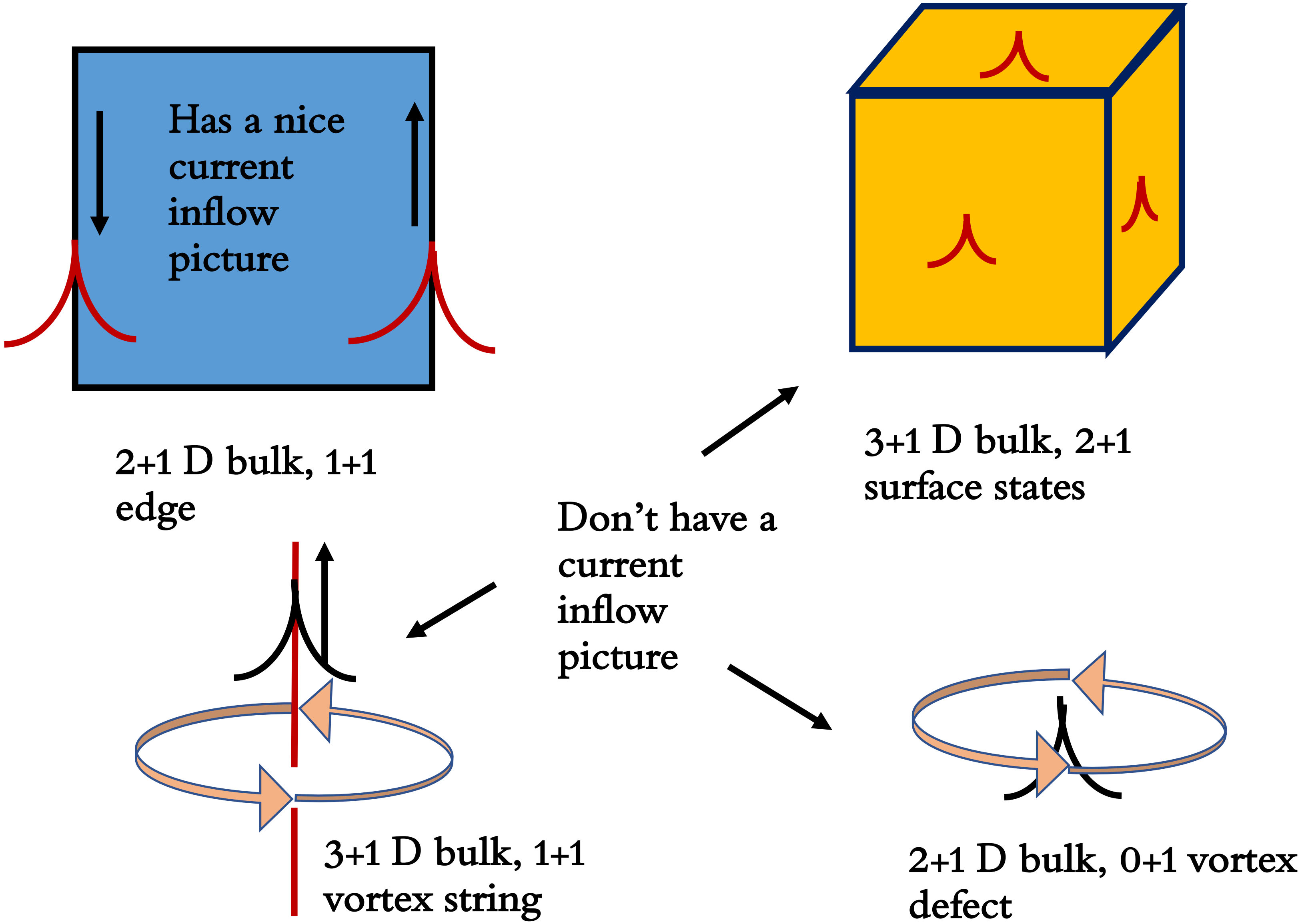


3+1 D bulk, 1+1 vortex string

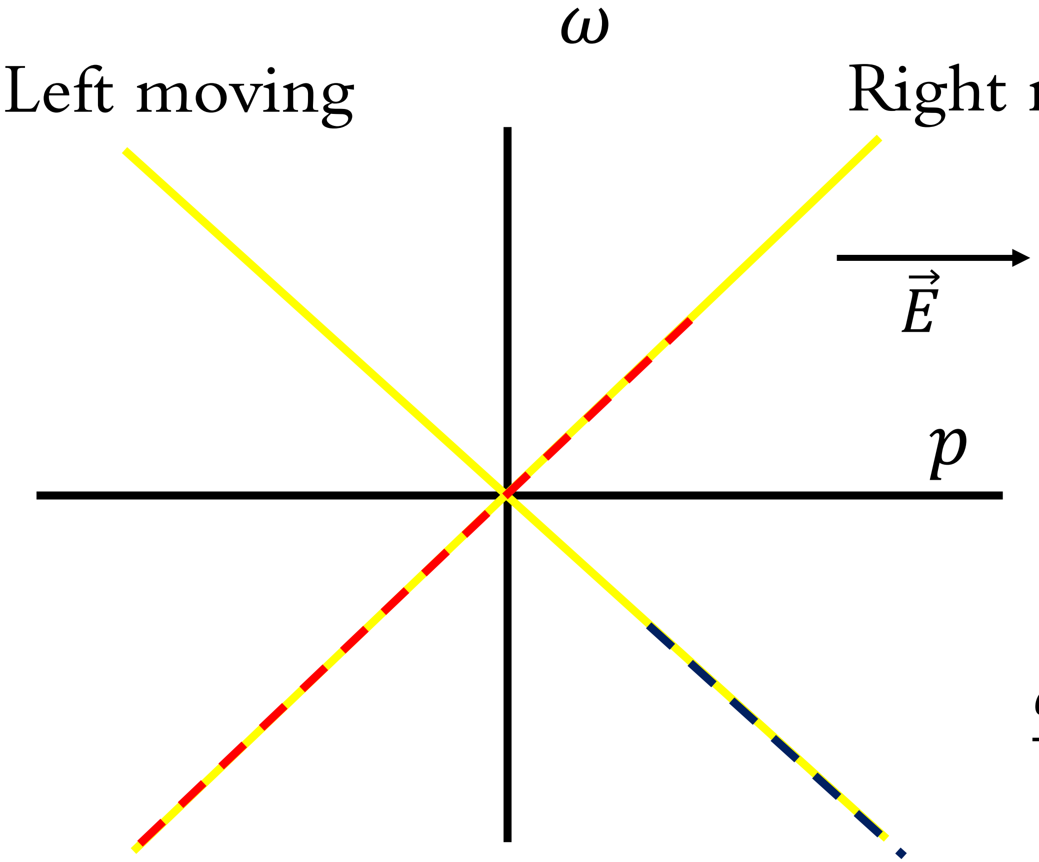


2+1 D bulk, 0+1 vortex defect

Fermion edge states



Current inflow picture: chiral anomaly



$$\frac{dn_R}{dt} = \frac{eE}{2\pi}$$

