The limits of the strong CP problem

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The aim:

Challenge the conventional view of the strong CP problem by showing that a careful **infinite 4d volume** limit implies that **QCD does not violate CP** regardless of the value of the $\boldsymbol{\theta}$ **angle**

The novelty:

The results would imply that there is no need to tune θ to explain the **absence of CP violation in the strong interactions**

The plan:

Fundamentals of the strong CP problem

Fermion correlators from cluster decomposition and the index theorem

Fundamentals of the strong CP problem

The QCD angle from the Lagrangian

$$S_{\text{QCD}} = \int d^4x \left[-\frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{g^2\theta}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} + \sum_{i=1}^{N_f} \overline{\psi}_i \left(i\gamma^\mu D_\mu - m_i e^{i\alpha_i \gamma_5} \right) \psi_i \right] .$$

θ-term is a total derivative and thus corresponds to a **boundary term**

Boundary terms never contribute in perturbation theory:

effects of θ are nonperturbative

S_θ is **CP-odd!**

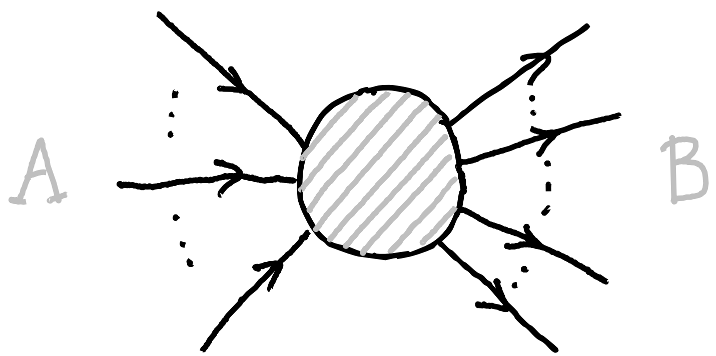
$$CP : A_0 \to -A_0, \quad A_i \to A_i \quad \Rightarrow \quad S_\theta \to -S_\theta$$

Yet no CP violation has been observed in the strong interactions: **Strong CP problem**

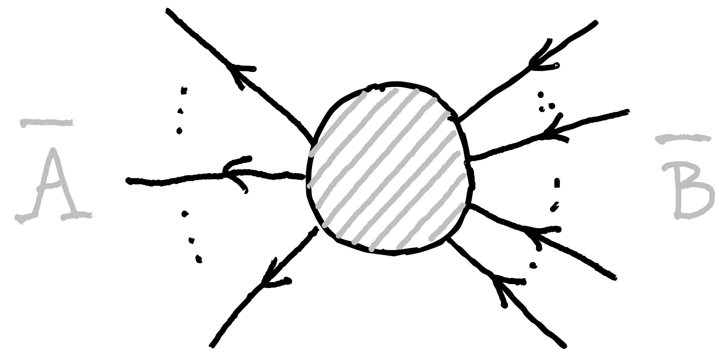
$$|d_n| < 1.8 \times 10^{-26} e \cdot cm$$ [nEDM collaboration 2020]

What do we need for CP violation?

Need **interfering contributions** to amplitudes with **misaligned phases**



$$|\mathcal{M}_{A\to B}|^2 = |c_0\hat{\mathcal{M}}_0 + c_1\hat{M}_1|^2$$



$$|\mathcal{M}_{\bar{A}\to \bar{B}}|^2 = |c_0^*\hat{\mathcal{M}}_0 + c_1^*\hat{M}_1|^2$$

$$|\mathcal{M}_{A\to B}|^2 - |\mathcal{M}_{\bar{A}\to \bar{B}}|^2 = 4\mathrm{Im}(c_0^* c_1)\mathrm{Im}(\mathcal{M}_0\mathcal{M}_1^*)$$

CP violation needs $c_0 \neq c_1$

Phases of perturbative contributions to correlators fixed by α_i

θ will lead to ~~CP~~ if it **affects phases** of **nonperturbative contributions** to **correlators.** Naively expected because $\exp(-S^{\mathrm{E}}_{\mathrm{QCD}}) \propto \exp(\mathrm{i}\Delta n\theta)$

Towards correlators: vacuum path integral

$$\langle \phi_f | e^{-\mathrm{i}HT} | \phi_i \rangle = \sum_n e^{-\mathrm{i}E_n T} \langle \phi_f | n \rangle \langle n | \phi_i \rangle = \int_{\phi_i, \phi_f, T} \left(\prod \mathcal{D}\phi \right) e^{\mathrm{i}S_T}$$

To get a **vacuum transition amplitude** we can take the **infinite *T* limit,** leading to **infinite 4d volume**

$$Z = \lim_{T\to\infty} \int_T \left(\prod \mathcal{D}\phi \right) e^{\mathrm{i}S_T} \sim \lim_{T\to\infty} \langle 0 | e^{-\mathrm{i}HT} | 0 \rangle$$

