$\mathcal{T}_{13}$ Flavor Symmetry for Quarks and Leptons

Moinul Hossain Rahat

University of Florida

DISCRETE 2020-2021
December 2, 2021

Outline

- Lepton Mixing Through "Cabibbo Haze"
  - bottom-up approach using $SU(5)$ GUT with Tribimaximal mixing
    `arXiv:1805.10684, M.H.R., P. Ramond, B. Xu`
- Yukawa Texture from a Discrete Flavor Symmetry
  - based on $\mathcal{T}_{13} = \mathcal{Z}_{13} \rtimes \mathcal{Z}_3$
    `arXiv:1907.10698, M.J. Pérez, M.H.R., P. Ramond, A.J. Stuart, B. Xu`
- Tribimaximal Seesaw Mixing from $\mathcal{T}_{13}$
  - predicts "normal ordering" of light neutrino masses
    `arXiv:2001.04019, M.J. Pérez, M.H.R., P. Ramond, A.J. Stuart, B. Xu`
- Leptogenesis in the $SU(5) \times \mathcal{T}_{13}$ Model
  - both high- and low-scale scenarios
    `arXiv:2008.04204, M.H.R.`; `arXiv:2103.14691, C.S. Fong, M.H.R., S.Saad`

A Marriage between the EW and Seesaw Sectors?

Lepton mixing is quite unlike quark mixing [PDG 2021]

$$\theta_{12} = 33.65^\circ, \quad \theta_{23} = 47.64^\circ, \quad \theta_{13} = 8.53^\circ$$

"Cabibbo Haze" [Datta, Everett, Ramond 2005]

$$\mathcal{U}_{PMNS} = \underbrace{\mathcal{U}^{(-1)^\dagger}}_{\text{EW sector}} \; \underbrace{\mathcal{U}_{Seesaw}}_{\text{Seesaw sector}}$$

$\mathcal{U}_{Seesaw}$: Tribimaximal mixing [Harrison, Perkins, Scott 2002]

$$\mathcal{U}_{TBM} \equiv \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}$$

Can the Electroweak sector generate enough "Cabibbo Haze"?

Hunting the 'Minimal' Texture

Standard Model Yukawas : $\quad Y^{(2/3)}, \quad Y^{(-1/3)}, \quad Y^{(-1)}$

SU(5) : $\quad Y^{(-1/3)} \sim Y^{(-1)T}$

Symmetric $Y^{(-1/3)} \sim Y^{(-1)T} \rightarrow \quad \theta_{13} \sim 3^\circ$

Asymmetric $Y^{(-1/3)} \sim Y^{(-1)T} \rightarrow \quad \theta_{13} \sim 12^\circ$

Introduce a phase in the TBM matrix!

$$\mathcal{U}_{PMNS} = \mathcal{U}^{(-1)\dagger} \, \mathrm{diag}(1, 1, e^{i\delta}) \, \mathcal{U}_{TBM}$$

The Asymmetric Texture arXiv:1805.10684

- Diagonal $Y^{(2/3)} \sim \mathrm{diag}(\lambda^8, \lambda^4, 1)$
- Minimal asymmetry in $(13) - (31)$

$$Y^{(-\frac{1}{3})} \sim \begin{pmatrix} \lambda^4 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda & \lambda^2 & 1 \end{pmatrix}, \quad Y^{(-1)} \sim \begin{pmatrix} \lambda^4 & \lambda^3 & \lambda \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}$$

- One phase to rule them all: $|\delta| = 78^\circ$ designed to fit θ_{13} brings the two large angles within 1σ of PDG 2021 fit

Prediction for leptonic CP violation

$$|\delta_{CP}| = 1.32\pi, \qquad |J_{CP}| = 0.028$$

consistent with PDG 2021 fit, to be measured at DUNE and Hyper-K in ~ 10 years [arXiv:1807.10334, 1805.04163].

Asymmetric Texture is compatible with TBM!

