The limits of the strong CP problem

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CP$^3$-Origins
The aim:

Challenge the conventional view of the strong CP problem by showing that a careful infinite 4d volume limit implies that QCD does not violate CP regardless of the value of the $\theta$ angle.

The novelty:

The results would imply that there is no need to tune $\theta$ to explain the absence of CP violation in the strong interactions.

The plan:

Fundamentals of the strong CP problem
Fermion correlators from cluster decomposition and the index theorem
Fundamentals of the strong CP problem
The QCD angle from the Lagrangian

\[ S_{\text{QCD}} = \int d^4x \left[ -\frac{1}{4g^2} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{g^2\theta}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a + \sum_{i=1}^{N_f} \bar{\psi}_i (i\gamma^\mu D_\mu - m_i e^{i\alpha_i} \gamma_5) \psi_i \right]. \]

\( \theta \)-term is a total derivative and thus corresponds to a boundary term.

Boundary terms never contribute in perturbation theory:

effects of \( \theta \) are nonperturbative

\( S_\theta \) is CP-odd!

\[ CP: A_0 \rightarrow -A_0, \quad A_i \rightarrow A_i \quad \Rightarrow \quad S_\theta \rightarrow -S_\theta \]

Yet no CP violation has been observed in the strong interactions: Strong CP problem

\[ |d_n| < 1.8 \times 10^{-26} \text{ e} \cdot \text{cm} \quad \text{[nEDM collaboration 2020]} \]
What do we need for CP violation?

Need **interfering contributions** to amplitudes with **misaligned phases**

\[ |\mathcal{M}_{A\to B}|^2 = |c_0 \hat{M}_0 + c_1 \hat{M}_1|^2 \]

\[ |\mathcal{M}_{\bar{A}\to \bar{B}}|^2 = |c_0^* \hat{M}_0 + c_1^* \hat{M}_1|^2 \]

\[ |\mathcal{M}_{A\to B}|^2 - |\mathcal{M}_{\bar{A}\to \bar{B}}|^2 = 4 \text{Im}(c_0^* c_1) \text{Im}(\mathcal{M}_0 \mathcal{M}_1^*) \]

CP violation needs \( c_0 \neq c_1 \)

Phases of perturbative contributions to correlators fixed by \( \alpha_i \)

\( \theta \) **will lead to CP** if it affects phases of **nonperturbative contributions** to correlators. Naively expected because \( \exp(-S^{E}_{\text{QCD}}) \propto \exp(i \Delta n \theta) \)
Towards correlators: vacuum path integral

$$\langle \phi_f | e^{-iHT} | \phi_i \rangle = \sum_n e^{-iE_n T} \langle \phi_f | n \rangle \langle n | \phi_i \rangle = \int_{\phi_i, \phi_f, T} \left( \prod \mathcal{D} \phi \right) e^{iS_T}$$

To get a vacuum transition amplitude we can take the infinite $T$ limit, leading to infinite 4d volume

$$Z = \lim_{T \to \infty} \int_T \left( \prod \mathcal{D} \phi \right) e^{iS_T} \sim \lim_{T \to \infty} \langle 0 | e^{-iHT} | 0 \rangle$$

To recover the vacuum amplitude for finite $T$, one would need to know the wave functional of the vacuum

$$\langle 0 | e^{-iHT} | 0 \rangle = \int [\mathcal{D} \phi_f]_{T/2} [\mathcal{D} \phi_i]_{-T/2} \langle 0 | \phi_f \rangle \langle \phi_f | e^{-iHT} | \phi_i \rangle \langle \phi_i | 0 \rangle$$

$$= \int [\mathcal{D} \phi_f]_{T/2} [\mathcal{D} \phi_i]_{-T/2} \langle 0 | \phi_f \rangle \langle \phi_f | 0 \rangle \int_{\phi_i, \phi_f, T} \left( \prod \mathcal{D} \phi \right) e^{iS}$$

To ensure projection into vacuum, we use path integral for infinite $VT$
Finite action constraints and topology

Euclidean path integral only receives contributions from finite action saddles and fluctuations about these

→ In infinite spacetime, gauge fields at saddles must be pure gauge transf. at $\infty$

→ Fields fall into homotopy classes with integer topological charge $\Delta n$

Atiyah-Singer’s index theorem:

$$\Delta n = \#(\text{Right-handed zero modes of } \mathcal{D}) - \#(\text{Left-handed zero modes of } \mathcal{D})$$

$$\mathcal{D}\psi_R = 0 \quad \mathcal{D}\psi_L = 0$$

The $\theta$-term is related to the topological charge!

$$-S^E_\theta = i\theta\Delta n$$

The $\theta$-term is only guaranteed to be $\propto$ to an integer in an infinite spacetime
Is $\theta$ physical?