To recover the vacuum amplitude for **finite *T***, one would **need to know the wave functional of the vacuum**

$$\langle 0 | e^{-iHT} | 0 \rangle = \int [\mathcal{D}\phi_f]_{T/2} [\mathcal{D}\phi_i]_{-T/2} \langle 0 | \phi_f \rangle \langle \phi_f | e^{-iHT} | \phi_i \rangle \langle \phi_i | 0 \rangle$$

$$= \int [\mathcal{D}\phi_f]_{T/2} [\mathcal{D}\phi_i]_{-T/2} \langle 0 | \phi_f \rangle \langle \phi_i | 0 \rangle \int_{\phi_i, \phi_f, T} \left(\prod \mathcal{D}\phi \right) e^{\mathrm{i}S}$$

To ensure projection into vacuum, we use path integral for infinite *VT*

Finite action constraints and topology

Euclidean path integral only receives contributions from **finite action saddles** and fluctuations about these

- In **infinite spacetime, gauge** fields at saddles must be **pure gauge transf. at ∞**

- Fields fall into **homotopy classes** with **integer topological charge Δn**

Atiyah-Singer's **index theorem:**

$$\Delta n = \#(\text{Right-handed zero modes of } \not{D}) - \#(\text{Left-handed zero modes of } \not{D})$$

$$\not{D}\psi_R = 0 \qquad \not{D}\psi_L = 0$$

The **θ-term** is related to the **topological charge!** $-S^E_\theta = i\theta\Delta n$

The θ-term is only guaranteed to be $\propto$ to an integer in an infinite spacetime

Is θ physical?

θ cannot be physical as it changes under **chiral field redefinitions** due to **anomaly**:

$$\partial_\mu \langle \sum_j \bar{\psi}_j \gamma^\mu \gamma_5 \psi_j \rangle = 2N_F \frac{g^2}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} + 2\sum_j \langle \bar{\psi}\gamma_5 m_j e^{i\alpha_j \gamma_5}\psi \rangle$$

$$\psi \to e^{i\beta\gamma_5}\psi$$
$$\bar{\psi} \to \bar{\psi} e^{i\beta\gamma_5}$$

$$Z(\theta, \alpha_j) \to Z(\theta - 2N_f\beta, \alpha_j + 2\beta)$$

fermion mass phases

Spurion symmetry: Z invariant under chiral transformations plus "spurion" transf:

$$\theta \to \theta + 2N_f\beta, \quad \mathfrak{m}_j = m_j e^{i\alpha_j} \to e^{-2i\beta}\mathfrak{m}_j$$

A **physical combination** is

$$\bar{\theta} \equiv \theta + \alpha, \quad \alpha = \sum_j \arg(\mathfrak{m}_j)$$

Strong CP problem:

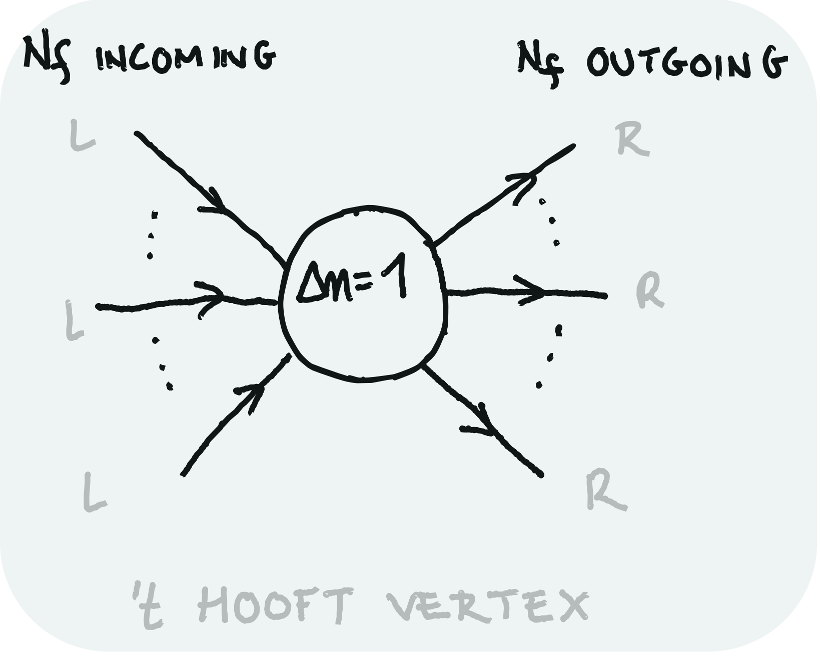
$$\bar{\theta} < 10^{-10}$$

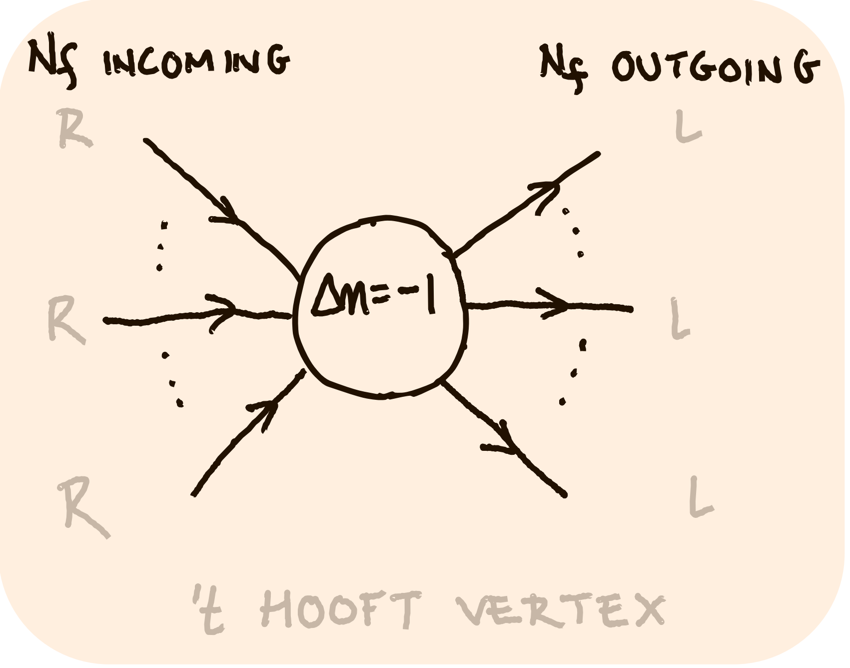
Nonperturbative effects in QCD

Integrating anomaly eq: $\Delta Q_5 = 2N_f \Delta n + \text{mass corrections}$

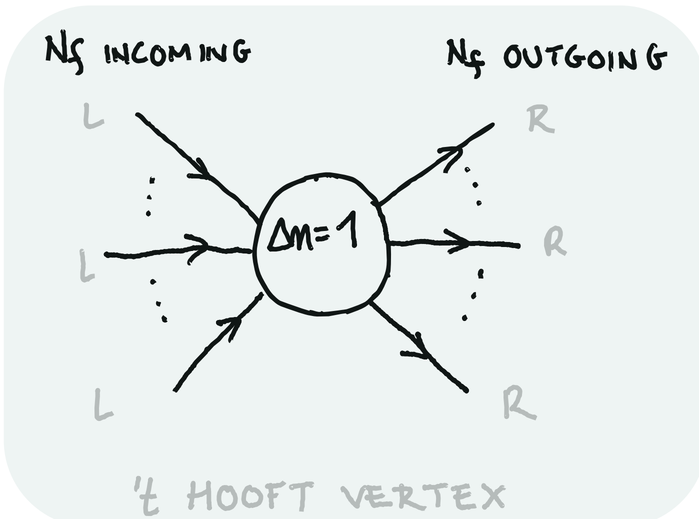
→ There are interactions that **violate chiral charge** by $2N_f \Delta n$ units

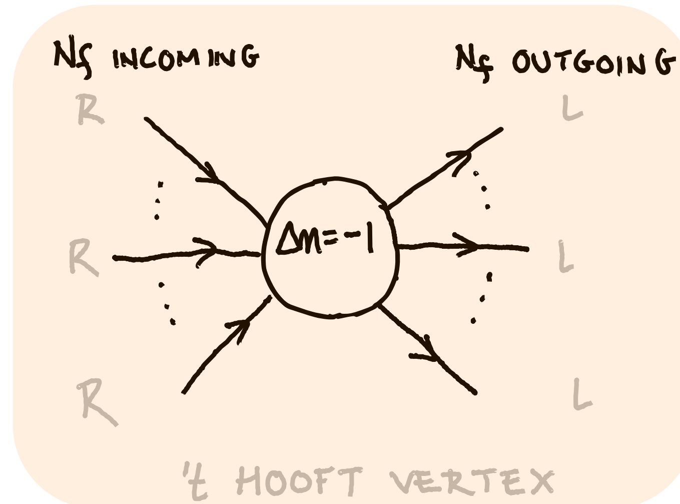
Can be recovered from nonperturbative contributions to the path integral around **saddle points** with nonzero Δn: **instantons** ['t Hooft]