Family Symmetry arXiv:1907.10698

Two properties: asymmetry of $Y^{(-1/3)} \sim Y^{(-1)}$, and diagonal $Y^{(2/3)}$

Asymmetry requires two inequivalent triplets

- $Y^{(-1/3)}$ and $Y^{(-1)}$ comes from $\bar{\mathbf{5}} \otimes \mathbf{10}$
- $SU(3)$: $3 \times 3 \Rightarrow$ either symmetric or antisymmetric
  At least two distinct triplets required!
- Candidates: $\mathcal{S}_4$ (order 24), $\Delta(27)$ (order 27), $\mathcal{T}_{13}$ (order 39), ...

Diagonals must separate from off-diagonals

- $Y^{(2/3)}$ comes from $\mathbf{10} \otimes \mathbf{10}$
- $\mathbf{3}_i \otimes \mathbf{3}_i = (\text{diagonal})_{\mathbf{r}} \oplus (\text{off-diagonal})_{\mathbf{s}}, \ \mathbf{r} \neq \mathbf{s}$

Smallest such group is $\mathcal{T}_{13} = \mathcal{Z}_{13} \rtimes \mathcal{Z}_3$

Model Building in Electroweak Sector arXiv:1907.10698

$Y^{(-\frac{1}{3})} \leftarrow \bar{\mathbf{5}} \otimes \mathbf{10} \otimes \text{Higgs} \otimes \text{Flavon(s)}$

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}_{\mathbf{3}_1} \otimes \begin{pmatrix} T_1 \\ T_3 \\ T_2 \end{pmatrix}_{\mathbf{3}_2} = \begin{pmatrix} F_3T_2 \\ F_1T_1 \\ F_2T_3 \end{pmatrix}_{\mathbf{3}_1} \oplus \begin{pmatrix} F_3T_1 \\ F_1T_3 \\ F_2T_2 \end{pmatrix}_{\bar{\mathbf{3}}_2} \oplus \begin{pmatrix} F_3T_3 \\ F_1T_2 \\ F_2T_1 \end{pmatrix}_{\mathbf{3}_2}$$

$\mathcal{T}_{13}$ can dial *individual* matrix elements!

$Y^{(\frac{2}{3})} \leftarrow \mathbf{10} \otimes \mathbf{10} \otimes \text{Higgs} \otimes \text{Flavon(s)}$

$$\begin{pmatrix} T_1 \\ T_3 \\ T_2 \end{pmatrix}_{\mathbf{3}_2} \otimes \begin{pmatrix} T_1 \\ T_3 \\ T_2 \end{pmatrix}_{\mathbf{3}_2} = \begin{pmatrix} T_3T_3 \\ T_2T_2 \\ T_1T_1 \end{pmatrix}_{\bar{\mathbf{3}}_1} \oplus \begin{pmatrix} T_3T_2 \\ T_2T_1 \\ T_1T_3 \end{pmatrix}_{\bar{\mathbf{3}}_2} \oplus \begin{pmatrix} T_2T_3 \\ T_1T_2 \\ T_3T_1 \end{pmatrix}_{\bar{\mathbf{3}}_2}$$

Diagonals are distinguished from off-diagonals!

Model Building in Seesaw Sector arXiv:2001.04019

- Introduce three right handed neutrinos $\bar{N} \sim (\mathbf{1}, \mathbf{3}_2)$
- Seesaw matrix, $\mathcal{S} = Y^{(0)}\ \mathcal{M}^{-1}\ Y^{(0)^T}$
- Both $Y^{(0)}$ and $\mathcal{M}$ are diagonalized by $\mathcal{U}_{TBM}$. How?
- Symmetric matrices diagonalized by TBM:

$$\mathcal{U}_{TBM}^T\ \mathcal{R}\ \mathcal{U}_{TBM} = -\mathrm{diag}(\alpha+\beta, \beta-2\alpha, \alpha-\beta)$$

$$\mathcal{R} = \alpha \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} + \beta \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- Both patterns arise from $\mathcal{T}_{13}$ Clebsch-Gordan coefficients!