$\theta$ cannot be physical as it changes under \textbf{chiral field redefinitions} due to \textbf{anomaly}:

$$
\partial_{\mu} \langle \sum_{j} \bar{\psi}_{j} \gamma^{\mu} \gamma_{5} \psi_{j} \rangle = 2N_{F} \frac{g^{2}}{64\pi^{2}} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^{a} F_{\rho\sigma}^{a} + 2 \sum_{j} \langle \bar{\psi} \gamma_{5} m_{j} e^{i\alpha_{j}} \gamma_{5} \psi \rangle
$$

\textbf{Spurion symmetry}: $Z$ invariant under chiral transformations plus “spurion” transf:

$$
\psi \rightarrow e^{i\beta \gamma_{5}} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{i\beta \gamma_{5}}
$$

$$
Z(\theta, \alpha_{j}) \rightarrow Z(\theta - 2N_{f} \beta, \alpha_{j} + 2\beta)
$$

\textbf{fermion mass phases}

$$
\theta \rightarrow \theta + 2N_{f} \beta, \quad m_{j} = m_{j} e^{i\alpha_{j}} \rightarrow e^{-2i\beta} m_{j}
$$

\textbf{A physical combination} is

$$
\bar{\theta} \equiv \theta + \alpha, \quad \alpha = \sum_{j} \text{arg}(m_{j})
$$

\textbf{Strong CP problem}:

$$
\bar{\theta} < 10^{-10}
$$
Nonperturbative effects in QCD

Integrating anomaly eq: \[ \Delta Q_5 = 2N_f \Delta n + \text{mass corrections} \]

There are interactions that violate chiral charge by \(2N_f \Delta n\) units.

Can be recovered from nonperturbative contributions to the path integral around saddle points with nonzero \(\Delta n\): instantons

[‘t Hooft]
Nonperturbative effects in QCD

Fermionic Green’s functions in instanton backgrounds can be captured by effective operators

\[ \mathcal{L}_{\text{eff}} \supset - \sum_j m_j \overline{\psi}_j (e^{-i\alpha_j} P_L + e^{i\alpha_j} P_R) \psi_j - \Gamma_{N_f} e^{i\xi} \prod_{j=1}^{N_f} (\overline{\psi}_j P_L \psi_j) - \Gamma_{N_f} e^{-i\xi} \prod_{j=1}^{N_f} (\overline{\psi}_j P_R \psi_j) \]

2 options compatible with spurion chiral symmetry:

- \( \xi = \theta \)  CP violation (phases not aligned)
- \( \xi = -\alpha \)  No CP violation (all phases aligned, can be removed)
How to resolve the ambiguity?

Must match effective `t Hooft vertices with QCD computations

Only real computation that we know of is `t Hooft’s, using dilute instanton gas and yielding $\xi = \theta$ (CP violation)

We have recomputed Green’s functions in the dilute instanton gas, in Euclidean and Minkowski spacetime, and found $\xi = -\alpha$ (no CP violation)

We also have a computation which does not rely on instantons, presented next
Fermion correlators from cluster decomposition and the index theorem
We want a derivation that does not rely on instantons

The aim is to constrain the functional dependence of the partition functions $Z_{\Delta n}$ on $VT$, $\Delta n$, $m_j = m_j e^{i\alpha_j}$.