Nonperturbative effects in QCD





Fermionic Green's functions in instanton backgrounds can be captured by **effective operators**

$$\mathcal{L}_{\text{eff}} \supset -\sum_j m_j \bar{\psi}_j (e^{-i\alpha_j} P_L + e^{i\alpha_j} P_R)\psi_j - \Gamma_{N_f} e^{i\xi} \prod_{j=1}^{N_f} (\bar{\psi}_j P_{\rm L} \psi_j) - \Gamma_{N_f} e^{-i\xi} \prod_{j=1}^{N_f} (\bar{\psi}_j P_{\rm R} \psi_j)$$

2 options compatible with spurion chiral symmetry:

$\xi = \theta$	CP violation (phases not aligned)
$\xi = -\alpha$	No CP violation (all phases aligned, can be removed)

How to resolve the ambiguity?

Must **match effective `t Hooft vertices** with **QCD computations**

Only real computation that we know of is **`t Hooft**'s, using **dilute instanton gas** and yielding $\boxed{\xi = \theta \quad \textbf{(CP violation)}}$

We have **recomputed Green's functions in the dilute instanton gas**, in Euclidean and Minkowski spacetime, and found $\boxed{\xi = -\alpha \quad \textbf{(no CP violation)}}$

We also have a **computation which does not rely on instantons,** presented next

Fermion correlators from cluster decomposition and the index theorem

Strategy

We want a **derivation that does not rely on instantons**

The aim is to **constrain** the **functional dependence** of the partition functions $Z_{\Delta n}$ on VT, Δn, $\mathfrak{m}_j = m_j e^{i\alpha_j}$

Fermion masses can be understood as **sources** for the integrated fermion correlators [Leutweyler & Smilga]

$$\mathcal{L} \supset \sum_j \left(\bar{\psi}_j (\mathfrak{m}_j^* P_L + \mathfrak{m}_j P_R) \psi_j \right)$$

$$\frac{\partial}{\partial \mathfrak{m}_i} Z_{\Delta n} = -\int d^4x \, \langle \bar{\psi}_i P_R \psi_i \rangle_\nu , \qquad \frac{\partial}{\partial \mathfrak{m}_i^*} Z_{\Delta n} = -\int d^4x \, \langle \bar{\psi}_i P_L \psi_i \rangle_\nu .$$

These correlators should be **sensitive to global CP-violating phases**

Cluster decomposition

Using Lagrangian without the θ angle, one can write expectation values by **weighing** over path integrals over the different **topological classes**

4d volume

$$\langle \mathcal{O} \rangle_\Omega = \frac{\sum_{\Delta n=-\infty}^{\infty} f(\Delta n) \int_{\Delta n} \mathcal{D}\phi \, \mathcal{O} \, \mathrm{e}^{-S_\Omega[\phi]}}{\sum_{\Delta n=-\infty}^{\infty} f(\Delta n) \int_{\Delta n} \mathcal{D}\phi \, \mathrm{e}^{-S_\Omega[\phi]}}$$

For a **local operator** $\mathcal{O}_1$ with support in a spacetime volume Ω_1

$$\langle \mathcal{O}_1 \rangle_\Omega = \frac{\sum_{\Delta n_1=-\infty}^{\infty} \sum_{\Delta n_2=-\infty}^{\infty} f(\Delta n_1 + \Delta n_2) \int_{\Delta n_1} \mathcal{D}\phi \, \mathcal{O}_1 \, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int_{\Delta n_2} \mathcal{D}\phi \, \mathrm{e}^{-S_{\Omega_2}[\phi]}}{\sum_{\Delta n_1=-\infty}^{\infty} \sum_{\Delta n_2=-\infty}^{\infty} f(\Delta n_1 + \Delta n_2) \int_{\Delta n_1} \mathcal{D}\phi \, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int_{\Delta n_2} \mathcal{D}\phi \, \mathrm{e}^{-S_{\Omega_2}[\phi]}}$$

If **physics** is **local**, fluctuations in Ω_2 must factor away (**cluster decomposition**)

$$f(\Delta n_1 + \Delta n_2) = f(\Delta n_1) f(\Delta n_2) \Rightarrow f(\Delta n) = \mathrm{e}^{\mathrm{i}\Delta n \theta}$$

Usual θ term recovered! [Weinberg]

Taking the clustering argument further

The denominators in the previous slide one assumed **factorization** of path integral

$$Z(\Omega) = \sum_{\Delta n} e^{i\Delta n \theta} \tilde{Z}_{\Delta n}(\Omega)$$

$$\tilde{Z}_{\Delta n}(\Omega = \Omega_1 + \Omega_2) = \sum_{\Delta n_1 = -\infty}^{\infty} \tilde{Z}_{\Delta n_1}(\Omega_1) \tilde{Z}_{\Delta n - \Delta n_1}(\Omega_2)$$