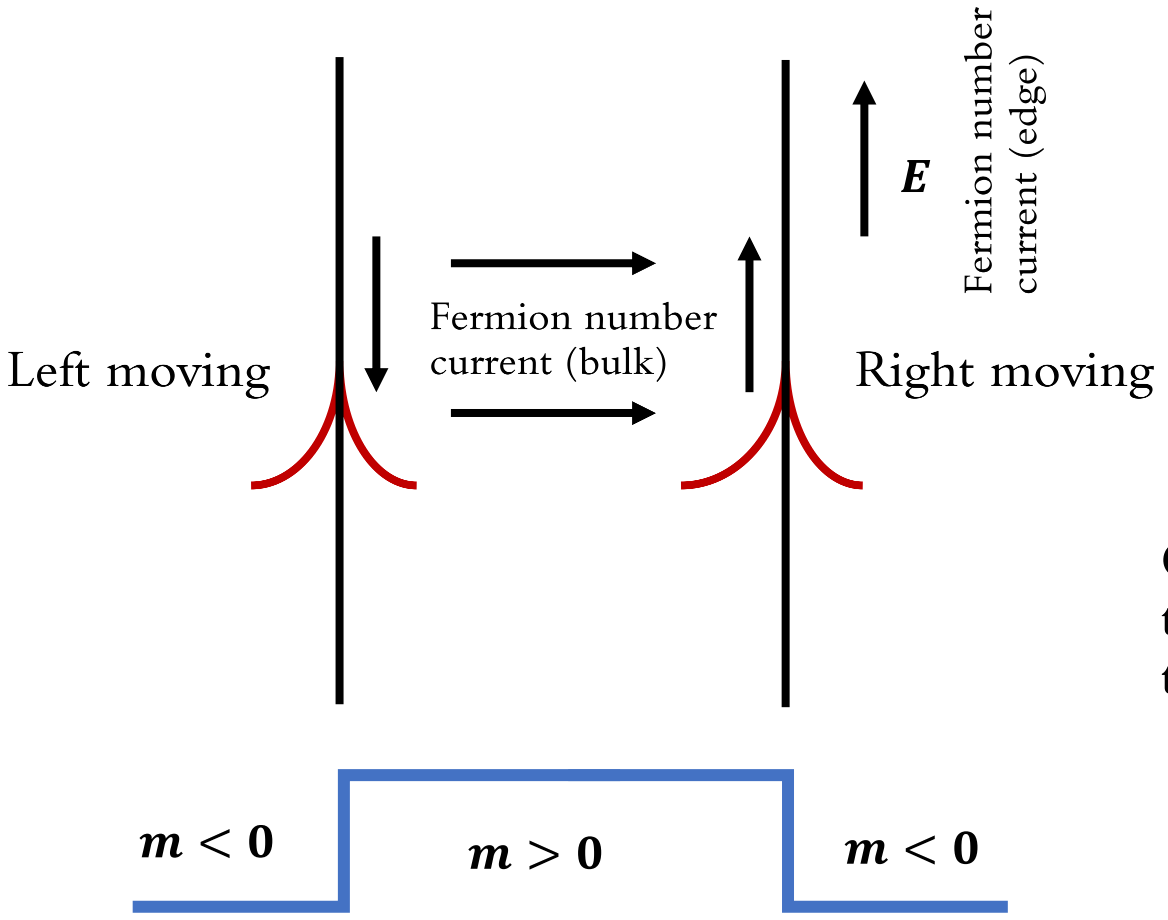
$$\frac{dn_L}{dt} = -\frac{eE}{2\pi}$$

$$\frac{d(n_R - n_L)}{dt} = \frac{eE}{\pi}$$

Also current $\propto E$

Place the anomalous theory in one higher dimensions: quantum Hall effect or domain wall fermions