Model Building in Seesaw Sector arXiv:2001.04019

- Introduce three right handed neutrinos $\bar{N} \sim (\mathbf{1}, \mathbf{3}_2)$
- Dirac Yukawa matrix, $Y^{(0)}$: $F\bar{N}\bar{H}_{\mathbf{5}}\varphi_{\mathcal{A}}$, $\varphi_{\mathcal{A}} \sim (\mathbf{1}, \bar{\mathbf{3}}_2)$

$$Y^{(0)} = \alpha \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} + \beta \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathcal{U}_{TBM}^T \, Y^{(0)} \, \mathcal{U}_{TBM} \sim \text{diag}(1, 1, -1)$$

- Majorana matrix, $\mathcal{M}$: $\bar{N}\bar{N}\varphi_{\mathcal{B}}$, $\varphi_{\mathcal{B}} \sim (\mathbf{1}, \mathbf{3}_2)$

$$\mathcal{M} = \alpha \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} + \beta \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathcal{U}_{TBM}^T \, \mathcal{M} \, \mathcal{U}_{TBM} \sim \text{diag}(1, -2, 1)$$

Model Building in Seesaw Sector arXiv:2001.04019

- Seesaw matrix $\mathcal{S} = \mathcal{U}_{TBM}\ \mathrm{diag}(1, -\frac{1}{2}, 1)\ \mathcal{U}_{TBM}^T$, light neutrino masses incompatible with neutrino oscillation data

- Introduce a fourth right handed neutrino $\bar{N}_4 \sim (\mathbf{1}, \mathbf{1})$ with interaction $F\bar{N}\bar{H}_{\mathbf{5}}\varphi_v$, with $\varphi_v \sim \bar{\mathbf{3}}_1$

- The phase originates from the vacuum expectation value of $\varphi_{\mathcal{A}}$ and φ_v, familons that generate $Y^{(0)}$

$$\mathcal{S} = \mathcal{U}_{TBM}(\delta)\ \mathrm{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3})\ \mathcal{U}_{TBM}^T$$

$$m_{\nu_1} < m_{\nu_2} = \frac{1}{2} m_{\nu_3}$$

Light Neutrino Masses arXiv:2001.04019

Prediction: normal ordering

$$m_{\nu_1} = 27.6, \quad m_{\nu_2} = 28.9, \quad m_{\nu_3} = 57.8 \text{ meV}$$

- $\sum_i |m_{\nu_i}| = 114.3$ meV compared to Planck: $\sum_i |m_{\nu_i}| < 120$ meV [arXiv:1807.06209].
- Combining data from Euclid and LSST to DESI and WFIRST, the error bound on $\sum_i |m_{\nu_i}|$ will be constrained to $8 - 11$ meV.
- Normal ordering is preferred above 3σ by Super-K, T2K and NOvA [arXiv:1710.09126].
- DUNE and Hyper-K will resolve the correct mass ordering beyond 5σ in $5 - 7$ yrs [arXiv:1807.10334, 1805.04163].

Neutrinoless Double Beta Decay arXiv:2001.04019

- Dirac $\not{CP}$ Jarlskog-Greenberg Invariant, $|\mathcal{J}| = 0.028$
- Majorana Invariants, $|\mathcal{I}_1| = 0.106, \quad |\mathcal{I}_2| = 0.011$

Prediction for $0\nu\beta\beta$

$$|m_{\beta\beta}| = 13.02 \text{ or } 25.21 \text{ meV}$$

compared to $|m_{\beta\beta}| \leqslant 61 - 165$ meV by KamLAND-Zen [arXiv:1605.02889]

- Our predictions are expected to be tested in several next generation experiments [J.Phys.Conf.Ser. 1390 (2019) 1, 012048]:

Experiment	Sensitivity (meV)	Experiment	Sensitivity (meV)
LEGEND	11 - 28	SNO+-II	20 - 70
nEXO	8 - 22	AMoRE-II	15 - 30
CUPID	6 - 17	PandaX-III	20 - 55