Fermion masses can be understood as sources for the integrated fermion correlators [Leutweyler & Smilga]

\[
\mathcal{L} \supset \sum_j \left( \bar{\psi}_j (m_j^* P_L + m_j P_R) \psi_j \right)
\]

\[
\frac{\partial}{\partial m_i} Z_{\Delta n} = - \int d^4 x \langle \bar{\psi}_i P_R \psi_i \rangle_\nu, \quad \frac{\partial}{\partial m_i^*} Z_{\Delta n} = - \int d^4 x \langle \bar{\psi}_i P_L \psi_i \rangle_\nu.
\]

These correlators should be sensitive to global CP-violating phases.
Using Lagrangian without the $\theta$ angle, one can write expectation values by weighing over path integrals over the different topological classes.

$$\langle \mathcal{O} \rangle_{\Omega} = \frac{\sum_{\Delta n=-\infty}^{\infty} f(\Delta n) \int \mathcal{D}\phi \mathcal{O} e^{-S_\Omega[\phi]}}{\sum_{\Delta n=-\infty}^{\infty} f(\Delta n) \int \mathcal{D}\phi e^{-S_\Omega[\phi]}}$$

For a local operator $\mathcal{O}_1$ with support in a spacetime volume $\Omega_1$:

$$\langle \mathcal{O}_1 \rangle_{\Omega} = \frac{\sum_{\Delta n_1=-\infty}^{\infty} \sum_{\Delta n_2=-\infty}^{\infty} f(\Delta n_1 + \Delta n_2) \int \mathcal{D}\phi \mathcal{O}_1 e^{-S_{\Omega_1}[\phi]} \int \mathcal{D}\phi e^{-S_{\Omega_2}[\phi]}}{\sum_{\Delta n_1=-\infty}^{\infty} \sum_{\Delta n_2=-\infty}^{\infty} f(\Delta n_1 + \Delta n_2) \int \mathcal{D}\phi e^{-S_{\Omega_1}[\phi]} \int \mathcal{D}\phi e^{-S_{\Omega_2}[\phi]}}$$

If physics is local, fluctuations in $\Omega_2$ must factor away (cluster decomposition):

$$f(\Delta n_1 + \Delta n_2) = f(\Delta n_1)f(\Delta n_2) \Rightarrow f(\Delta n) = e^{i\Delta n \theta}$$

Usual $\theta$ term recovered! [Weinberg]
The denominators in the previous slide one assumed factorization of path integral

\[ Z(\Omega) = \sum_{\Delta n} e^{i\Delta n \theta} \tilde{Z}_{\Delta n}(\Omega) \]

\[ \tilde{Z}_{\Delta n}(\Omega = \Omega_1 + \Omega_2) = \sum_{\Delta n_1 = -\infty}^{\infty} \tilde{Z}_{\Delta n_1}(\Omega_1) \tilde{Z}_{\Delta n - \Delta n_1}(\Omega_2) \]

Assume complex phases in \( Z_{\Delta n} \) fixed as in one-loop determinants:

- **phases of nonzero modes cancel** (related by parity)
- **global phase** determined by fermion zero modes \( \Rightarrow \) index theorem!

\[ \#(\text{Right-handed zero modes of } \mathcal{D}) - \#(\text{Left-handed zero modes of } \mathcal{D}) = \Delta n \]

\[ \tilde{Z}_{\Delta n}(\Omega) = e^{i\Delta n \alpha} g_{\Delta n}(\Omega) \Rightarrow g_{\Delta n}(\Omega_1 + \Omega_2) = \sum_{\Delta n_1 = -\infty}^{\infty} g_{\Delta n_1}(\Omega_1) g_{\Delta n - \Delta n_1}(\Omega_2) \]

Real
Parity changes sign of $\Delta n$ and $\alpha$. This and solving the relations for $\Omega = 0$ motivates the Ansatz

$$g_{\Delta n}(\Omega) = \Omega^{\frac{\Delta n}{2}} f_{\Delta n}(\Omega^2), \quad f_{\Delta n}(0) \neq 0.$$ 

Assuming analiticity in $\Omega$ there is a unique solution with free parameter $\beta$ !

$$f_{\Delta n}(\Omega) = I_{\Delta n}(2\beta\Omega)$$
$$Z_{\Delta n} = e^{i\Delta n(\theta+\alpha)} I_{\Delta n}(2\beta\Omega)$$

c.f. [Leutweyler & Smilga]
Mass dependence and correlation functions

As the \( g_{\Delta n} \) are real:

\[
Z_{\Delta n}(\Omega) = e^{i\Delta n(\theta + \alpha)} I_{\Delta n}(2\beta(m_k m_k^*) \Omega) = e^{i\Delta n(\theta - i/2 \sum_j \log(m_j/m_j^*))} I_{\Delta n}(2\beta(m_k m_k^*) \Omega)
\]