Current inflow picture



$$L_{2+1} = \bar{\psi}(i\gamma\partial - m)\psi$$

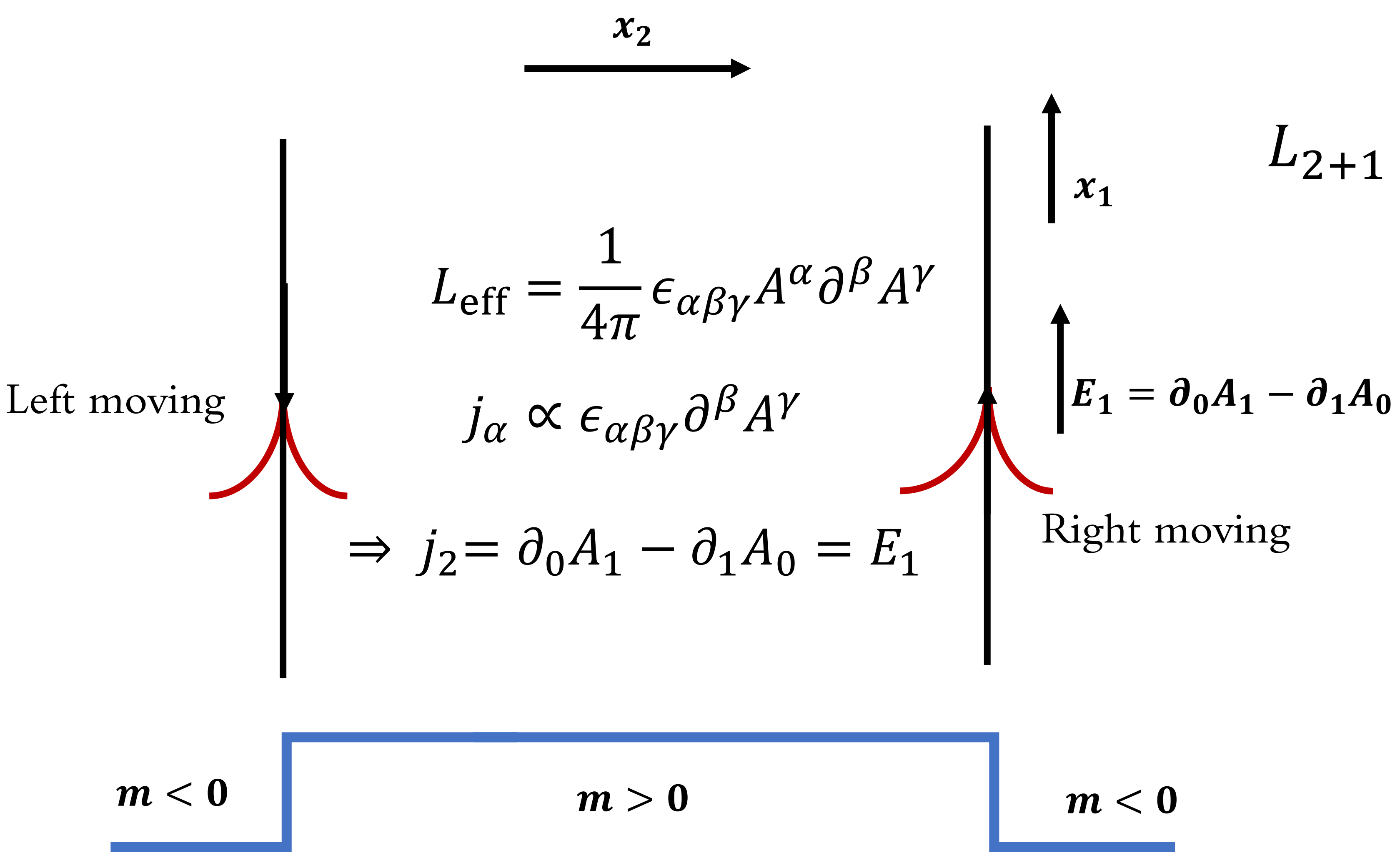
Step function in Dirac mass m in x_3 (domain walls)

Chern-Simons theory in the bulk: current flow to the wall

Chiral fermions on the edge: current along the wall

Current conservation

A bit more detail about inflowing current

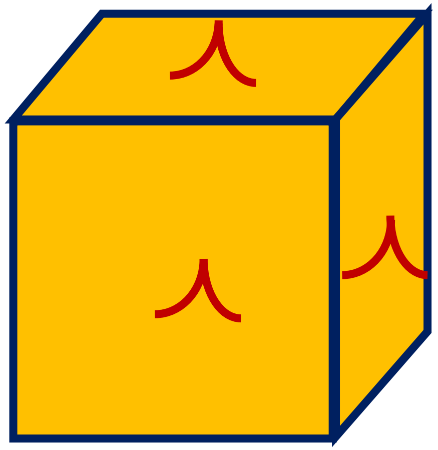


$\partial + i\,A$

$$L_{2+1} = \bar{\psi}(i\gamma D - m)\psi$$

Step function in Dirac mass m in x_3 (domain walls)

No current inflow picture



3+1 D bulk, 2+1 surface states

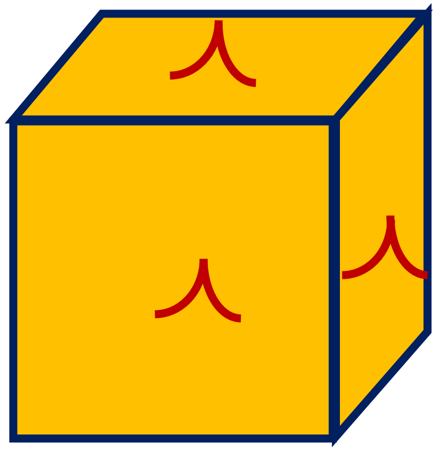
Odd dimensional defect fermions: No chiral anomaly, no current inflow.

