Anatomy of a top-down approach to discrete and modular flavor symmetry

Andreas Trautner

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based on:

PLB 786 (2018) 283-287	1808.07060	w/ H.P. Nilles, M. Ratz, P. Vaudrevange
PLB 795 (2019) 7-14	1901.03251	w/ A. Baur, H.P. Nilles, P. Vaudrevange
NPB 947 (2019) 114737	1908.00805	w/ A. Baur, H.P. Nilles, P. Vaudrevange
NPB 971 (2021) 115534	2105.08078	w/ H.P. Nilles, S. Ramos-Sánchez, P. Vaudrevange
to appear	2112.xxxxx	w/ A.Baur, H.P. Nilles, S. Ramos-Sánchez, P. Vaudrevange

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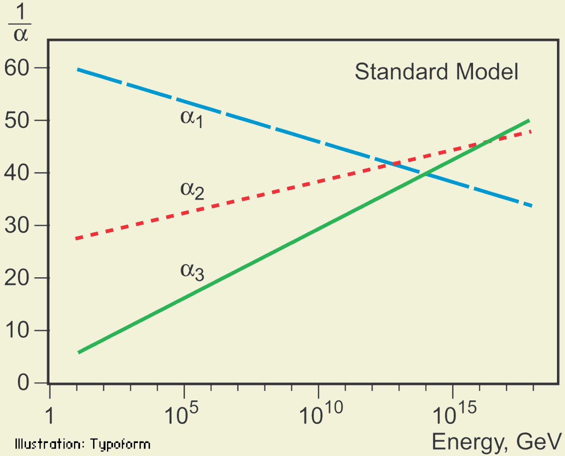
MAX-PLANCK-GESELLSCHAFT

MAX-PLANCK-INSTITUT FÜR KERNPHYSIK HEIDELBERG

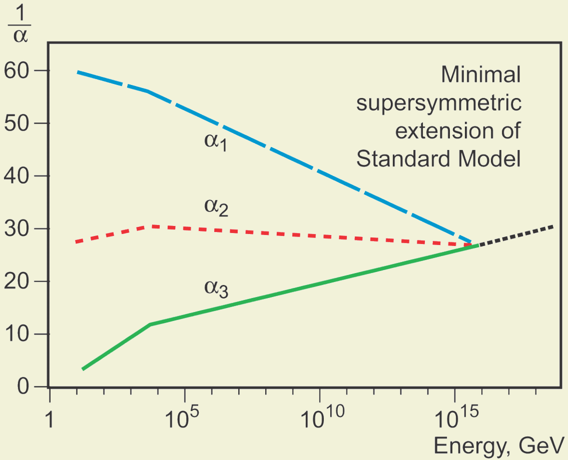
Outline: Flavor symmetry

- Bottom-up and top-down approaches to flavor symmetry
- Eclectic flavor symmetry…
- …from heterotic string theory
- Breaking of the Eclectic flavor symmetry
- Summary

Unification – bottom-up vs. top-down



Unification – bottom-up vs. top-down



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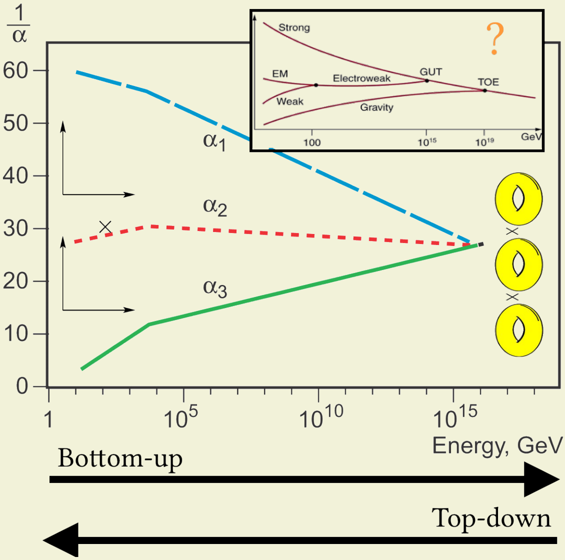
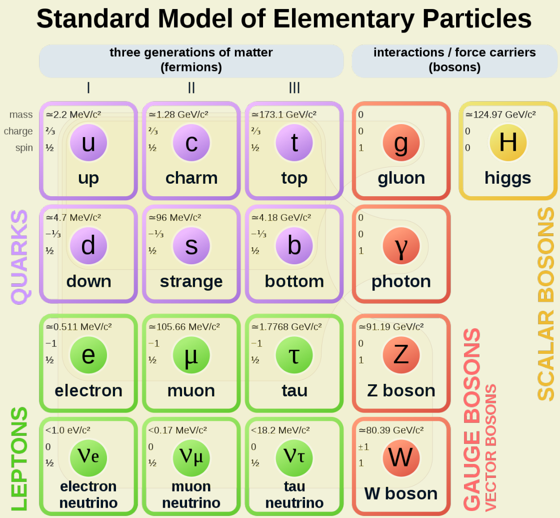


See e.g. "Supersymmetric standard model from the heterotic string"
[Buchmüller, Hamaguchi, Lebedev, Ratz '05]

Is everything unified?



No "theory of everything" without Flavor!

Types of (discrete) flavor symmetries

Schematically for the example of $\mathcal{N} = 1$ SUSY.

x: spacetime, θ: superspace, Φ : (Super-)fields, T: modulus.
$K(T,\Phi)$: Kähler potential, $W(T,\Phi)$: Superpotential

$$\mathcal{S} = \int d^4x\, d^2\theta\, d^2\overline{\theta} K(T,\overline{T},\Phi,\overline{\Phi}) + \int d^4x\, d^2\theta\, W(T,\Phi) + \int d^4x\, d^2\overline{\theta}\, \overline{W}(\overline{T},\overline{\Phi})\ .$$

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- **“traditional” Flavor symmetries** $\Phi\mapsto\rho(\mathrm{g})\Phi\,,\quad \mathrm{g}\in G$

  for a review, see e.g. [King & Luhn ’13]

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- "traditional" Flavor symmetries $G_{\mathrm{traditional}}$
- **modular Flavor symmetries** [Feruglio '17]

$$\Phi \xrightarrow{\gamma} (c\,T+d)^n \rho(\gamma)\Phi \;, \quad T \xrightarrow{\gamma} \frac{a\,T+b}{c\,T+d} \;, \quad \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;.$$

Couplings are modular forms: $Y = Y(T)$, $Y(\gamma T) = (c\,T+d)^{k_Y}\,\rho_Y(\gamma)\,Y(T)$.

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- modular Flavor symmetries G_{modular}
- **R symmetries** for non-Abelian discrete R flavor symmetries see [Chen, Ratz, AT '13]

$$\Phi(x,\theta)=\boldsymbol{\phi}(x)+\sqrt{2}\boldsymbol{\theta}\,\boldsymbol{\psi}(x)+\theta\theta F(x)\;,\;\Longrightarrow\;\phi\mapsto e^{iq_\Phi\alpha}\phi,\;\psi\mapsto e^{i(q_\Phi-q_\theta)\alpha}\psi\,.$$

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- **general CP(-like) symmetries** [Novichkov, Penedo et al. '19],[Baur et al. '19]

$$\Phi \xmapsto{\bar{\gamma}} (c\overline{T}+d)^n\rho(\bar{\gamma})\overline{\Phi}\ ,\quad T \xmapsto{\bar{\gamma}} \frac{a\overline{T}+b}{c\overline{T}+d}\ ,\quad \det\left[\,\bar{\gamma}\in \mathrm{GL}(2,\mathbb{Z})\,\right]=-1\ .$$

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- general $\mathcal{CP}$(-like) symmetries $\mathcal{CP}$

From the bottom-up: All kinds known, individually!