Assume **complex phases in** $Z_{\Delta n}$ **fixed** as in one-loop determinants:

- **phases** of **nonzero modes cancel** (related by **parity**)
- **global phase** determined by **fermion zero modes** → **index theorem**!

$$\#(\text{Right-handed zero modes of } \not{D}) - \#(\text{Left-handed zero modes of } \not{D}) = \Delta n$$

→ $\tilde{Z}_{\Delta n}(\Omega) = e^{i\Delta n \alpha} \underbrace{g_{\Delta n}(\Omega)}_{\text{Real}} \Rightarrow g_{\Delta n}(\Omega_1 + \Omega_2) = \sum_{\Delta n_1 = -\infty}^{\infty} g_{\Delta n_1}(\Omega_1) g_{\Delta n - \Delta n_1}(\Omega_2)$

Clustering revisited

Parity changes sign of Δn and α. This and solving the relations for $\Omega = 0$ motivates the **Ansatz**

$$g_{\Delta n}(\Omega) = \Omega^{|\Delta n|} f_{|\Delta n|}(\Omega^2), \quad f_{|\Delta n|}(0) \neq 0.$$

Assuming **analiticity** in Ω there is a **unique solution** with free parameter β !

$$f_{\Delta n}(\Omega) = I_{\Delta n}(2\beta\Omega)$$

$$Z_{\Delta n} = \mathrm{e}^{\mathrm{i}\Delta n(\theta+\alpha)} I_{\Delta n}(2\beta\Omega)$$

c.f. [Leutweyler & Smilga]

Mass dependence and correlation functions

As the $g_{\Delta n}$ are real:

$$Z_{\Delta n}(\Omega) = e^{\mathrm{i}\Delta n(\theta+\alpha)} I_{\Delta n}(2\beta(\mathfrak{m}_k\mathfrak{m}_k^*)\,\Omega) = $$
$$= e^{\mathrm{i}\Delta n(\theta - \mathrm{i}/2\sum_j \log(\mathfrak{m}_j/\mathfrak{m}_j^*))} I_{\Delta n}(2\beta(\mathfrak{m}_k\mathfrak{m}_k^*)\,\Omega)$$

Taking derivatives with respect to $\mathfrak{m},\ \mathfrak{m}^*$ gives **averaged integrated correlators**

Spurion chiral charge +2

$$\frac{1}{VT}\int d^4x\,\langle\bar\psi_i P_R \psi_i\rangle_{\Delta n} = -\,e^{\mathrm{i}\Delta n(\theta+\bar\alpha)}\Bigg(-\frac{\beta}{2\mathfrak{m}_i}(I_{\Delta n+1}(2\beta\Omega) - I_{\Delta n-1}(2\beta\Omega)) + \mathfrak{m}_i^*(I_{\Delta n+1}(2\beta\Omega) + I_{\Delta n-1}(2\beta\Omega))\frac{\partial}{\partial(\mathfrak{m}_i\mathfrak{m}_i^*)}\,\beta(\mathfrak{m}_k\mathfrak{m}_k^*)\Bigg)$$

Spurion chiral charge -2

$$\frac{1}{VT}\int d^4x\,\langle\bar\psi_i P_L \psi_i\rangle_{\Delta n} = -\,e^{\mathrm{i}\Delta n(\theta+\bar\alpha)}\Bigg(\frac{\beta}{2\mathfrak{m}_i^*}(I_{\Delta n+1}(2\beta\Omega) - I_{\Delta n-1}(2\beta\Omega)) + \mathfrak{m}_i(I_{\Delta n+1}(2\beta\Omega) + I_{\Delta n-1}(2\beta\Omega))\frac{\partial}{\partial(\mathfrak{m}_i\mathfrak{m}_i^*)}\,\beta(\mathfrak{m}_k\mathfrak{m}_k^*)\Bigg)$$

Summing over topological sectors

$$\frac{1}{VT}\int d^4x\,\langle\bar\psi_i P_R \psi_i\rangle = \lim_{N\to\infty}\lim_{VT\to\infty}\frac{\sum_{\Delta n=-N}^{N}\frac{1}{VT}\int d^4x\,\langle\bar\psi_i P_R\psi_i\rangle_{\Delta n}}{\sum\limits_{\Delta m=-N}^{N} Z_{\Delta m}} = 2\mathfrak{m}_i^*\,\partial_{\mathfrak{m}_i\mathfrak{m}_i^*}\beta(\mathfrak{m}_k\mathfrak{m}_k^*),$$

$$\frac{1}{VT}\int d^4x\,\langle\bar\psi_i P_L \psi_i\rangle = \lim_{N\to\infty}\lim_{VT\to\infty}\frac{\sum_{\Delta n=-N}^{N}\frac{1}{VT}\int d^4x\,\langle\bar\psi_i P_L\psi_i\rangle_{\Delta n}}{\sum\limits_{\Delta m=-N}^{N} Z_{\Delta m}} = 2\mathfrak{m}_i\,\partial_{\mathfrak{m}_i\mathfrak{m}_i^*}\beta(\mathfrak{m}_k\mathfrak{m}_k^*).$$