Theories without continuous symmetries: No current to flow anywhere.

$$L_{2+1} = \bar{\psi}(\gamma\partial - m)\psi$$

Add a $\psi\psi$ term here: fermion number is broken down to Z_2 … so no current inflow. But the chiral edge states survive.

No current inflow picture



3+1 D bulk, 2+1 surface states

Odd dimensional defect fermions: No chiral anomaly, no current inflow.

Theories without continuous symmetries: No current to flow anywhere.

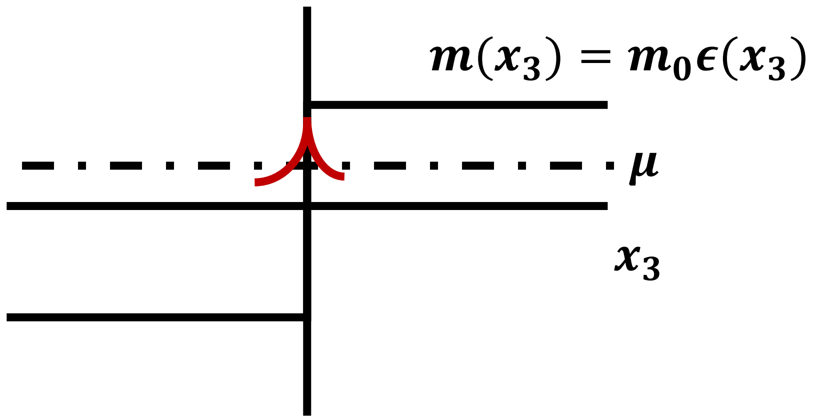
$$L_{2+1} = \bar{\psi}(\gamma\partial - m)\psi$$

Add a $\psi\psi$ term here: fermion number is broken down to Z_2 … so no current inflow. But the chiral edge states survive.

Goal

- We want a unifying current inflow picture for all edge fermions.
- Present irrespective of what symmetries the original theory has and whether there is a chiral anomaly on the defect.
- Example: Majorana edge states on the boundary of a system with only discrete symmetries.

DWF with a Majorana mass (2+1 D)