$\rightarrow$ See talks by Penedo, Feruglio, de Medeiros Varzielas.

for an up-to-date review see [Feruglio&Romanino '19]

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- general $\mathcal{CP}$(-like) symmetries $\mathcal{CP}$

From the top-down: *all, at the same time*!

$$G_{\textbf{eclectic}} = G_{\mathrm{traditional}} \cup G_{\mathrm{modular}} \cup G_{\mathrm{R}} \cup \mathcal{CP}\ ,$$

see works by [Baur, Nilles, AT, Vaudrevange ’19; Nilles, Ramos-Sánchez, Vaudrevange ’20]

→ See also talk by Ramos-Sánchez.

Origin of eclectic flavor symmetry in heterotic orbifolds

Narain lattice formulation of heterotic string theory: [Narain '86]

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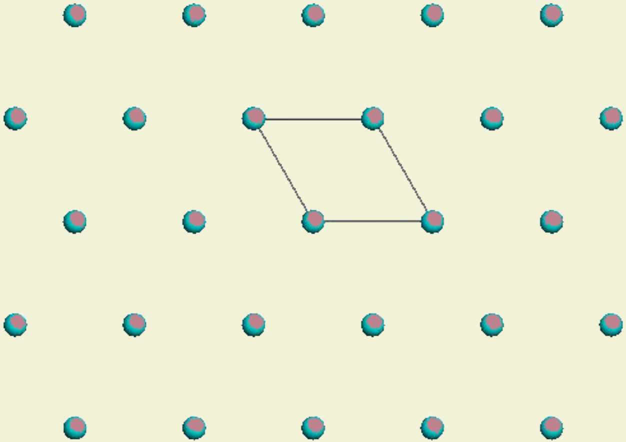

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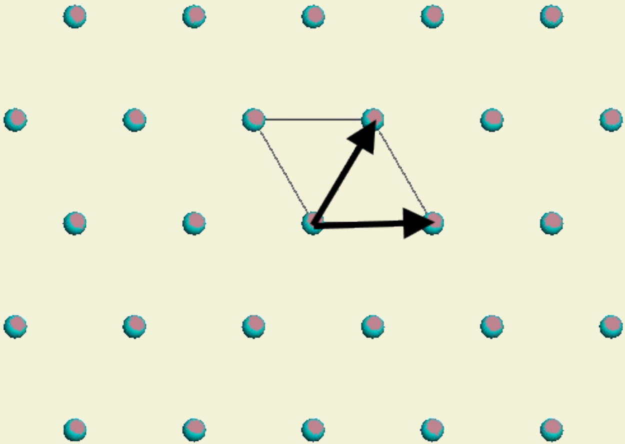

discrete translations

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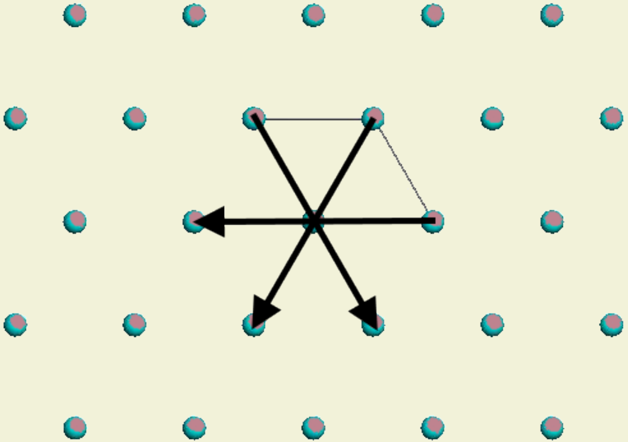

reflections / inversions

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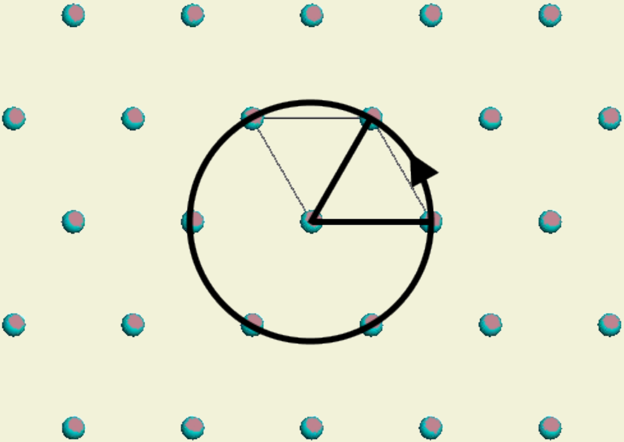

discrete rotations

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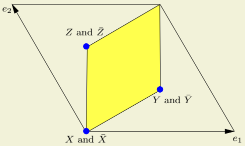
Lattice can have symmetries. Symmetries can have fixed points.
e.g. $\mathbb{T}^2/\mathbb{Z}_3$ (with $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$)



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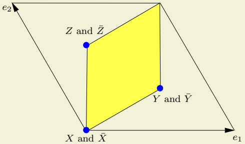

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"Symmetries of symmetries" [AT'16]

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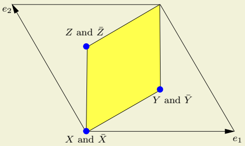

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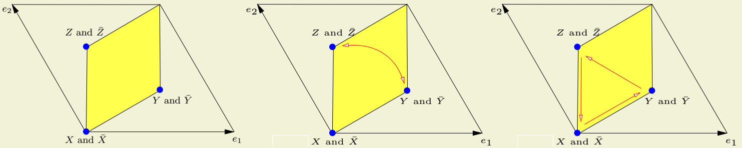

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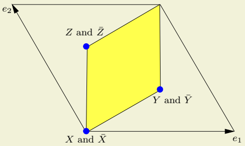

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Here, these leave the lattice symmetries invariant, but act non-trivially on the fixed points.

New insight: Flavor symmetries are given by outer automorphisms of the Narain lattice space group! [Baur, Nilles, AT, Vaudrevange '19]

In this way we can unambiguously compute them in the top-down approach.

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- Bosonic string coordinates, D right- and D left-moving, $y_{\mathrm{R,L}}$, *compactified* on $2D$ torus:

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- $\Theta^K = \mathbb{1}$, is an "orbifold twist" with $\theta_{\mathrm{R,L}} \in \mathrm{SO}(D)$.
- "Narain lattice":

$$\Gamma = \{E\,\hat{N} \mid \hat{N} \in \mathbb{Z}^{2D}\}$$

(Γ is even, self-dual lattice with metric $\eta = \mathrm{diag}(-\mathbb{1}_D, \mathbb{1}_D)$.)

- $\hat{N} = (n, m) \in \mathbb{Z}^{2D}$, n: winding number, m: Kaluza-Klein number of string boundary condition.
- E: "Narain vielbein", depends on moduli of the torus; $E^{\mathrm{T}} E \equiv \mathcal{H} = \mathcal{H}(T, U)$.

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$$\mathcal{H}(T,U) \;=\; \frac{1}{\mathrm{Im}\,T\,\mathrm{Im}\,U} \begin{pmatrix} |T|^2 & |T|^2\,\mathrm{Re}\,U & \mathrm{Re}\,T\,\mathrm{Re}\,U & -\mathrm{Re}\,T \\ |T|^2\,\mathrm{Re}\,U & |T\,U|^2 & |U|^2\,\mathrm{Re}\,T & -\mathrm{Re}\,T\,\mathrm{Re}\,U \\ \mathrm{Re}\,T\,\mathrm{Re}\,U & |U|^2\,\mathrm{Re}\,T & |U|^2 & -\mathrm{Re}\,U \\ -\mathrm{Re}\,T & -\mathrm{Re}\,T\,\mathrm{Re}\,U & -\mathrm{Re}\,U & 1 \end{pmatrix}.$$

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Narain space group $g = (\Theta^k, E\,\hat{N}) \in S_{\mathrm{Narain}}$ is given by multiplicative closure of all twist and shifts

$$S_{\mathrm{Narain}} := \langle\, (\Theta, 0)\ ,\ (\mathbb{1}, E_i)\ \text{ for }\ i \in \{1, \ldots, 2D\}\, \rangle\,.$$

Top down flavor symmetries

- We identify points $Y \sim gY$ with $g \in S_{\text{Narain}} \Rightarrow$ fixed points.
- g constitutes boundary condition for closed strings

$\Rightarrow$ "Strings are localized at fixed points." [Dixon, Harvey, Vafa, Witten '85,'86]

- Each fixed point corresponds to a whole conjugacy class $[g] = \{f\, g\, f^{-1} \mid f \in S_{\text{Narain}}\}$ of space group elements
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  - Non-trivial: outer auts of $S_{\text{Narain}} \Leftrightarrow$ permutation of c.c.'s

  $\Rightarrow$ non-trivial maps between strings at different f.p.'s!