Taking derivatives with respect to \( m, m^* \) gives averaged integrated correlators

Spurion chiral charge +2

\[
\frac{1}{V T} \int d^4 x \langle \bar{\psi}_i P_R \psi_i \rangle_{\Delta n} = -e^{i\Delta n(\theta + \alpha)} \left( \frac{\beta}{2m_i} (I_{\Delta n+1}(2\beta \Omega) - I_{\Delta n-1}(2\beta \Omega)) \right. \\
+ m_i^* (I_{\Delta n+1}(2\beta \Omega) + I_{\Delta n-1}(2\beta \Omega)) \left. \frac{\partial}{\partial (m_i m_i^*)} \beta(m_k m_k^*) \right)
\]

Spurion chiral charge -2

\[
\frac{1}{V T} \int d^4 x \langle \bar{\psi}_i P_L \psi_i \rangle_{\Delta n} = -e^{i\Delta n(\theta + \alpha)} \left( \frac{\beta}{2m^*_i} (I_{\Delta n+1}(2\beta \Omega) - I_{\Delta n-1}(2\beta \Omega)) \right. \\
+ m_i (I_{\Delta n+1}(2\beta \Omega) + I_{\Delta n-1}(2\beta \Omega)) \left. \frac{\partial}{\partial (m_i m_i^*)} \beta(m_k m_k^*) \right)
\]
Summing over topological sectors

\[
\frac{1}{VT} \int d^4x \langle \bar{\psi}_i P_R \psi_i \rangle = \lim_{N \to \infty} \lim_{VT \to \infty} \sum_{\Delta n = -N}^N \frac{1}{VT} \int d^4x \langle \bar{\psi}_i P_R \psi_i \rangle_{\Delta n} \sum_{\Delta m = -N}^N Z_{\Delta m} = 2m_i^* \partial_{m_i m_i^*} \beta(m_k m_k^*),
\]

\[
\frac{1}{VT} \int d^4x \langle \bar{\psi}_i P_L \psi_i \rangle = \lim_{N \to \infty} \lim_{VT \to \infty} \sum_{\Delta n = -N}^N \frac{1}{VT} \int d^4x \langle \bar{\psi}_i P_L \psi_i \rangle_{\Delta n} \sum_{\Delta m = -N}^N Z_{\Delta m} = 2m_i \partial_{m_i m_i^*} \beta(m_k m_k^*).
\]

Topological classification only enforced in infinite volume, which fixes ordering

\[
\frac{1}{VT} \int d^4x \langle \bar{\psi}_i \psi_i \rangle = 2m_i e^{-i\alpha_i \gamma_5} \partial_{m_i m_i^*} \beta(m_k m_k^*).
\]

Only a single phase: no CP violation

Result also valid for more general correlators. Opposite order of limits yields CP-violation.
Conclusions
QCD with an arbitrary $\theta$ does not predict CP violation, as long as the sum over topological sectors is performed at infinite volume.

This ordering of limits is the correct one because the topological classification is only enforced for an infinite volume.

Further reading in our paper

- For local observables one can recover CP-conserving expectation values from path integrals in a finite subvolume without $\theta$ dependence.
- No conflict with nonzero topological susceptibility in the lattice and $\eta'$ mass.
Thank you!
Additional material
The **chiral Lagrangian** at lowest order has the form

\[ \mathcal{L} = f_\pi^2 \text{Tr} \partial_\mu U \partial^\mu U^\dagger + af_\pi^3 \text{Tr}MU + bf_\pi^4 \text{det}U + h.c. \]

Captures t’ Hooft vertices \( U \sim \bar{\psi} P_R \psi \sim e^{i \frac{\eta}{\sqrt{2} f_\pi}} \)

There are again **2 options compatible with spurion chiral symmetry**

\[ b \propto e^{-i\theta} \quad \text{Usual option, assumed by [Baluni, Crewther et al]} \]

\[ b \propto e^{i\alpha} = e^{i \sum_j \text{arg}(m_j)} \]

No CP violation!
No CP violation in the chiral Lagrangian

\[ \mathcal{L} = f_\pi^2 \text{Tr} \partial_\mu U \partial^\mu U^\dagger + \alpha f_\pi^3 \text{Tr} MU + |b| e^{i\xi} f_\pi^4 \text{det} U + \text{h.c.} \]