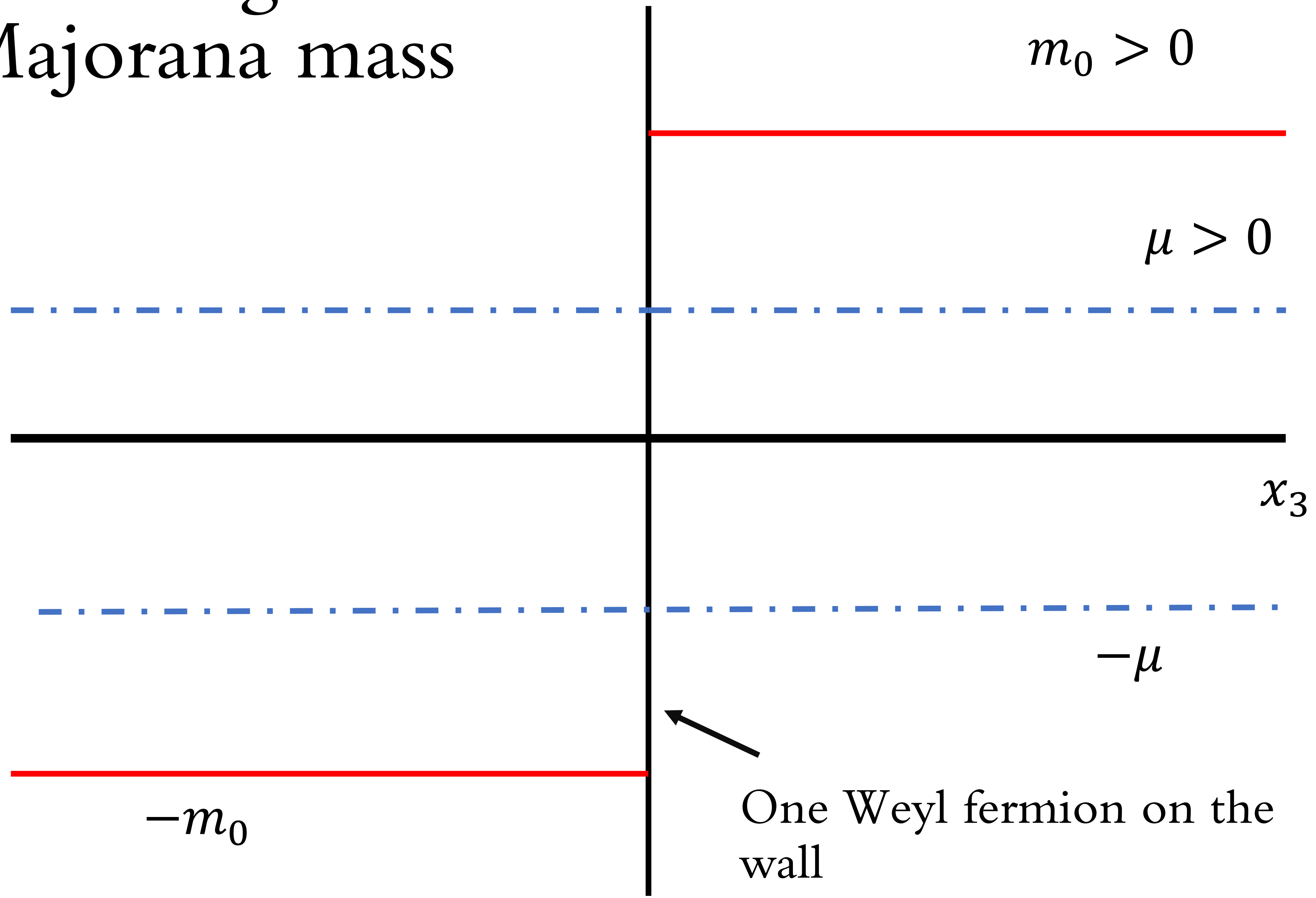


$$L_{2+1} = \bar{\psi}(i\,\gamma\partial - m)\psi - i\frac{\mu}{2}\psi^T\sigma_2\psi - i\frac{\mu^*}{2}\,\bar{\psi}\,\sigma_2\,\bar{\psi}^T$$

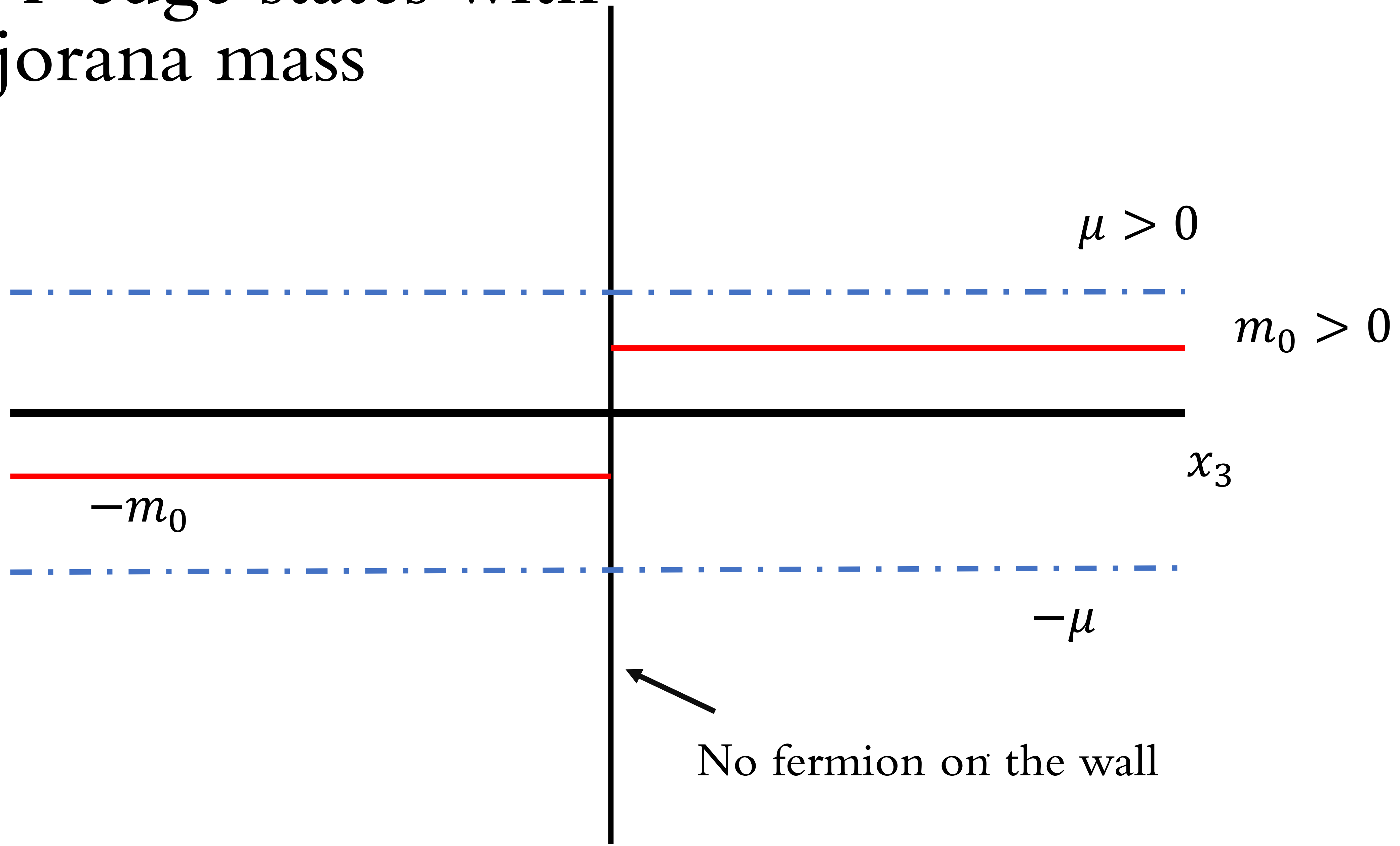
For a small enough majorana mass nothing changes for the chiral edge state. It's stable. Yet, no current inflow picture for any nonzero μ.