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- The thus derived flavor symmetries automatically contain the so-called "space-group selection rules". [Hamidi and Vafa '86]
- They agree with previously derived non-Abelian flavor symmetries. [Kobayashi, Nilles, Plöger, Raby, Ratz '06]

Outs of the Narain lattice

Maps beween Narain lattice Γ to an equivalent lattice Γ' are given by outer automorphisms of the **Narain lattice**

$$\mathrm{O}_{\hat{\eta}}(D,D,\mathbb{Z}) \;:=\; \left\langle\,\hat{\Sigma}\;\middle|\;\hat{\Sigma}\;\in\;\mathrm{GL}(2D,\mathbb{Z})\quad\text{with}\quad\hat{\Sigma}^{\mathrm{T}}\hat{\eta}\,\hat{\Sigma}\;=\;\hat{\eta}\,\right\rangle\,.$$

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For example, specializing to $D=2$, $\curvearrowright$ d.o.f. in E are Kähler (T) and complex strucutre moduli (U). Outs of Narain lattice:

$$\mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z}) \cong \left[(\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)\right]/\mathbb{Z}_2 \;.$$

With $\mathrm{SL}(2,\mathbb{Z})$ and action on any modulus $M=\{T,U\}$ given by

$$\mathrm{SL}(2,\mathbb{Z}) = \left\langle\, \mathsf{s},\mathsf{t} \,\middle|\, \mathsf{s}^4 = 1, \mathsf{s}^2 = \mathsf{s}\mathsf{t}^3 \,\right\rangle .$$

$$\mathsf{s} : M \mapsto -\frac{1}{M} \qquad \text{and} \qquad \mathsf{t} : M \mapsto M+1 \;,$$

Outs of the Narain lattice

Maps beween Narain lattice Γ to an equivalent lattice Γ' are given by outer automorphisms of the **Narain lattice**

$$\mathrm{O}_{\hat{\eta}}(D,D,\mathbb{Z}) \;:=\; \left\langle\, \hat{\Sigma} \;\middle|\; \hat{\Sigma} \,\in\, \mathrm{GL}(2D,\mathbb{Z}) \quad \text{with} \quad \hat{\Sigma}^{\mathrm{T}}\hat{\eta}\,\hat{\Sigma} \;=\; \hat{\eta} \,\right\rangle .$$

For example, specializing to $D=2$, $\curvearrowright$ d.o.f. in E are Kähler (T) and complex strucutre moduli (U). Outs of Narain lattice:

$$\mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z}) \cong \left[(\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)\right]/\mathbb{Z}_2 \;.$$

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$$\mathsf{s} : M \mapsto -\frac{1}{M} \qquad \text{and} \qquad \mathsf{t} : M \mapsto M+1 \;,$$

Outer automorphisms of Γ contain the **modular transformations**, including T-duality transformations, $T \leftrightarrow U$ mirror symmetry and a $\mathcal{CP}$-like transformation $M \mapsto -\overline{M}$.

[Baur, Nilles, AT, Vaudrevange '19]

Outs of the Narain space group

For the full **Narain space group**, the outer automorphisms are given by transformations $h := (\hat{\Sigma}, \hat{T}) \notin S_{\text{Narain}}$ such that

$$g \stackrel{h}{\mapsto} h\, g\, h^{-1} \stackrel{!}{\in} S_{\text{Narain}}\;.$$

Outs are given by the solutions to the **consistency conditions**

$$\begin{aligned} \hat{\Sigma}\,\Theta^k\,\hat{\Sigma}^{-1} &\stackrel{!}{=} \Theta^{k'}\;, \\ \left(\mathbb{1} - \hat{\Sigma}\,\Theta^k\,\hat{\Sigma}^{-1}\right)\hat{T} &\stackrel{!}{=} \hat{N}'\;. \end{aligned}$$

Solutions yield a set of generators of the Out group as

$$\left\{(\hat{\Sigma}_1, 0),\ (\hat{\Sigma}_2, 0),\ \ldots,\ (\mathbb{1}, \hat{T}_1),\ (\mathbb{1}, \hat{T}_2),\ \ldots\right\}\;.$$

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Note: These Outs also act on the moduli. $M \equiv T, U$

$$M \overset{h}{\longmapsto} M' = M \qquad \rightarrow \text{“traditional flavor trafo”}$$

$$M \overset{h}{\longmapsto} M' \neq M \qquad \rightarrow \text{“modular flavor trafo”}$$

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Outer automorphisms of Narain space group unify flavor symmetries with **modular transformations**, including $\mathcal{CP}$-like transformations.

The eclectic flavor symmetry of $\mathbb{T}^2/\mathbb{Z}_3$

(For this specific orbifold, $\langle U \rangle = \exp(2\pi\mathrm{i}/3)$.)

The outer automorphisms of the corresponding Narain space group yield the following symmetries:

[Baur, Nilles, AT, Vaudrevange '19; Nilles, Ramos-Sánchez, Vaudrevange '20]

- a $\Delta(54)$ traditional flavor symmetry,
- an $\mathrm{SL}(2,\mathbb{Z})_T$ modular symmetry which acts as a $\Gamma_3' \cong T'$ finite modular symmetry on matter fields and their couplings,
- a $\mathbb{Z}_9^R$ discrete R-symmetry as remnant of $\mathrm{SL}(2,\mathbb{Z})_U$, and
- a $\mathbb{Z}_2^{\mathcal{CP}}$ $\mathcal{CP}$-like transformation.

$$G_{\textbf{eclectic}} = G_{\text{traditional}} \cup G_{\text{modular}} \cup G_{\mathrm{R}} \cup \mathcal{CP}\,,$$

Together, the full eclectic group of this setting is of order 3888 given by

$$G_{\text{eclectic}} = \Omega(2) \rtimes \mathbb{Z}_2^{\mathcal{CP}}\,, \qquad \text{with} \quad \Omega(2) \cong [1944, 3448]\,.$$

The eclectic flavor symmetry of $\mathbb{T}^2/\mathbb{Z}_3$

nature of symmetry		outer automorphism of Narain space group	flavor groups				
eclectic	modular	rotation $\mathrm{S} \in \mathrm{SL}(2,\mathbb{Z})_T$	$\mathbb{Z}_4$	T'			$\Omega(2)$
		rotation $\mathrm{T} \in \mathrm{SL}(2,\mathbb{Z})_T$	$\mathbb{Z}_3$				
	traditional flavor	translation A	$\mathbb{Z}_3$	$\Delta(27)$	$\Delta(54)$	$\Delta'(54,2,1)$	
		translation B	$\mathbb{Z}_3$				
		rotation $\mathrm{C} = \mathrm{S}^2 \in \mathrm{SL}(2,\mathbb{Z})_T$	$\mathbb{Z}_2^R$				
		rotation $\mathrm{R} \in \mathrm{SL}(2,\mathbb{Z})_U$	$\mathbb{Z}_9^R$				

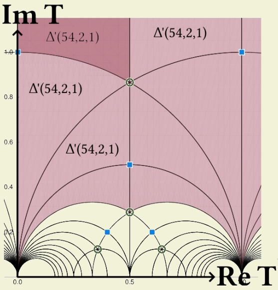
table from [Nilles, Ramos-Sánchez, Vaudrevange '20]

Action on the T modulus as

$$\mathrm{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathrm{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\mathrm{K}_*^{\mathcal{CP}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

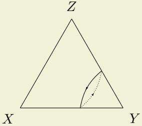
$\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{R}$: trivial!