Topological classification only enforced in infinite volume, which fixes ordering

$$\frac{1}{VT}\int d^4x\,\langle\bar\psi_i\psi_i\rangle = 2m_i e^{-\mathrm{i}\alpha_i\gamma_5}\partial_{\mathfrak{m}_i\mathfrak{m}_i^*}\beta(\mathfrak{m}_k\mathfrak{m}_k^*)$$

Only a single phase: no CP violation

Result also valid for more general correlators. Opposite order of limits yields CP-violation

Conclusions

QCD with an arbitrary θ **does not predict CP violation**, as long as the sum over topological sectors is performed at **infinite volume**

This **ordering of limits** is the correct one because the topological classification is only enforced for an infinite volume

Further reading in our paper

- For **local observables** one can recover CP-conserving expectation values from **path integrals in a finite subvolume without θ dependence**
- **No conflict** with nonzero **topological susceptibility** in the lattice and **η' mass**

Thank you!

Additional material

Phase ambiguity in the chiral Lagrangian

The **chiral Lagrangian** at lowest order has the form

$$\mathcal{L} = f_\pi^2 \mathrm{Tr}\partial_\mu U \partial^\mu U^\dagger + a f_\pi^3 \mathrm{Tr} M U + b f_\pi^4 \det U + \mathrm{h.c.}$$

Captures t' Hooft vertices $U \sim \bar{\psi} P_R \psi \sim e^{\mathrm{i}\frac{\Pi^a \sigma^a}{\sqrt{2} f_\pi}}$

There are again **2 options compatible with spurion chiral symmetry**

$b \propto e^{-i\theta}$

$b \propto e^{i\alpha} = e^{i\sum_j \arg(\mathfrak{m}_j)}$

Usual option, **assumed** by **[Baluni, Crewther et al]** → CP violation

No CP violation!

No CP violation in the chiral Lagrangian

$$\mathcal{L} = f_\pi^2 \mathrm{Tr} \partial_\mu U \partial^\mu U^\dagger + a f_\pi^3 \mathrm{Tr} M U + |b| e^{\mathrm{i}\xi} f_\pi^4 \mathrm{det} U + \mathrm{h.c.}$$

Minimizing the potential for the pions leads to

$$\langle U \rangle = U_0 = \mathrm{diag}\left(\mathrm{e}^{\mathrm{i}\varphi_u}, \mathrm{e}^{\mathrm{i}\varphi_d}, \mathrm{e}^{\mathrm{i}\varphi_s}\right) .$$

$$m_i \varphi_i = \frac{m_u m_d m_s (\xi + \alpha_u + \alpha_d + \alpha_s)}{m_u m_d + m_d m_s + m_s m_u} = \tilde{m}(\xi + \alpha_u + \alpha_d + \alpha_s) .$$

Adding field *N* containing neutron and proton, the *CP*-violating neutron-pion interactions are of the form

$$\frac{c_+ \tilde{m}(\xi + \alpha_u + \alpha_d + \alpha_s)}{2 f_\pi} \bar{N} \Phi N$$

(ϕ containing $U, U^\dagger$ and gamma matrices) which cancel for $\xi = -\alpha$ leading to no CP violation

Baluni's CP-violating effective Lagrangian

Baluni's CP-violating Lagrangian (used by [Crewther et al]) is based on searching for field redefinitions that minimize the QCD mass term

$$\mathcal{L}_M(U_{R,L}) = \bar{\psi} U_R^\dagger M U_L \psi_L + \text{h.c.}, \quad U_{R,L} \in SU_{R,L}(3)$$

$$\langle 0|\delta\mathcal{L}|0\rangle = \min_{U_{R,L}} \langle 0|\mathcal{L}_M(U_{R,L})|0\rangle$$

However, there is an **extra assumption**: that the **phase of the fermion condensate is aligned with θ**

$$\langle \bar{\psi}_R \psi_L \rangle = \Delta e^{\mathrm{i}c\theta} \mathbb{I}$$

This assumption **does not hold** for the chiral Lagrangian with $\xi = -\alpha$ as seen in previous slide