Minimizing the potential for the pions leads to

\[ \langle U \rangle = U_0 = \text{diag}(e^{i\varphi_u}, e^{i\varphi_d}, e^{i\varphi_s}) . \]

\[ m_i \varphi_i = \frac{m_u m_d m_s (\xi + \alpha_u + \alpha_d + \alpha_s)}{m_u m_d + m_d m_s + m_s m_u} = \tilde{m}(\xi + \alpha_u + \alpha_d + \alpha_s) . \]

Adding field \( N \) containing neutron and proton, the \( CP \)-violating neutron-pion interactions are of the form

\[ \frac{c_+ \tilde{m}(\xi + \alpha_u + \alpha_d + \alpha_s)}{2 f_\pi} \bar{N} \Phi N \]

\( (\Phi \text{ containing } U, U^\dagger \text{ and gamma matrices}) \text{ which cancel for } \xi = -\alpha \text{ leading to no } CP \text{ violation} \)
Baluni’s CP-violating effective Lagrangian

Baluni’s CP-violating Lagrangian (used by [Crewther et al]) is based on searching for field redefinitions that minimize the QCD mass term

\[ \mathcal{L}_M(U_{R,L}) = \overline{\psi}_R U_R^\dagger M U_L \psi_L + \text{h.c.,} \quad U_{R,L} \in SU_{R,L}(3) \]

\[ \langle 0 | \delta \mathcal{L} | 0 \rangle = \min_{U_{R,L}} \langle 0 | \mathcal{L}_M(U_{R,L}) | 0 \rangle \]

However, there is an extra assumption: that the phase of the fermion condensate is aligned with \( \theta \)

\[ \langle \overline{\psi}_R \psi_L \rangle = \Delta e^{i \epsilon \theta} \]

This assumption does not hold for the chiral Lagrangian with \( \xi = -\alpha \) as seen in previous slide
The $\eta'$ mass

Chiral Lagrangian with alignment in the phases of mass terms and anomalous terms still predicts a **nonzero value of the $\eta'$ mass**

$$\mathcal{L} = f_\pi^2 \text{Tr} \partial_\mu U \partial^\mu U^\dagger + a f_\pi^3 \text{Tr} M U + |b| e^{i \arg \det M} f_\pi^4 \det U + \text{h.c.}$$

$$m_{\eta'}^2 = 8 |b| f_\pi^2$$

Can be seen to be **proportional** to the **topological susceptibility** over **finite volumes** of the **pure gauge theory**, in line with [Witten, Di Vecchia & Veneziano]

**Classic arguments linking topological susceptibility to CP violation** ([Shifman et al]) rely on analytic expansions in $\theta$ which **don’t apply** with our limiting procedure

$Z$ becomes **non-analytic in $\theta$**. This possibility has been mentioned by [Witten]
the physics is of order $e^{-N}$, contrary to the basic assumptions of this paper, or else the physics is non-analytic as a function of $\theta$. In the latter case, which is quite plausible, the singularities would probably be at $\theta = \pm \pi$, as Coleman found for the massive Schwinger model [10]. It is also quite plausible that $\theta$ is not really an angular variable.)

To write a formal expression for $d^2E/d\theta^2$, let us think of the path integral formulation of the theory:

$$Z = \int dA_\mu \exp i \int \text{Tr} \left[ -\frac{1}{4} F_{\mu\nu} + \frac{g^2 \theta}{16\pi^2 N} F_{\mu\nu} F_{\mu\nu} \right]. \quad (5)$$
Partition function and analyticity

Usual partition function is analytic in $\theta$

$$Z_{\text{usual}} = \lim_{VT \to \infty} \lim_{N \to \infty} \sum_{\Delta n = -N}^{N} Z_{\Delta n} = e^{2i\kappa_{N_f} VT \cos(\bar{\alpha} + \theta + N_f \pi)}$$

$\theta$-dependence of observables (giving CP violation) is usually obtained by expanding action inside path integral in powers of $\theta$. E.g.

$$\frac{\langle \Delta n \rangle}{\Omega} = i(\theta - \theta_0) \left( \frac{\langle \Delta n^2 \rangle}{\Omega} \right)_{\theta_0} + \mathcal{O}(\theta - \theta_0)^2$$