Transformation of massless matter fields

sector	matter fields	eclectic flavor group $\Omega(2)$								
		modular T' subgroup				traditional $\Delta(54)$ subgroup				$\mathbb{Z}_9^R$
	Φ_n	irrep $\boldsymbol{s}$	$\rho_{\boldsymbol{s}}(\mathrm{S})$	$\rho_{\boldsymbol{s}}(\mathrm{T})$	n	irrep $\boldsymbol{r}$	$\rho_{\boldsymbol{r}}(\mathrm{A})$	$\rho_{\boldsymbol{r}}(\mathrm{B})$	$\rho_{\boldsymbol{r}}(\mathrm{C})$	R
bulk	Φ_0	$\mathbf{1}$	1	1	0	$\mathbf{1}$	1	1	$+1$	0
	Φ_{-1}	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3
θ	$\Phi_{-2/3}$	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-2/3$	$\mathbf{3}_2$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$+\rho(\mathrm{C})$	1
	$\Phi_{-5/3}$	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-5/3$	$\mathbf{3}_1$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$-\rho(\mathrm{C})$	-2
θ^2	$\Phi_{-1/3}$	$\mathbf{2}''\oplus\mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$-1/3$	$\bar{\mathbf{3}}_1$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$-\rho(\mathrm{C})$	2
	$\Phi_{+2/3}$	$\mathbf{2}''\oplus\mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$+2/3$	$\bar{\mathbf{3}}_2$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$+\rho(\mathrm{C})$	5
super-potential	$\mathcal{W}$	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3

table from [Nilles, Ramos-Sánchez, Vaudrevange '20]



$$\left(\omega := \mathrm{e}^{2\pi\mathrm{i}/3}\right)$$

$$\rho(\mathrm{S}) = \frac{\mathrm{i}}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2\end{pmatrix}, \quad \rho(\mathrm{T}) = \begin{pmatrix}\omega^2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$\rho(\mathrm{A}) = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}, \quad \rho(\mathrm{B}) = \begin{pmatrix}1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^2\end{pmatrix}, \quad \rho(\mathrm{C}) = -\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} = \rho(\mathrm{S})^2.$$

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	Φ_n	irrep $\boldsymbol{s}$	$\rho_{\boldsymbol{s}}(\mathrm{S})$	$\rho_{\boldsymbol{s}}(\mathrm{T})$	n	irrep $\boldsymbol{r}$	$\rho_{\boldsymbol{r}}(\mathrm{A})$	$\rho_{\boldsymbol{r}}(\mathrm{B})$	$\rho_{\boldsymbol{r}}(\mathrm{C})$	R
bulk	Φ_0	$\mathbf{1}$	1	1	0	$\mathbf{1}$	1	1	$+1$	0
	Φ_{-1}	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3
θ	$\Phi_{-2/3}$	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-2/3$	$\mathbf{3}_2$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$+\rho(\mathrm{C})$	1
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θ^2	$\Phi_{-1/3}$	$\mathbf{2}''\oplus\mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$-1/3$	$\bar{\mathbf{3}}_1$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$-\rho(\mathrm{C})$	2
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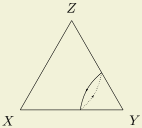

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MSSM charge assignments in explicit models

Model	ℓ	$\bar{e}$	$\bar{\nu}$	q	$\bar{u}$	$\bar{d}$	H_u	H_d	flavons
A	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	Φ_0	Φ_0	$\Phi_{-2/3,-1}$
B	$\Phi_{-1/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-1/3}$	Φ_{-1}	Φ_0	$\Phi_{-2/3,-1}$
C	$\Phi_{-2/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-2/3}$	Φ_{-1}	Φ_{-1}	$\Phi_{-1/3,-1}$
D	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{\pm 2/3,0}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	Φ_0	$\Phi_{-1,0}$	$\Phi_{\pm 2/3,-1}$
E	$\Phi_{-2/3,-1/3}$	$\Phi_{-2/3,0}$	$\Phi_{0,-2/3,-1/3,-5/3}$	$\Phi_{-1,-2/3}$	$\Phi_{-2/3}$	$\Phi_{0,-2/3}$	Φ_0	Φ_0	$\Phi_{-2/3,-1/3,-5/3,-1}$

for methodology, see [Carballo-Pérez, Peinado, Ramos-Sánchez '16; Ramos-Sánchez '17]

$\rightarrow$ See talk by Ramos-Sánchez.

For example, superpotential in model A:

$$W = \phi^0 \left[\left(\phi_u^0\,\varphi_u\right) Y_u\, H_u\, \overline{u}\, q + \left(\phi_d^0\,\varphi_d\right) Y_d\, H_d\, \overline{d}\, q + \left(\phi_e^0\,\varphi_d\right) Y_\ell\, H_d\, \overline{e}\, \ell\right] + \left(\phi^0\varphi_\nu\right) Y_\nu\, H_u\, \overline{\nu}\, \ell + \phi_M^0\, \varphi_d\, \overline{\nu}\, \overline{\nu}\,.$$

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B	$\Phi_{-1/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-1/3}$	Φ_{-1}	Φ_0	$\Phi_{-2/3,-1}$
C	$\Phi_{-2/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-2/3}$	Φ_{-1}	Φ_{-1}	$\Phi_{-1/3,-1}$
D	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{\pm 2/3,0}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	Φ_0	$\Phi_{-1,0}$	$\Phi_{\pm 2/3,-1}$
E	$\Phi_{-2/3,-1/3}$	$\Phi_{-2/3,0}$	$\Phi_{0,-2/3,-1/3,-5/3}$	$\Phi_{-1,-2/3}$	$\Phi_{-2/3}$	$\Phi_{0,-2/3}$	Φ_0	Φ_0	$\Phi_{-2/3,-1/3,-5/3,-1}$

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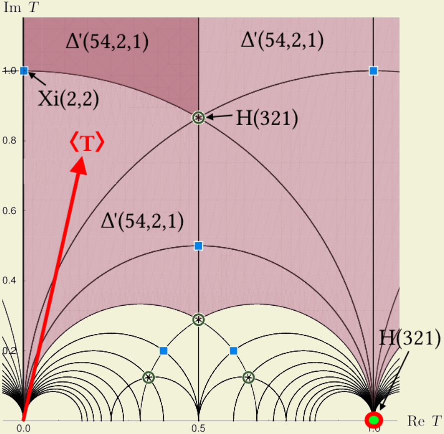
Observations:

- All Yukawas are **integer weight** modular forms.
- All eclectic charge assignments are $1:1$ with modular weights!
  ("representation under eclectic symmetry completely fixed by modular weights").

Conjecture: Is this a general top-down feature?

for other examples, see [Ishiguro, Kobayashi, Otsuka '21], [Kikuchi, Kobayahsi, Uchida '21] [Almumin, Chen, Knapp-Pérez, Ramos-Sánchez, Ratz, Shukla '21]

Sources of breaking of the eclectic symmetry



1. VEV of the moduli, $\langle T\rangle$.

Special points w/ enhanced symmetry:

$\langle T\rangle = \mathrm{i}, \qquad \Rightarrow \Xi(2,2)$

$\langle T\rangle = \omega, -\omega^2 \qquad \Rightarrow H(3,2,1)$

$\langle T\rangle = \mathrm{i}\infty$ dual to $\langle T\rangle = 1$,
$\Rightarrow H(3,2,1)$.