$$\psi(x) = \begin{pmatrix} i\,c_{1,>}\,e^{-(m_0-\mu)x_3} + c_{2,>}\,e^{-(m_0+\mu)x_3} \\ 0 \end{pmatrix}\theta(x_3) + \begin{pmatrix} i\,c_{2,>}\,e^{(m_0-\mu)x_3} + i\,c_{1,>}\,e^{(m_0+\mu)x_3} \\ 0 \end{pmatrix}\theta(-x_3)$$

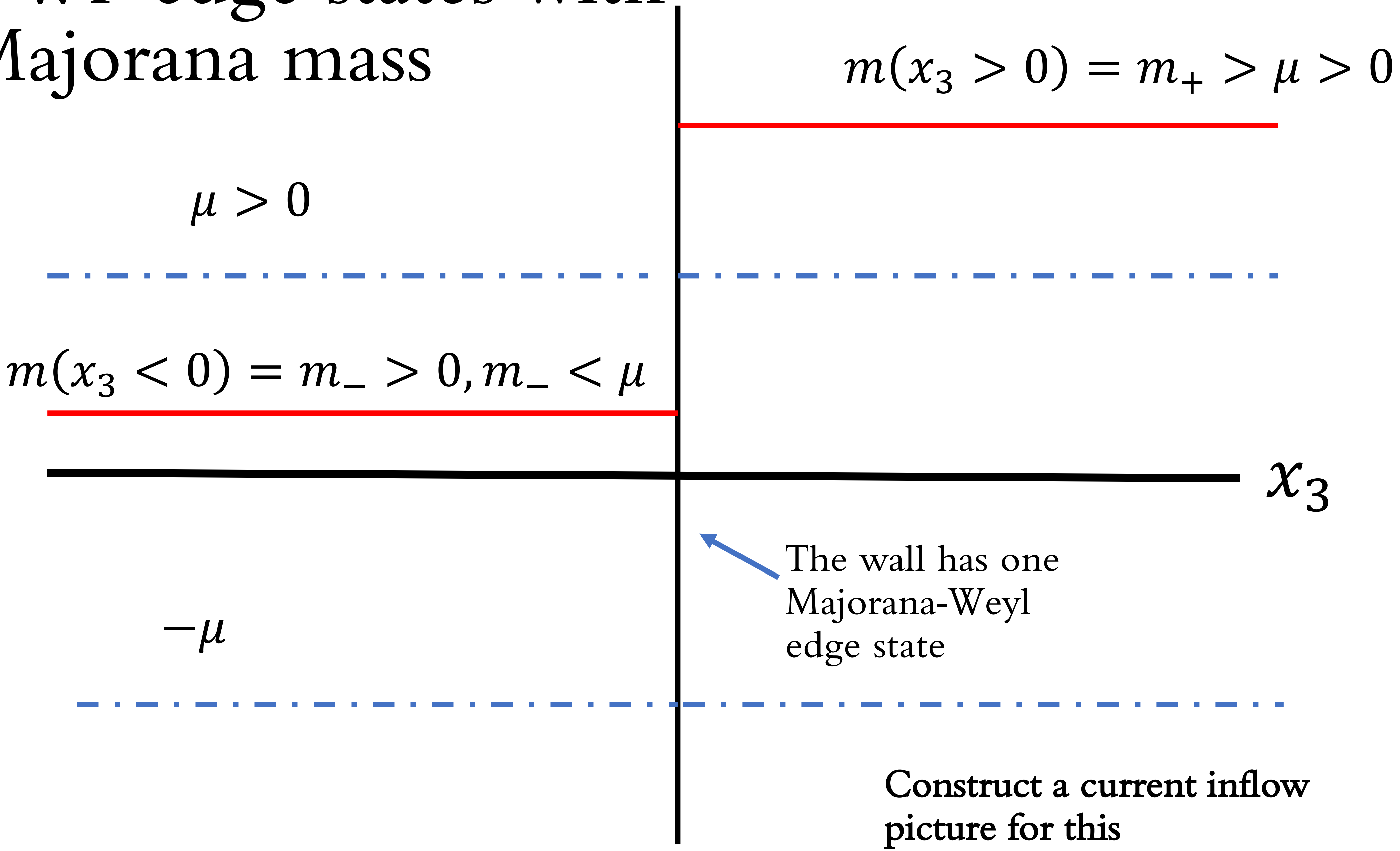
DWF edge states with Majorana mass



DWF edge states with Majorana mass

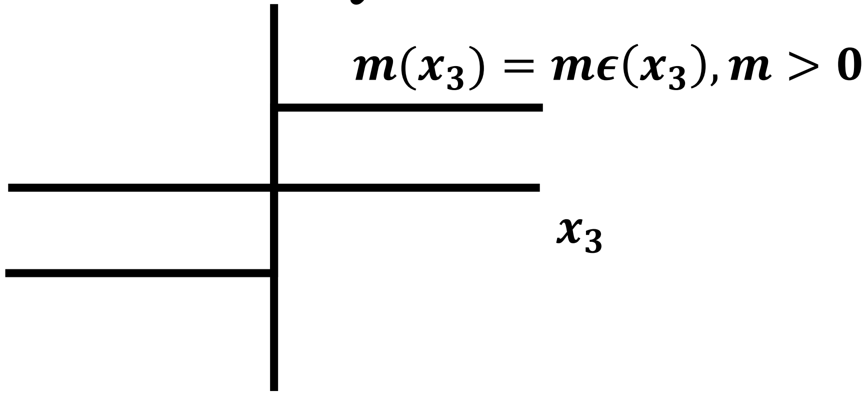


DWF edge states with Majorana mass



Construct a current inflow picture for this

Look at 1+1 domain walls in 2+1 more carefully



Massless edge states with

$$L = \bar{\psi}\big(i\gamma_\mu\partial_\mu + m(x_3)\big)\psi$$

Solve for massless edge states with:

$$\big(\mathrm{i}\,\gamma_\mu\partial_\mu + m(x_3)\big)\psi = 0$$

$$\Rightarrow \big(\underbrace{i\,\partial_1 - i\,\sigma_3\partial_2}_{} + \underbrace{i\sigma_2\partial_3 + m(x_3)}_{}\big)\psi(x_1, x_2, x_3) = 0$$

$$\psi = \phi(x_1, x_2)\chi(x_3)$$

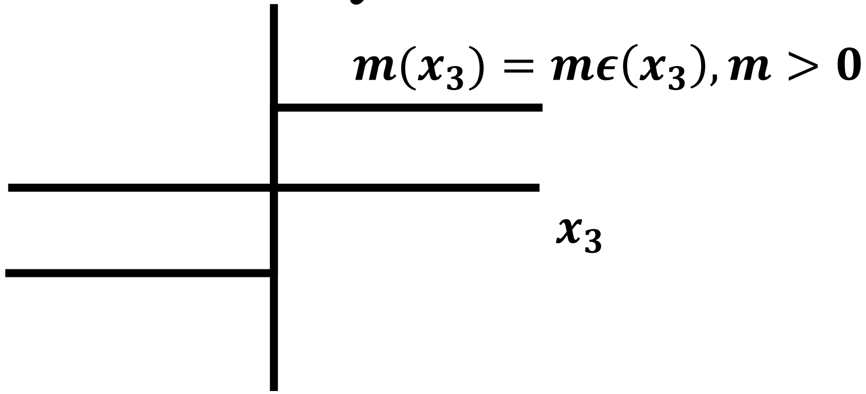
$$= 0 \qquad = 0$$

$$\begin{pmatrix} 0 & \overbrace{\partial_3 + m\epsilon(x_3)}^{D} \\ \underbrace{-\partial_3 + m\epsilon(x_3)}_{= D^\dagger} & 0 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0$$

Solutions

$$\begin{cases} \phi(x_1, x_2) = e^{ik_1x_1 + ik_2x_2}, \\ \chi_2(x_3) = e^{-m|x_3|} \\ \chi_1 = 0 \end{cases}$$

Look at 1+1 domain walls in 2+1 more carefully



Massless edge states with

$D\chi_2 = 0 \Rightarrow D$ has a one normalizable solution with zero eigenvalue. $D^\dagger$ has none.

Number of chiral fermions on wall ⇔ # of zero eigenvalues of D −# of zero eigenvalues of $D^\dagger$

$$L = \bar{\psi}\big(i\gamma_\mu\partial_\mu + m(x_3)\big)\psi$$

Solve for massless edge states with:

$$\big(\mathrm{i}\,\gamma_\mu\partial_\mu + m(x_3)\big)\psi = 0$$

$$\Rightarrow \big(i\,\partial_1 - i\,\sigma_3\partial_2 + i\sigma_2\partial_3 + m(x_3)\big)\psi(x_1, x_2, x_3) = 0$$

$$\psi = \phi(x_1, x_2)\chi(x_3)$$

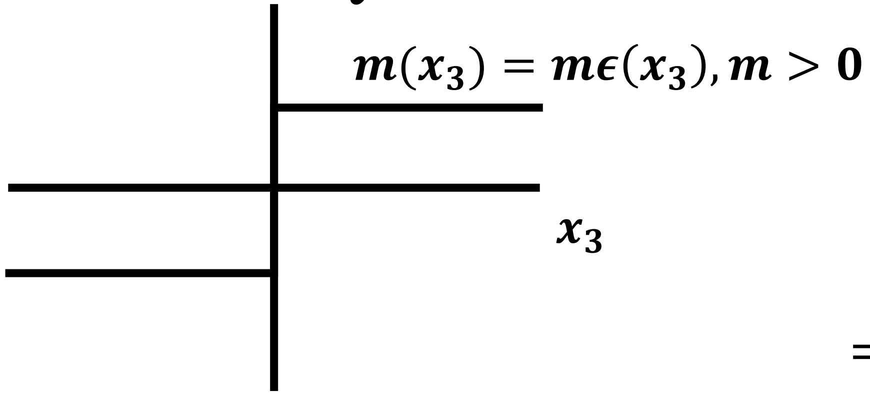
$$= 0 \qquad = 0$$

$$\begin{pmatrix} 0 & \partial_3 + m\epsilon(x_3) \\ -\partial_3 + m\epsilon(x_3) & 0 \end{pmatrix}\begin{pmatrix}\chi_1 \\ \chi_2\end{pmatrix} = 0$$

$\partial_3 + m\epsilon(x_3) \rightarrow D$

$-\partial_3 + m\epsilon(x_3) \rightarrow\; = D^\dagger$

Look at 1+1 domain walls in 2+1 more carefully



Massless edge states with

$D\chi_2 = 0 \Rightarrow D$ has a one normalizable solution with zero eigenvalue. $D^\dagger$has none.