[Shifman et al]

topological susceptibility

In our limiting procedure the former is not valid, as $Z$ becomes nonanalytic in $\theta$

$$Z = \lim_{N \to \infty} \lim_{VT \to \infty} \sum_{\Delta n = -N}^{N} Z_{\Delta n} = I_0(2i\kappa_{N_f} VT) \lim_{N \to \infty} \sum_{N \in N \ |\Delta n| \leq N} e^{i\Delta n(\bar{\alpha} + \theta + N_f \pi)}$$

$\theta$ drops out from observables, there is no CP violation
Finite volumes in an infinite spacetime

Even in an infinite spacetime, we can express expectation values of local observables in terms over path integration over finite volume.

This can help make contact with lattice computations.

Assume local operator $\mathcal{O}_1$ with support in finite spacetime volume $\Omega_1$

$$
\langle \mathcal{O}_1 \rangle_{\Omega} = \frac{\sum_{\Delta n = -\infty}^{\infty} f(\Delta n) \int \mathcal{D} \phi \mathcal{O}_1 e^{-S_{\Omega}[\phi]}}{\sum_{\Delta n = -\infty}^{\infty} f(\Delta n) \int \mathcal{D} \phi e^{-S_{\Omega}[\phi]}}
$$

$$
= \frac{\sum_{\Delta n = -\infty}^{\infty} \sum_{\Delta n_1 = -\infty}^{\infty} f(\Delta n) \int \mathcal{D} \phi \mathcal{O}_1 e^{-S_{\Omega_1}[\phi]} \int \mathcal{D} \phi e^{-S_{\Omega_2}[\phi]}}{\sum_{\Delta n = -\infty}^{\infty} \sum_{\Delta n_1 = -\infty}^{\infty} f(\Delta n) \int \mathcal{D} \phi e^{-S_{\Omega_1}[\phi]} \int \mathcal{D} \phi e^{-S_{\Omega_2}[\phi]}}.
$$
Finite volumes in an infinite spacetime

Path integrations over $\Omega_2$ give just the **partition functions** we calculated before.

In the **infinite volume** limit the **Bessel functions tend to common value** and dependence on $\Delta n$ factorizes out and cancels:

$$
\langle O_1 \rangle_\Omega = \frac{\sum_{\Delta n_1 = -\infty}^{\infty} \int D\phi (-1)^{-N_f \Delta n_1} e^{-i \alpha \Delta n_1} O_1 e^{-S_{\Omega_1}[\phi]}}{\sum_{\Delta n_1 = -\infty}^{\infty} \int D\phi (-1)^{-N_f \Delta n_1} e^{-i \alpha \Delta n_1} e^{-S_{\Omega_1}[\phi]}}.
$$

We recover a **path integration** over a **finite volume, without $\theta$ dependence**.

**Extra phases** precisely cancel those from fermion determinants in.

This **removes interferences** between different **topological sectors**.
Hamiltonian is zero for pure gauge transformations, with integer $n_{\text{CS}}$: Expect degenerate pre-vacua $|n_{\text{CS}}\rangle \equiv |n\rangle$

The true vacuum $|\omega\rangle$ is a linear combination of prevacua

$$|\omega\rangle = \sum_n f(n)|n\rangle$$

Demanding invariance up to a phase under gauge transformations in the $\Delta n$ class

$$U_{\Delta n}|\omega\rangle = \sum_n f(n)|n + \Delta n\rangle = e^{i\Delta n\theta}|\omega\rangle \Rightarrow f(n) = e^{-i n \theta}$$

Constructing the partition function as a transition amplitude from $|\omega\rangle$ onto itself

$$Z(\theta) = \langle \omega | e^{-HT} | \omega \rangle = \sum_m \sum_n \langle m | e^{-HT} e^{i\theta (m-n)} | n \rangle = N \sum_{\Delta n} \langle n + \Delta n | e^{-HT} e^{i\theta \Delta n} | n \rangle$$

$$= N \sum_{\Delta n} \int_{\Delta n} \mathcal{D}\phi e^{-S_\theta + \ldots}$$
Can one use $\theta$ the vacuum at finite volume?

Bloch wave function in QM:

Too naive! Have to use path integral in infinite 4D volume to project into vacuum.