$\Xi(2,2) \cong [324, 111]$
$H(3,2,1) \cong [486, 125]$

2. In addition to $\langle T\rangle$, breaking of eclectic symmetry by flavon vevs:

$$\langle\Phi_{-2/3}\rangle \sim \langle\mathbf{3}_2\rangle, \qquad \langle\Phi_{-5/3}\rangle \sim \langle\mathbf{3}_1\rangle, \qquad \langle\Phi_{-1}\rangle \sim \langle\mathbf{1}'\rangle\ ,$$

$$\langle\Phi_{-1/3}\rangle \sim \langle\overline{\mathbf{3}}_1\rangle, \qquad \langle\Phi_{+2/3}\rangle \sim \langle\overline{\mathbf{3}}_2\rangle.$$

Example: Breakdown of $H(3,2,1)$ at $\langle T\rangle = \omega$

$H(3,2,1)$ subgroup	branchings $\Phi_{-2/3}$	$\Phi_{-5/3}$	subgroup generator(s)	corresponding vevs $\langle\Phi_{-2/3}\rangle$	$\langle\Phi_{-5/3}\rangle$
$\mathrm{S}_3^{(2)}\times\mathbb{Z}_3^{(3)}$	$\mathbf{1}_1'\oplus\mathbf{2}_c$	$\mathbf{1}\oplus\mathbf{2}_c$	$\mathrm{C},\mathrm{AB^2A},\mathrm{AB^2AR(ST)}$	–	$(\omega^2,1,1)^\mathrm{T}$
$\mathbb{Z}_3^{(2)}\times\mathbb{Z}_3^{(3)}$	$\mathbf{1}\oplus\mathbf{1}_{\omega,1}\oplus\mathbf{1}_{\omega^2,\omega}$	$\mathbf{1}\oplus\mathbf{1}_{\omega^2,1}\oplus\mathbf{1}_{\omega,\omega^2}$	$\mathrm{AB^2A},\mathrm{AB^2AR(ST)}$	$(\omega^2,1,1)^\mathrm{T}$	$(\omega^2,1,1)^\mathrm{T}\oplus\langle\Phi_{-1}\rangle$
$\mathbb{Z}_3^{(3)}$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}_\omega$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}_{\omega^2}$	$\mathrm{AB^2AR(ST)}$	$(0,1,-\omega^2)^\mathrm{T}$ $+\alpha(1,0,-\omega^2)^\mathrm{T}$	$(1,-1,0)^\mathrm{T}$ $+\alpha(0,-\omega,1)^\mathrm{T}$
$\mathrm{S}_3^{(1)}$	$\mathbf{1}'\oplus\mathbf{2}$	$\mathbf{1}\oplus\mathbf{2}$	C,A	–	$(1,1,1)^\mathrm{T}$
$\mathbb{Z}_3^{(1)}$	$\mathbf{1}\oplus\mathbf{1}_\omega\oplus\mathbf{1}_{\omega^2}$	$\mathbf{1}\oplus\mathbf{1}_\omega\oplus\mathbf{1}_{\omega^2}$	A	$(1,1,1)^\mathrm{T}$	$(1,1,1)^\mathrm{T}\oplus\langle\Phi_{-1}\rangle$
$\mathbb{Z}_6$	$\mathbf{1}\oplus\mathbf{1}_{-1}\oplus\mathbf{1}_{-\omega}$	$\mathbf{1}\oplus\mathbf{1}_{-1}\oplus\mathbf{1}_{\omega}$	$\mathrm{B^2ACR^2(ST)^2}$	$(1,-1,0)^\mathrm{T}$	$(1,1,-2\omega^2)^\mathrm{T}$
$\mathbb{Z}_3^{(3)}$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}_{\omega^2}$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}_{\omega^2}$	$\mathrm{AB^2AR(ST)}$	$(0,1,-\omega^2)^\mathrm{T}$ $+\alpha(1,0,-\omega^2)^\mathrm{T}$	$(1,-1,0)^\mathrm{T}$ $+\alpha(0,-\omega,1)^\mathrm{T}$
$\mathbb{Z}_3^{(4)}$	$\mathbf{1}\oplus\mathbf{1}_\omega\oplus\mathbf{1}_{\omega^2}$	$\mathbf{1}\oplus\mathbf{1}_\omega\oplus\mathbf{1}_{\omega^2}$	$\mathrm{BR(ST)^2}$	$(1,a,b)^\mathrm{T}$	$(1,a,b)^\mathrm{T}$
$\mathbb{Z}_2$	$\mathbf{1}\oplus\mathbf{1}_{-1}\oplus\mathbf{1}_{-1}$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}_{-1}$	C	$(0,1,-1)^\mathrm{T}$ (preserves $\mathbb{Z}_6^{(2)}$)	$(1,0,0)^\mathrm{T}$ $+\alpha(0,1,1)^\mathrm{T}$

Representation matrices of the flavor group of twisted matter fields $\Phi_{-2/3}$ and $\Phi_{-5/3}$

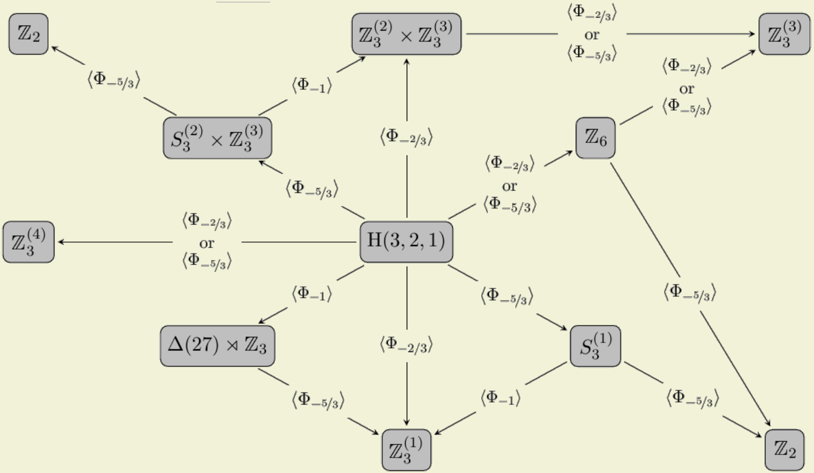
$$\Phi_{-2/3}:\quad \rho_{\mathbf{3}_2,\omega}(\mathrm{A})=\rho(\mathrm{A}),\qquad \rho_{\mathbf{3}_2,\omega}(\mathrm{B})=\rho(\mathrm{B}),\qquad \rho_{\mathbf{3}_2,\omega}(\mathrm{C})=\rho(\mathrm{C}),$$

$$\rho_{\mathbf{3}_2,\omega}(\mathrm{R})=\mathrm{e}^{2\pi\mathrm{i}/9}\,\mathbb{1}_3,\qquad \rho_{\mathbf{3}_2,\omega}(\mathrm{ST})=\mathrm{e}^{2\pi\mathrm{i}\,2/9}\,\rho(\mathrm{ST}),\qquad \text{and}$$

$$\Phi_{-5/3}:\quad \rho_{\mathbf{3}_1,\omega}(\mathrm{A})=\rho(\mathrm{A}),\qquad \rho_{\mathbf{3}_1,\omega}(\mathrm{B})=\rho(\mathrm{B}),\qquad \rho_{\mathbf{3}_1,\omega}(\mathrm{C})=-\rho(\mathrm{C}),$$

$$\rho_{\mathbf{3}_1,\omega}(\mathrm{R})=\mathrm{e}^{-4\pi\mathrm{i}/9}\,\mathbb{1}_3,\qquad \rho_{\mathbf{3}_1,\omega}(\mathrm{ST})=\mathrm{e}^{2\pi\mathrm{i}\,5/9}\,\rho(\mathrm{ST}).$$

Example: Breakdown of $H(3,2,1)$ at $\langle T\rangle = \omega$



Residual symmetries help to generate hierarchies in masses and mixing matrix elements.

see e.g. talks by Penedo and Feruglio.