Right-Handed Neutrino Masses from Leptogenesis

- Seesaw $\mathcal{S} = Y^{(0)} \frac{1}{\mathcal{M}} Y^{(0)T}$ is ignorant of the right-handed neutrino mass scale, since

$$Y^{(0)} \sim \sqrt{M}, \qquad \mathcal{M} \sim M$$

- Right-handed neutrino masses can be determined requiring the observed baryon asymmetry is generated via leptogenesis
- Both high and low-scale leptogenesis scenarios are probable
- High-scale leptogenesis yields right-handed neutrino masses $\mathcal{O}(10^{11-12})$ GeV [arXiv:2008.04204]
- Low-scale resonant leptogenesis is viable for right-handed neutrino masses as low as a few GeV [arXiv:2103.14691]

Summary

- An $SU(5) \times \mathcal{T}_{13}$ model that covers both quarks and leptons, and is consistent with TBM seesaw mixing
- Features of the electroweak textures uniquely single out $\mathcal{T}_{13}$ flavor symmetry
- Tribimaximal seesaw mixing comes out naturally from $\mathcal{T}_{13}$
- Prediction for CP violation: $|\delta_{CP}| = 1.32\pi$
- Prediction for light neutrino masses: Normal ordering
  $m_{\nu_1} = 27.6,\ m_{\nu_2} = 28.9,\ m_{\nu_1} = 57.8$ meV
- Prediction for $0\nu\beta\beta$: $|m_{\beta\beta}| = 13.02$ or 25.21 meV

Thank You!

Backup slides

- $\mathcal{T}_{13} = \mathcal{Z}_{13} \rtimes \mathcal{Z}_3$ has two generators a and b
- Presentation: $\langle a, b \mid a^{13} = b^3 = I, bab^{-1} = a^3 \rangle$
- Order $= 13 \times 3 = 39$
- Three one dimensional irreps $\mathbf{1}$, $\mathbf{1'}$, $\bar{\mathbf{1}}'$ and four three dimensional irreps $\mathbf{3_1}$, $\bar{\mathbf{3}}_\mathbf{1}$, $\mathbf{3_2}$, $\bar{\mathbf{3}}_\mathbf{2}$

  $1^2 + 1^2 + 1^2 + 3^2 + 3^2 + 3^2 + 3^2 = 39$

- $\mathbf{1} : a = 1, b = 1$; $\mathbf{1'} : a = 1, b = \omega$; $\bar{\mathbf{1}}' : a = 1, b = \omega^2$, where $\omega = e^{i2\pi/3}$

$\mathcal{T}_{13}$ Group Theory (contd.)

- b permutes the components of the triplets
- a assigns $\mathcal{Z}_{13}$ charges:

$$\mathbf{3_1}: \quad (\rho,\, \rho^3,\, \rho^9), \qquad \mathbf{3_2}: \quad (\rho^2,\, \rho^6,\, \rho^5)$$

  where $\rho^{13} = 1$.

- Diagonal elements are naturally singled out in Kronecker products of triplets:

$$\begin{aligned}
\mathbf{3_1} \otimes \mathbf{3_1} &= \mathbf{\bar{3}_1} \oplus \mathbf{3_2} \oplus \mathbf{\bar{3}_1}, \\
\mathbf{3_2} \otimes \mathbf{3_2} &= \mathbf{\bar{3}_2} \oplus \mathbf{\bar{3}_1} \oplus \mathbf{\bar{3}_2}, \\
\mathbf{3_1} \otimes \mathbf{3_2} &= \mathbf{3_2} \oplus \mathbf{3_1} \oplus \mathbf{\bar{3}_2}.
\end{aligned}$$

$\mathcal{T}_{13}$ Group Theory (contd.)

Consider a triplet $\psi = \{\psi^1, \psi^2, \psi^3\} \sim \mathbf{3_1}$

Under the generator b,
$\{\psi^1, \psi^2, \psi^3\} \to \{\psi^2, \psi^3, \psi^1\}$ cyclic permutation of the components

Under the generator a,
$\{\psi^1, \psi^2, \psi^3\} \to \{\rho\psi^1, \rho^3\psi^2, \rho^9\psi^3\}$ the components gain $\mathcal{Z}_{13}$ charges

We can determine the Clebsch-Gordan decompositions by simply following the action of the a generator!