The η' mass

Chiral Lagrangian with alignment in the phases of mass terms and anomalous terms still predicts a **nonzero value of the η' mass**

$$\mathcal{L} = f_\pi^2 \mathrm{Tr}\partial_\mu U \partial^\mu U^\dagger + a f_\pi^3 \mathrm{Tr} M U + |b| e^{\mathrm{i}\arg\det M} f_\pi^4 \det U + \mathrm{h.c.}$$

$$m_{\eta'}^2 = 8|b| f_\pi^2$$

Can be seen to be **proportional** to the **topological susceptibility** over **finite volumes** of the **pure gauge theory**, in line with [Witten, Di Vecchia & Veneziano]

Classic arguments linking topological susceptibility to CP violation ([Shifman et al]) rely on analytic expansions in θ which **don't apply** with our limiting procedure

***Z* becomes non-analytic in θ.** This possibility has been mentioned by [Witten]

Witten, Nucl. Phys. B 156 (1979)

the physics is of order e^{-N}, contrary to the basic assumptions of this paper, or else the physics is non-analytic as a function of θ, In the latter case, which is quite plausible, the singularities would probably be at $\theta = \pm\pi$, as Coleman found for the massive Schwinger model [10]. It is also quite plausible that θ is not really an angular variable.)

To write a formal expression for $d^2E/d\theta^2$, let us think of the path integral formulation of the theory:

$$Z = \int dA_\mu \exp i \int \mathrm{Tr}\left[-\tfrac{1}{4}F_{\mu\nu} + \frac{g^2\theta}{16\pi^2 N} F_{\mu\nu}\tilde{F}_{\mu\nu}\right]. \tag{5}$$

Partition function and analyticity

Usual partition function is analytic in θ

$$Z_{\text{usual}} = \lim_{VT\to\infty} \lim_{\substack{N\to\infty \\ N\in N}} \sum_{\Delta n=-N}^{N} Z_{\Delta n} = \mathrm{e}^{2\mathrm{i}\kappa_{N_f} VT \cos(\bar{\alpha}+\theta+N_f\pi)}$$

θ-dependence of observables (giving CP violation) is usually obtained by expanding action inside path integral in powers of θ. E.g.

$$\frac{\langle \Delta n \rangle}{\Omega} = \mathrm{i}\,(\theta-\theta_0) \underbrace{\frac{\langle \Delta n^2 \rangle}{\Omega}}_{\text{topological susceptibility}} \Bigg|_{\theta_0} + \mathcal{O}(\theta-\theta_0)^2$$

[Shifman et al]

In our limiting procedure the former is not valid, as Z becomes nonanalytic in θ

$$Z = \lim_{\substack{N\to\infty \\ N\in N}} \lim_{VT\to\infty} \sum_{\Delta n=-N}^{N} Z_{\Delta n} = I_0(2\mathrm{i}\kappa_{N_f} VT) \lim_{\substack{N\to\infty \\ N\in N}} \sum_{|\Delta n|\leq N} \mathrm{e}^{\mathrm{i}\Delta n(\bar{\alpha}+\theta+N_f\pi)}$$

θ drops out from observables, there is no CP violation

Finite volumes in an infinite spacetime

Even in an infinite spacetime, we can express expectation values of local observables in terms over **path integration over finite volume**.

This can help make **contact with lattice computations**

Assume **local operator** $\mathcal{O}_1$ with **support** in finite spacetime volume Ω_1

$$\langle \mathcal{O}_1 \rangle_\Omega = \frac{\sum\limits_{\Delta n=-\infty}^{\infty} f(\Delta n) \int\limits_{\Delta n} \mathcal{D}\phi \, \mathcal{O}_1 \, \mathrm{e}^{-S_\Omega[\phi]}}{\sum\limits_{\Delta n=-\infty}^{\infty} f(\Delta n) \int\limits_{\Delta n} \mathcal{D}\phi \, \mathrm{e}^{-S_\Omega[\phi]}}$$

$$= \frac{\sum\limits_{\Delta n=-\infty}^{\infty} \sum\limits_{\Delta n_1=-\infty}^{\infty} f(\Delta n) \int\limits_{\Delta n_1} \mathcal{D}\phi \, \mathcal{O}_1 \, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int\limits_{\Delta n_2 = \Delta n - \Delta n_1} \mathcal{D}\phi \, \mathrm{e}^{-S_{\Omega_2}[\phi]}}{\sum\limits_{\Delta n=-\infty}^{\infty} \sum\limits_{\Delta n_1=-\infty}^{\infty} f(\Delta n) \int\limits_{\Delta n_1} \mathcal{D}\phi \, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int\limits_{\Delta n_2 = \Delta n - \Delta n_1} \mathcal{D}\phi \, \mathrm{e}^{-S_{\Omega_2}[\phi]}} .$$

Finite volumes in an infinite spacetime

Path integrations over Ω_2 give just the **partition functions** we calculated before