Possible lessons for bottom-up model building

Empirical observations:

- Modular flavor symmetries do not arise alone;
  They are generically accompanied by (partly overlapping!)
  - "traditional" discrete flavor symmetries (& flavons),
  - discrete (non-Abelian) R symmetries,
  - $\mathcal{CP}$-type symmetries.

$$G_{\mathbf{eclectic}} = G_{\text{traditional}} \cup G_{\text{modular}} \cup G_{\text{R}} \cup \mathcal{CP}\,.$$

- Modular weights of matter fields are fractional,
  Modular weights of (Yukawa) couplings are integer.
- Modular weights potentially $1:1$ "locked" to other flavor symmetry representations.
- Different sectors of the theory may have different moduli / different residual symmetries $\Leftrightarrow$ "local flavor unification".

The modular flavor "swampland" may be bigger than anticipated.

Many open questions

- Extra tori?
- Other possible realistic configurations?

  "Size of the 'landscape' "?
- Moduli stabilization?
- Flavon potential?
- Restrictions on Kähler potential?

  see [Chen, Ramos-Sanchez, Ratz '19]
  [Chen, Knapp-Perez, Ramos-Hamud, Ramos-Sanchez, Ratz, Shukla '21]

Summary

- There are explicit models of heterotic string theory that reproduce, at low energies, the

  MSSM + (modular) flavor symmetry + flavons.

- The complete flavor symmetry can unambiguously be derived by the outer automorphisms of the Narain space group.
- One finds an "eclectic" flavor symmetry that non-trivially unifies:

$$G_{\textbf{eclectic}} = G_{\text{traditional}} \cup G_{\text{modular}} \cup G_{\text{R}} \cup \mathcal{CP}\,.$$

- This symmetry is broken by
  - Expectation values of the moduli, e.g. $\langle U \rangle$, $\langle T \rangle$.
  - Expectation values of the flavon fields.
- (Approximate) residual symmetries are common, and can help to naturally generate hierachies in masses and mixing matrix elements.

To come: explicit fit of this specific model to the observed SM flavor structure.

Thank You

Backup slides

“Modular Flavor”, general idea

Even w/o thoughts about UV completions: Very attractive framework.

Very predictive (few parameters), CP violation “built in”!

- Neutrinos/Leptons
  [Kobayashi, Tanaka, Tatsuishi ’18], [Penedo, Petcov ’18], [Criado, Feruglio ’18], [Kobayashi, Omoto, Shimizu, Takagi, Tanimoto, Tatsuishi ’18], [Novichkov, Penedo, Petcov, Titov ’18 (2x)], [Novichkov, Petcov, Tanimoto ’18]
- Quark sector (including CP violation)
  [Okada, Tanimoto ’18],[Kobayashi, Shimizu, Takagi, Tanimoto, Tatsuishi, Uchida ’18]
- Within GUTs [de Anda, King, Perdomo ’18]

Also very attractive from top-down UV complete viewpoint: Modular invariance controls orbifold compactifications of heterotic string.

- Couplings among twisted-sector states are modular forms
  [Ibañez ’86],[Hamdi, Vafa ’87],...
- In particular, for Yukawa couplings! ..., [Kobayashi, Lebedev ’03]

↷ Explore relation of new approach to string theory compactifications [Kobayashi, Nagamoto, Takada, Tamba, Tatsuishi ’18],[Kobayashi, Tamba ’18] [Baur, Nilles, AT, Vaudrevange ’19]

Flavor and Modular Symmetries

Feruglio: "Are neutrino masses modular forms?" [Feruglio '17]

General (bottom-up) idea:

- Supersymmetric (say $N = 1$) theory.
- Ask for **modular invariance**: [Ferrara, Lüst, (Shapere), Theisen '89(x2)]

$$\tau \mapsto \gamma\tau = \frac{a\tau + b}{c\tau + d} , \qquad \varphi^{(I)} \mapsto (c\tau + d)^{-k_I} \, \rho^{(I)}(\gamma)\varphi^{(I)} .$$

$$\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N) , \quad \{a, b, c, d\} \in \mathbb{Z} , \qquad \Phi := (\tau, \varphi) .$$

- *EITHER* $W(\Phi)$, $K(\Phi, \overline{\Phi})$ invariant (K up to Kähler transf.), ⟶global SUSY
  OR compensating against each other. ⟶SUGRA
- In any case, Yukawa couplings must be *modular forms*:

$$W(\Phi) = \sum_n Y_{I_1 \ldots I_n}(\tau) \, \varphi^{(I_1)} \ldots \varphi^{(I_n)} ,$$

$$Y_{I_1 \ldots I_n}(\tau) \mapsto Y_{I_1 \ldots I_n}(\gamma\tau) \stackrel{!}{=} \left[\mathrm{e}^{\mathrm{i}\alpha(\gamma)} \right] (c\tau + d)^{k_Y(n)} \, Y_{I_1 \ldots I_n}(\tau)$$

- $\tau \to \langle\tau\rangle$ breaks modular symmetry $\Longleftrightarrow$ τ takes rôle of flavon!

Comments on relation to bottom-up constructions

- We connect to Super*gravity* theories. To relate in bottom-up approach, one should **not** take invariant superpotential, but

$$W(\Phi) \mapsto \mathrm{e}^{\mathrm{i}\alpha(\gamma)}\,(c\tau+d)^{-h}\,W(\Phi)\;.$$

(it has to cancel non-trivially transforming $K(\Phi,\overline{\Phi})$, see [Feruglio '17])

- Traditional flavor symmetry $\Delta'(54,2,1)$ is *universal* in moduli space, i.e. it is never broken by $\langle T\rangle$. (for a realistic pheno, it must be broken by some other mechanism)
- In particular, not *all* of the flavor symmetry arises from subgroups of the modular group $\mathrm{SL}(2,\mathbb{Z})_T$.

(exactly the parts corresponding to Narain lattice translations)

Narain vielbein

The Narain vielbein can be parameterized as (in absence of Wilson lines)

$$E \;:=\; \frac{1}{\sqrt{2}}\begin{pmatrix} \dfrac{e^{-\mathrm{T}}}{\sqrt{\alpha'}}\,(G-B) & -\sqrt{\alpha'}\,e^{-\mathrm{T}} \\ \dfrac{e^{-\mathrm{T}}}{\sqrt{\alpha'}}\,(G+B) & \sqrt{\alpha'}\,e^{-\mathrm{T}} \end{pmatrix}\;.$$

In this definition of the Narain vielbein, e denotes the vielbein of the D-dimensional geometrical torus $\mathbb{T}^D$ with metric $G := e^{\mathrm{T}}e$, $e^{-\mathrm{T}}$ corresponds to the inverse transposed matrix of e, B is the anti-symmetric background B-field ($B = -B^{\mathrm{T}}$), and α' is called the Regge slope.