$\mathcal{T}_{13}$ Clebsch-Gordan Decomposition Table

$$\mathbf{3_1} \otimes \mathbf{3_1} \to \begin{pmatrix} \psi^{11} \\ \psi^{22} \\ \psi^{33} \end{pmatrix}_{\mathbf{3_2}}, \quad \begin{pmatrix} \psi^{23} \\ \psi^{31} \\ \psi^{12} \end{pmatrix}_{\mathbf{\bar{3}_1}}, \quad \begin{pmatrix} \psi^{32} \\ \psi^{13} \\ \psi^{21} \end{pmatrix}_{\mathbf{\bar{3}_1}}$$

$$\mathbf{3_2} \otimes \mathbf{3_2} \to \begin{pmatrix} \psi^{22} \\ \psi^{33} \\ \psi^{11} \end{pmatrix}_{\mathbf{\bar{3}_1}}, \quad \begin{pmatrix} \psi^{23} \\ \psi^{31} \\ \psi^{12} \end{pmatrix}_{\mathbf{\bar{3}_2}}, \quad \begin{pmatrix} \psi^{32} \\ \psi^{13} \\ \psi^{21} \end{pmatrix}_{\mathbf{\bar{3}_2}}$$

$$\mathbf{3_1} \otimes \mathbf{\bar{3}_1} \to \begin{pmatrix} \psi^{1}_{2} \\ \psi^{2}_{3} \\ \psi^{3}_{1} \end{pmatrix}_{\mathbf{\bar{3}_2}}, \quad \begin{pmatrix} \psi^{2}_{1} \\ \psi^{3}_{2} \\ \psi^{1}_{3} \end{pmatrix}_{\mathbf{3_2}}, \quad (\psi^1_1 + \psi^2_2 + \psi^3_3)_{\mathbf{1}},$$

$$(\psi^1_1 + \omega\psi^2_2 + \omega^2\psi^3_3)_{\mathbf{1'}}, \quad (\psi^1_1 + \omega^2\psi^2_2 + \omega\psi^3_3)_{\mathbf{\bar{1}'}}$$

$\mathcal{T}_{13}$ Clebsch-Gordan Decomposition Table

$$\mathbf{3_2} \otimes \bar{\mathbf{3}}_\mathbf{2} \to \begin{pmatrix} \psi_2^3 \\ \psi_3^1 \\ \psi_1^2 \end{pmatrix}_{\bar{\mathbf{3}}_\mathbf{1}}, \quad \begin{pmatrix} \psi_3^2 \\ \psi_1^3 \\ \psi_2^1 \end{pmatrix}_{\mathbf{3_1}}, \quad (\psi_1^1 + \psi_2^2 + \psi_3^3)_\mathbf{1},$$

$$(\psi_1^1 + \omega\psi_2^2 + \omega^2\psi_3^3)_{\mathbf{1}'}, \quad (\psi_1^1 + \omega^2\psi_2^2 + \omega\psi_3^3)_{\bar{\mathbf{1}}'}$$

$$\mathbf{3_1} \otimes \mathbf{3_2} \to \begin{pmatrix} \psi^{33} \\ \psi^{11} \\ \psi^{22} \end{pmatrix}_{\mathbf{3_1}}, \quad \begin{pmatrix} \psi^{31} \\ \psi^{12} \\ \psi^{23} \end{pmatrix}_{\bar{\mathbf{3}}_\mathbf{2}}, \quad \begin{pmatrix} \psi^{32} \\ \psi^{13} \\ \psi^{21} \end{pmatrix}_{\mathbf{3_2}}$$

$$\mathbf{3_1} \otimes \bar{\mathbf{3}}_\mathbf{2} \to \begin{pmatrix} \psi_1^1 \\ \psi_2^2 \\ \psi_3^3 \end{pmatrix}_{\bar{\mathbf{3}}_\mathbf{1}}, \quad \begin{pmatrix} \psi_3^2 \\ \psi_1^3 \\ \psi_2^1 \end{pmatrix}_{\bar{\mathbf{3}}_\mathbf{2}}, \quad \begin{pmatrix} \psi_1^2 \\ \psi_2^3 \\ \psi_3^1 \end{pmatrix}_{\mathbf{3_1}}$$