Number of chiral fermions on wall ⟺ # of zero eigenvalues of D – # of zero eigenvalues of $D^\dagger$

Index of D

$$L = \bar{\psi}\left(i\gamma_\mu \partial_\mu + m(x_3)\right)\psi$$

Solve for massless edge states with:

$$\left(\mathrm{i}\, \gamma_\mu \partial_\mu + m(x_3)\right)\psi = 0$$

$$\Rightarrow \left(i\,\partial_1 - i\,\sigma_3 \partial_2 + i\sigma_2 \partial_3 + \sigma_1 m(x_3)\right)\psi(x_1, x_2, x_3) = 0$$

$$\psi = \phi(x_1, x_2)\chi(x_3)$$

$= 0 \quad = 0$

$$\begin{pmatrix} 0 & \partial_3 + m\epsilon(x_3) \\ -\partial_3 + m\epsilon(x_3) & 0 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0$$

$\partial_3 + m\epsilon(x_3)$ → D

$-\partial_3 + m\epsilon(x_3)$ → $= D^\dagger$

Index

Index of D, $\mathrm{Lim}_{M\to 0}\, I(M) = \mathrm{Lim}_{M\to 0}\,\mathrm{Tr}\left[\frac{M^2}{D^\dagger D+M^2} - \frac{M^2}{DD^\dagger+M^2}\right]$

$$= \mathrm{Lim}_{M\to 0}\,\mathrm{Tr}\left[\Gamma_\chi \frac{M}{M+K}\right]$$

$$K = \begin{pmatrix} 0 & -D^\dagger \\ D & 0 \end{pmatrix} \qquad \Gamma_\chi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

What if we use a Euclidean Dirac operator for D ?

Use that D, the Minkowski edge states of which we are interested in.

Index to doubled theory

Index of D, $\mathrm{Lim}_{M\to 0}\, I(M) = \mathrm{Lim}_{M\to 0}\mathrm{Tr}\left[\Gamma_\chi \frac{1}{M+K} M\right]$

Looks like a fermion propagator

$$S = \int d^{d+1}x\, \bar{\Psi}(K+M)\Psi \longleftarrow$$

Spinors double the size of the original spinors: has its own fermion number symmetry and chiral symmetry in the $M \to 0$ limit.

In this new theory $I(M) = M \int d^{d+1}x \langle \bar{\Psi}(x)\Gamma_\chi \Psi(x)\rangle$

Index of D

Just because the Minkowski theory has massless edge states, does not imply the Euclidean Dirac operator will have nonzero index.

No reason to expect a localized zero mode.

We will turn on diagnostic background fields needed to localize the state to turn it into a zero mode.

Thus, every time there is a massless edge state in the Minkowski theory, there will be a corresponding index in the Euclidean theory.

Doubled theory

$$S = \int d^{d+1}x\, \bar{\Psi}(K + M)\Psi$$

This doubled theory amounts to introducing an extra dimension… But the fields don't depend on it.

Gamma matrices of this theory:

$$\frac{\delta(K\Psi)}{\delta\, \partial_\mu \Psi} = \Gamma_\mu \Psi$$

Define an axial current: $J_\mu = \bar{\Psi}\Gamma_\mu \Gamma_\chi \Psi$

Satisfies a Ward-Takahashi identity $\partial_\mu J_\mu^\chi = 2M\, \bar{\Psi}\Gamma_\chi \Psi + \mathcal{A}$ → Variance of the measure of the path integral (ends up being zero)

Doubled theory

$$S = \int d^{d+1}x\, \bar{\Psi}(K + M)\Psi$$

This doubled theory amounts to introducing an extra dimension… But the fields don't depend on it.

Gamma matrices of this theory:

$$\frac{\delta(K\Psi)}{\delta\, \partial_\mu \Psi} = \Gamma_\mu \Psi$$

Define an axial current: $J_\mu = \bar{\Psi}\Gamma_\mu \Gamma_\chi \Psi$

What we want! The divergence of a current.

Satisfies a Ward-Takahashi identity

$$\partial_\mu J^\chi_\mu = 2M\, \bar{\Psi}\Gamma_\chi \Psi + \mathcal{A}$$

Index.

Variance of the measure of the path integral (ends up being zero)

Current in the DWF theory with a Majorana mass

Compute the current with background gauge field

$$J_\mu^\chi = -\frac{\eta}{2\pi}\epsilon_{\mu\nu\rho}F_{\nu\rho} \qquad \eta = \begin{cases} 1, & |\mu| < m \\ 0, & -|\mu| < m < |\mu| \\ -1, & m < -|\mu| \end{cases}$$

$$\mathcal{I}(0) = \nu \times \begin{cases} 2 & m_\pm > |\mu| \\ 1 & m_- < |\mu| \text{ and } m_+ > |\mu| \\ 1 & m_- > |\mu| \text{ and } m_+ < |\mu| \\ 0 & m_\pm < |\mu| \end{cases}$$

Counting Majorana-Weyl edge states

Two Majorana-Weyl edge state: One Weyl fermion

Conclusion

- We have been able to construct a current inflow picture for edge states in the absence of a continuous symmetry.
- There are other examples where the same current inflow construction works. E.g. 0+1 dimensional domain wall in 1+1 dimensional fermionic theory.
- This generalized Hall current unifies the physics of quantum Hall edge states with other types of fermionic edge states with no inflowing Hall current picture.

Questions to explore in the future

- Can the generalized Hall current be observable in experiments or is this only a theoretical tool?

- Can we interpret the extra dimension as extra flavors of fermions?

- How general is our construction? Can all types of fermion edge states be related to a generalized Hall current?