World-sheet modular invariance requires E to span even, self-dual lattice $\Gamma = \{E\,\hat{N} \mid \hat{N} \in \mathbb{Z}^{2D}\}$ with metric η of signature (D, D). Consequently, one can always choose E such that

$$E^{\mathrm{T}}\eta\,E \;=\; \hat{\eta}\;, \quad \text{where} \quad \eta \;:=\; \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad \text{and} \quad \hat{\eta} \;:=\; \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\;.$$

Transformation of moduli

To compute the transformation properties of the moduli T and U we use the generalized metric $\mathcal{H} = E^{\mathrm{T}}E$. As the Narain vielbein depends on the moduli $E = E(T, U)$ so does the generalized metric $\mathcal{H} = \mathcal{H}(T, U)$. It transforms as

$$\mathcal{H}(T,U) \overset{\hat{\Sigma}}{\longmapsto} \mathcal{H}(T',U') = \hat{\Sigma}^{-\mathrm{T}}\mathcal{H}(T,U)\hat{\Sigma}^{-1} .$$

This equation can be used to read off the transformations of the moduli

$$T \overset{\hat{\Sigma}}{\longmapsto} T' = T'(T,U) \quad \text{and} \quad U \overset{\hat{\Sigma}}{\longmapsto} U' = U'(T,U) .$$

For a two-torus $\mathbb{T}^2$, the generalized metric in terms of the torus moduli reads

$$\mathcal{H}(T,U) = \frac{1}{\operatorname{Im} T \operatorname{Im} U} \begin{pmatrix} |T|^2 & |T|^2 \operatorname{Re} U & \operatorname{Re} T \operatorname{Re} U & -\operatorname{Re} T \\ |T|^2 \operatorname{Re} U & |T\,U|^2 & |U|^2 \operatorname{Re} T & -\operatorname{Re} T \operatorname{Re} U \\ \operatorname{Re} T \operatorname{Re} U & |U|^2 \operatorname{Re} T & |U|^2 & -\operatorname{Re} U \\ -\operatorname{Re} T & -\operatorname{Re} T \operatorname{Re} U & -\operatorname{Re} U & 1 \end{pmatrix} .$$

Explicit generators of $\Omega(2)$ for $\mathbb{T}^2/\mathbb{Z}_3$

$\mathrm{SL}(2,\mathbb{Z})_T$ modular generators S and T arise from rotational outer automorphisms and act on the modulus via

$$\mathrm{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

Reflectional outer automorphism coresponding to $\mathbb{Z}_2^{\mathcal{CP}}$ $\mathcal{CP}$-like transformation:

$$\mathrm{K}_* = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\rho(\mathrm{S}) = \frac{\mathrm{i}}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \quad \text{and} \quad \rho(\mathrm{T}) = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

The traditional flavor symmetry $\Delta(54)$ is generated by two translational outer automorphisms of the Narain space group A and B, together with the $\mathbb{Z}_2$ rotational outer automorphism $\mathrm{C} := \mathrm{S}^2$.

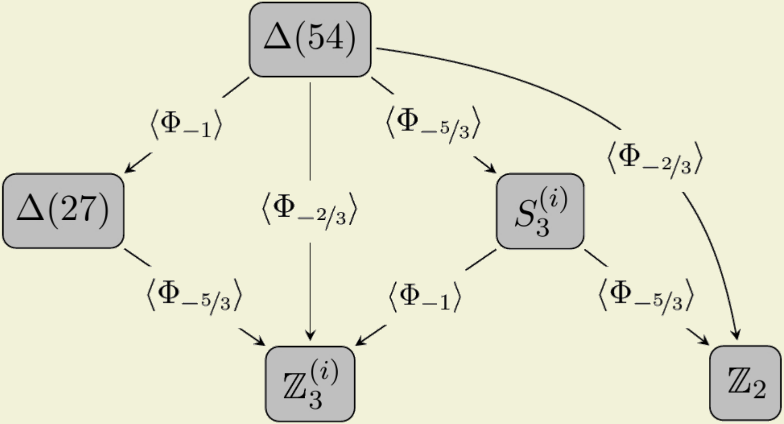
$$\rho(\mathrm{A}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \ \rho(\mathrm{B}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \text{ and } \rho(\mathrm{C}) = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \rho(\mathrm{S})^2,$$

Explicit generators of $\Omega(2)$ for $\mathbb{T}^2/\mathbb{Z}_3$

The complex structure modulus with vev $\langle U \rangle = \exp(2\pi\mathrm{i}/3)$ breaks the $\mathrm{SL}(2,\mathbb{Z})_U$ of $\mathbb{T}^2$ to a discrete remnant that acts as a $\mathbb{Z}_9^R$ symmetry on matter fields.
The traditional flavor symmetry is enhanced to $\Delta(54) \cup \mathbb{Z}_9^R \cong \Delta'(54,2,1) \cong [162,44]$.
Altogether, the full eclectic group is a group of order 3888 given by

$$G_{\mathrm{eclectic}} \;=\; \Omega(2) \rtimes \mathbb{Z}_2^{\mathcal{CP}}\,, \qquad \text{where} \quad \Omega(2) \cong [1944, 3448]\,. \tag{1}$$

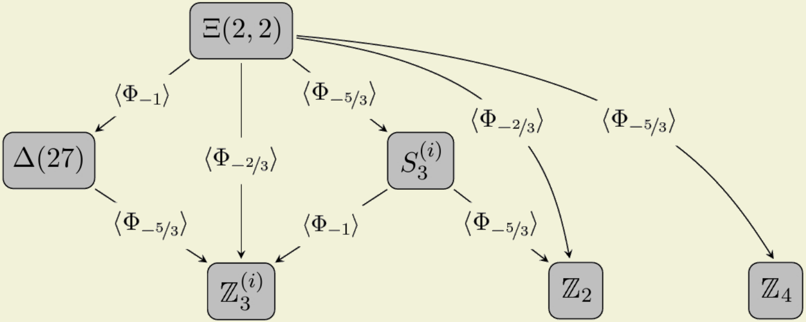
Delta(54), $\langle T \rangle$ universal



Delta(54), $\langle T \rangle$ universal

$\Delta(54)$ subgroup	Φ_{-1}	branchings $\Phi_{-2/3}$	$\Phi_{-5/3}$	subgroup generator(s)	corresponding vevs $\langle \Phi_{-2/3} \rangle$	$\langle \Phi_{-5/3} \rangle$
$\Delta(27)$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	A, B	–	–
$\mathrm{S}_3^{(1)}$	$\mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{2}$	$\mathbf{1} \oplus \mathbf{2}$	A, C	–	$(1,1,1)^{\mathrm{T}}$
$\mathbb{Z}_3^{(1)}$	$\mathbf{1}$	$\mathbf{1} \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$		A	$(1,1,1)^{\mathrm{T}}$	$(1,1,1)^{\mathrm{T}} \oplus \langle \Phi_{-1} \rangle$
$\mathrm{S}_3^{(2)}$	$\mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{2}$	$\mathbf{1} \oplus \mathbf{2}$	B, C	–	$(1,0,0)^{\mathrm{T}}$
$\mathbb{Z}_3^{(2)}$	$\mathbf{1}$	$\mathbf{1} \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$		B	$(1,0,0)^{\mathrm{T}}$	$(1,0,0)^{\mathrm{T}} \oplus \langle \Phi_{-1} \rangle$
$\mathrm{S}_3^{(3)}$	$\mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{2}$	$\mathbf{1} \oplus \mathbf{2}$	ABA, C	–	$(\omega,1,1)^{\mathrm{T}}$
$\mathbb{Z}_3^{(3)}$	$\mathbf{1}$	$\mathbf{1} \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$		ABA	$(\omega,1,1)^{\mathrm{T}}$	$(\omega,1,1)^{\mathrm{T}} \oplus \langle \Phi_{-1} \rangle$
$\mathrm{S}_3^{(4)}$	$\mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{2}$	$\mathbf{1} \oplus \mathbf{2}$	$\mathrm{AB}^2\mathrm{A}, \mathrm{C}$	–	$(\omega^2,1,1)^{\mathrm{T}}$
$\mathbb{Z}_3^{(4)}$	$\mathbf{1}$	$\mathbf{1} \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$		$\mathrm{AB}^2\mathrm{A}$	$(\omega^2,1,1)^{\mathrm{T}}$	$(\omega^2,1,1)^{\mathrm{T}} \oplus \langle \Phi_{-1} \rangle$
$\mathbb{Z}_2$	$\mathbf{1}_{-1}$	$\mathbf{1} \oplus \mathbf{1}_{-1} \oplus \mathbf{1}_{-1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}_{-1}$	C	$(0,1,-1)^{\mathrm{T}}$	$(1,0,0)^{\mathrm{T}}$ $+\alpha(0,1,1)^{\mathrm{T}}$