In the **infinite volume** limit the **Bessel functions tend to common value** and dependence on Δn factorizes out and cancels:

$$\langle \mathcal{O}_1 \rangle_\Omega = \frac{\sum_{\Delta n_1=-\infty}^{\infty} \int_{\Delta n_1} \mathcal{D}\phi \, (-1)^{-N_f \Delta n_1} \mathrm{e}^{-\mathrm{i}\,\alpha \Delta n_1} \mathcal{O}_1 \, \mathrm{e}^{-S_{\Omega_1}[\phi]}}{\sum_{\Delta n_1=-\infty}^{\infty} \int_{\Delta n_1} \mathcal{D}\phi \, (-1)^{-N_f \Delta n_1} \mathrm{e}^{-\mathrm{i}\,\alpha \Delta n_1} \mathrm{e}^{-S_{\Omega_1}[\phi]}} \,.$$

We recover a **path integration** over a **finite volume, without θ dependence**

Extra phases precisely **cancel those from fermion determinants in**

This **removes interferences** between different **topological sectors**

The QCD angle from the vacuum state

Hamiltonian is zero for pure gauge transformations, with integer $n_{\rm CS}$: Expect **degenerate pre-vacua** $|n_{\rm CS}\rangle \equiv |n\rangle$

The **true vacuum** $|\omega\rangle$ is a linear combination of prevacua

$$|\omega\rangle = \sum_n f(n)|n\rangle$$

Demanding **invariance up to a phase** under **gauge transformations** in the Δn class

$$U_{\Delta n}|\omega\rangle = \sum_n f(n)|n+\Delta n\rangle = e^{i\Delta n\theta}|\omega\rangle \Rightarrow f(n) = e^{-in\theta}$$

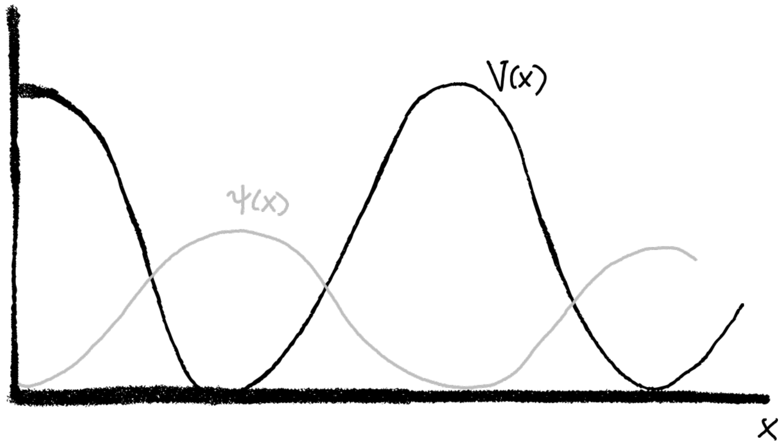
Constructing the partition function as a transition amplitude from $|\omega\rangle$ onto itself

$$Z(\theta) = \langle\omega|e^{-HT}|\omega\rangle = \sum_m\sum_n\langle m|e^{-HT}e^{i\theta(m-n)}|n\rangle = \mathcal{N}\sum_{\Delta n}\langle n+\Delta n|e^{-HT}e^{i\theta\Delta n}|n\rangle$$

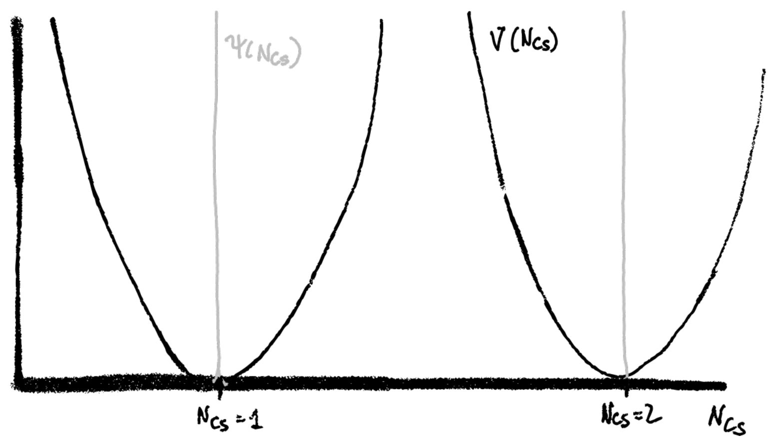
$$= \mathcal{N}\sum_{\Delta n}\int_{\Delta n}\mathcal{D}\phi\, e^{-S_\theta+\dots}$$

Can one use θ the vacuum at finite volume?

Bloch wave function in QM:



vs θ vacuum



Too naive! Have to use path integral in infinite 4D volume to project into vacuum