$\mathcal{T}_{13}$ Clebsch-Gordan Decompositions

$\mathbf{1}' \otimes \mathbf{1}' = \bar{\mathbf{1}}'$
$\bar{\mathbf{1}}' \otimes \bar{\mathbf{1}}' = \mathbf{1}'$
$\mathbf{1}' \otimes \bar{\mathbf{1}}' = \mathbf{1}$
$\mathbf{3_1} \otimes \mathbf{3_1} = \bar{\mathbf{3}}_\mathbf{1} \oplus \bar{\mathbf{3}}_\mathbf{1} \oplus \mathbf{3_2}$
$\mathbf{3_2} \otimes \mathbf{3_2} = \bar{\mathbf{3}}_\mathbf{2} \oplus \bar{\mathbf{3}}_\mathbf{1} \oplus \bar{\mathbf{3}}_\mathbf{2}$
$\mathbf{3_1} \otimes \bar{\mathbf{3}}_\mathbf{1} = \mathbf{1} \oplus \mathbf{1}' \oplus \bar{\mathbf{1}}' \oplus \mathbf{3_2} \oplus \bar{\mathbf{3}}_\mathbf{2}$
$\mathbf{3_2} \otimes \bar{\mathbf{3}}_\mathbf{2} = \mathbf{1} \oplus \mathbf{1}' \oplus \bar{\mathbf{1}}' \oplus \mathbf{3_1} \oplus \bar{\mathbf{3}}_\mathbf{1}$
$\mathbf{3_1} \otimes \mathbf{3_2} = \bar{\mathbf{3}}_\mathbf{2} \oplus \mathbf{3_1} \oplus \mathbf{3_2}$
$\mathbf{3_1} \otimes \bar{\mathbf{3}}_\mathbf{2} = \bar{\mathbf{3}}_\mathbf{2} \oplus \mathbf{3_1} \oplus \bar{\mathbf{3}}_\mathbf{1}$
$\mathbf{3_2} \otimes \bar{\mathbf{3}}_\mathbf{1} = \mathbf{3_2} \oplus \mathbf{3_1} \oplus \bar{\mathbf{3}}_\mathbf{1}$

Table 1: $Z_{13} \rtimes Z_3$ Kronecker products

Why do F and T transform differently under $\mathcal{T}_{13}$?

$$SU(5)\times\mathcal{T}_{13}: \quad \overbrace{(\mathbf{10},\mathbf{3_2})}^{T}\oplus\overbrace{(\bar{\mathbf{5}},\mathbf{3_1})}^{F}\oplus\overbrace{(\mathbf{1},\mathbf{3_2})}^{\bar{N}}\oplus\overbrace{(\mathbf{1},\mathbf{1})}^{\bar{N}_4}$$

$$SO(10)\times\mathcal{T}_{13}: \qquad (\mathbf{16},\mathbf{3_2})\oplus(\mathbf{10},\mathbf{3_1})\oplus(\mathbf{1},\mathbf{1}).$$

$$\mathbf{16}=\mathbf{10_{-1}}\oplus\bar{\mathbf{5}}_{\mathbf{3}}\oplus\mathbf{1_{-5}}, \qquad \mathbf{10}=\mathbf{5_3}\oplus\bar{\mathbf{5}}_{\mathbf{-3}},$$

$$E_6\supset SO(10)\times U(1): \quad \mathbf{27}=\mathbf{16_1}\oplus\mathbf{10_{-2}}\oplus\mathbf{1_4},$$

Does the $U(1)$ charge dictate which is $\mathbf{3}_1$ and which is $\mathbf{3}_2$?