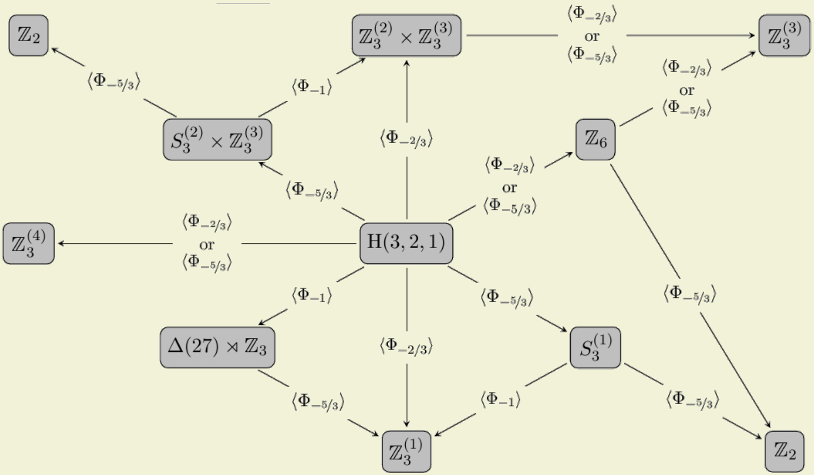
$\Xi(2,2)$, $\langle T \rangle = \mathrm{i}$



$\Xi(2,2)$, $\langle T \rangle = \mathrm{i}$

$\Xi(2,2)$ subgroup	branchings $\Phi_{-2/3}$	$\Phi_{-5/3}$	subgroup generator(s)	corresponding vevs $\langle\Phi_{-2/3}\rangle$	$\langle\Phi_{-5/3}\rangle$
$S_3^{(1)}$	$\mathbf{1}' \oplus \mathbf{2}$	$\mathbf{1} \oplus \mathbf{2}$	A, C	–	$(1,1,1)^{\mathrm{T}}$
$\mathbb{Z}_3^{(1)}$	$\mathbf{1} \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$	$\mathbf{1} \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$	A	$(1,1,1)^{\mathrm{T}}$	$(1,1,1)^{\mathrm{T}} \oplus \langle\Phi_{-1}\rangle$
$S_3^{(2)}$	$\mathbf{1}' \oplus \mathbf{2}$	$\mathbf{1} \oplus \mathbf{2}$	ABA, C	–	$(\omega,1,1)^{\mathrm{T}}$
$\mathbb{Z}_3^{(2)}$	$\mathbf{1} \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$	$\mathbf{1} \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$	ABA	$(\omega,1,1)^{\mathrm{T}}$	$(\omega,1,1)^{\mathrm{T}} \oplus \langle\Phi_{-1}\rangle$
$\mathbb{Z}_4$	$\mathbf{1}_{-1} \oplus \mathbf{1}_{\mathrm{i}} \oplus \mathbf{1}_{-\mathrm{i}}$	$\mathbf{1} \oplus \mathbf{1}_{-1} \oplus \mathbf{1}_{\mathrm{i}}$	$\mathrm{AB^2ACS}$	–	$(1, 1+\sqrt{3}, 1)^{\mathrm{T}}$
$\mathbb{Z}_2$	$\mathbf{1} \oplus \mathbf{1}_{-1} \oplus \mathbf{1}_{-1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}_{-1}$	C	$(0,1,-1)^{\mathrm{T}}$	$(1,0,0)^{\mathrm{T}}$ $+\alpha(0,1,1)^{\mathrm{T}}$

$H(3,2,1)$, $\langle T \rangle = \omega, 1, \mathrm{i}\infty$



$H(3,2,1)$, $\langle T\rangle = \omega, 1, \mathrm{i}\infty$

$H(3,2,1)$	branchings		subgroup	corresponding vevs	
subgroup	$\Phi_{-2/3}$	$\Phi_{-5/3}$	generator(s)	$\langle\Phi_{-2/3}\rangle$	$\langle\Phi_{-5/3}\rangle$
$\mathrm{S}_3^{(2)} \times \mathbb{Z}_3^{(3)}$	$\mathbf{1}_1' \oplus \mathbf{2}_c$	$\mathbf{1} \oplus \mathbf{2}_c$	C, AB^{2}A, AB^{2}AR(ST)	–	$(\omega^2, 1, 1)^{\mathrm{T}}$
$\mathbb{Z}_3^{(2)} \times \mathbb{Z}_3^{(3)}$	$\mathbf{1} \oplus \mathbf{1}_{\omega,1} \oplus \mathbf{1}_{\omega^2,\omega}$	$\mathbf{1} \oplus \mathbf{1}_{\omega^2,1} \oplus \mathbf{1}_{\omega,\omega^2}$	AB^{2}A, AB^{2}AR(ST)	$(\omega^2, 1, 1)^{\mathrm{T}}$	$(\omega^2, 1, 1)^{\mathrm{T}} \oplus \langle\Phi_{-1}\rangle$
$\mathbb{Z}_3^{(3)}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}_{\omega}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}_{\omega^2}$	AB^{2}AR(ST)	$(0, 1, -\omega^2)^{\mathrm{T}}$ $+\alpha(1, 0, -\omega^2)^{\mathrm{T}}$	$(1, -1, 0)^{\mathrm{T}}$ $+\alpha(0, -\omega, 1)^{\mathrm{T}}$
$\mathrm{S}_3^{(1)}$	$\mathbf{1}' \oplus \mathbf{2}$	$\mathbf{1} \oplus \mathbf{2}$	C, A	–	$(1, 1, 1)^{\mathrm{T}}$
$\mathbb{Z}_3^{(1)}$	$\mathbf{1} \oplus \mathbf{1}_{\omega} \oplus \mathbf{1}_{\omega^2}$	$\mathbf{1} \oplus \mathbf{1}_{\omega} \oplus \mathbf{1}_{\omega^2}$	A	$(1, 1, 1)^{\mathrm{T}}$	$(1, 1, 1)^{\mathrm{T}} \oplus \langle\Phi_{-1}\rangle$
$\mathbb{Z}_6$	$\mathbf{1} \oplus \mathbf{1}_{-1} \oplus \mathbf{1}_{-\omega}$	$\mathbf{1} \oplus \mathbf{1}_{-1} \oplus \mathbf{1}_{\omega}$	B^{2}ACR^2(ST)^2	$(1, -1, 0)^{\mathrm{T}}$	$(1, 1, -2\omega^2)^{\mathrm{T}}$
$\mathbb{Z}_3^{(3)}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}_{\omega^2}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}_{\omega^2}$	AB^{2}AR(ST)	$(0, 1, -\omega^2)^{\mathrm{T}}$ $+\alpha(1, 0, -\omega^2)^{\mathrm{T}}$	$(1, -1, 0)^{\mathrm{T}}$ $+\alpha(0, -\omega, 1)^{\mathrm{T}}$
$\mathbb{Z}_3^{(4)}$	$\mathbf{1} \oplus \mathbf{1}_{\omega} \oplus \mathbf{1}_{\omega^2}$	$\mathbf{1} \oplus \mathbf{1}_{\omega} \oplus \mathbf{1}_{\omega^2}$	BR(ST)^2	$(1, a, b)^{\mathrm{T}}$	$(1, a, b)^{\mathrm{T}}$
$\mathbb{Z}_2$	$\mathbf{1} \oplus \mathbf{1}_{-1} \oplus \mathbf{1}_{-1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}_{-1}$	C	$(0, 1, -1)^{\mathrm{T}}$ (preserves $\mathbb{Z}_6^{(2)}$)	$(1, 0, 0)^{\mathrm{T}}$ $+\alpha(0, 1, 1)^{\mathrm{T}}$