Anomalous Currents and Constitutive Relations of a Chiral Hadronic Superfluid

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Based on: J.L.Mañes, EM, M.Valle, M.A.V.M, JHEP1811 ('18), JHEP1912 ('19).

Other references: K.Landsteiner, EM, F.Pena-Benitez, PRL107('11);

Lect. Notes Phys. 81 ('13); EM, M.Valle, JHEP1411 ('14).

Issues

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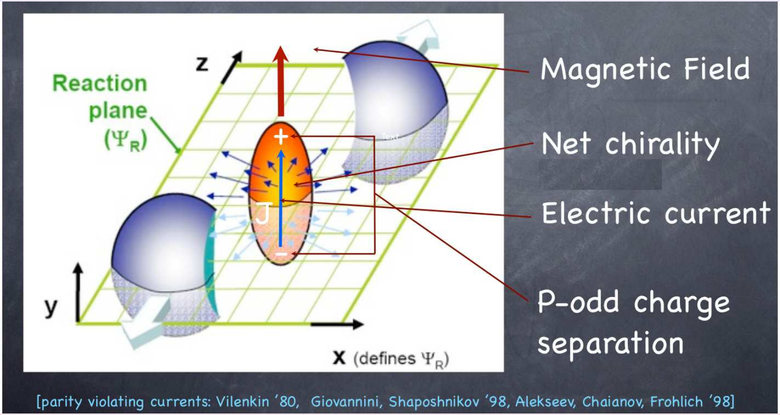
The Chiral Magnetic Effect (CME)

[Kharzeev, McLerran, Warringa '07]



Strong Magnetic field induces a $\mathcal{P}$-odd charge separation $\Longrightarrow$
$\Longrightarrow$ Electric current: $\vec{J} = \sigma^{\mathcal{B}} \vec{B}$.

Issues

Hydrodynamics of Relativistic Fluids

[Son,Surowka], [Eling,Neiman,Oz], [Erdmenger et al.], [Banerjee et al.], [Loganayagam], [Kharzeev, Yee], [Sadovyev et al.], [Landsteiner, EM, Pena-Benitez], . . .

$$\langle T^{\mu\nu}\rangle = \underbrace{(\varepsilon+\mathcal{P})u^\mu u^\nu + \mathcal{P}g^{\mu\nu}}_{\text{Ideal Hydro}} + \underbrace{\langle T^{\mu\nu}\rangle_{\text{diss \& anom}}}_{\text{Dissipative \& Anomalous}} ,$$

$$\langle J^\mu\rangle = \underbrace{nu^\mu}_{\text{Ideal Hydro}} + \underbrace{\langle J^\mu\rangle_{\text{diss \& anom}}}_{\text{Dissipative \& Anomalous}} .$$

- Landau frame: $\langle T^{0i}\rangle \sim u^i$

$$\langle T^{\mu\nu}\rangle_{\text{diss \& anom}} = -\eta P^{\mu\alpha}P^{\nu\beta}\left(D_\alpha u_\beta + D_\beta u_\alpha - \frac{2}{3}g_{\alpha\beta}D^\lambda u_\lambda\right) - \zeta P^{\mu\nu}D^\alpha u_\alpha + \cdots$$

$$\langle J^\mu\rangle_{\text{diss \& anom}} = -\sigma T P^{\mu\nu}D_\nu\left(\frac{\mu}{T}\right) + \sigma E^\mu + \underbrace{\sigma^{\mathcal{B}}B^\mu}_{\text{CME}} + \underbrace{\sigma^{\mathcal{V}}\omega^\mu}_{\text{CVE}} + \cdots$$

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, and
vorticity: $\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}u_\nu D_\rho u_\lambda$ → Chiral Vortical Effect (CVE).

Parity and Time Reversal Properties

- Hydrodynamics at 1st order in derivative expansion:

$$\text{CME}: \quad \underbrace{\langle\vec{J}\rangle_1}_{\mathcal{P}-odd,\mathcal{T}-odd} = \underbrace{\sigma^{\mathcal{B}}}_{\mathcal{P}-odd,\mathcal{T}-even} \cdot \underbrace{\vec{B}}_{\mathcal{P}-even,\mathcal{T}-odd}$$

$$\text{CVE}: \quad \overbrace{\langle\vec{J}\rangle_1} = \overbrace{\sigma^{\mathcal{V}}} \cdot \overbrace{\vec{\omega}}$$

$\mathcal{T}-even \Longrightarrow \sigma^{\mathcal{B}}$ and $\sigma^{\mathcal{V}}$ are non dissipative, i.e. they cannot contribute to entropy production:

$$\frac{\partial}{\partial t}s > 0 \qquad \text{(Only } \mathcal{T}\text{-odd contributions in } s\text{)}$$

Electric conductivity is dissipative: $\langle\vec{J}\rangle_1 = \sigma\vec{E} \Longrightarrow \partial_t s > 0$.

Issues

Equilibrium Partition Function

[Banerjee et al '12], [Jensen et al '13], [Bhattacharyya '14], [EM, Valle '14]

- Relativistic Invariant Quantum Field Theory on the manifold

$$ds^2 = -e^{2\sigma(\vec{x})}(dt + a_i(\vec{x})dx^i)^2 + g_{ij}(x)dx^i dx^j$$

  and time independent background $U(1)$ gauge connection:

$$\mathcal{A} = \mathcal{A}_0(\vec{x})dx^0 + \mathcal{A}_i(\vec{x})dx^i \, .$$

- Partition function of the system:

$$Z = \mathrm{Tr}\, e^{-\frac{H-\mu_0 Q}{T_0}}$$

  ⟶ Dependence of Z on σ, g_{ij} and a_i?

- Most general partition function consistent with:
  - 3-dim diffeomorphism invariance.
  - Kaluza-Klein (KK) invariance: $t \to t + \phi(\vec{x})\,,\ \vec{x} \to \vec{x}$.
  - $U(1)$ *time-independent* gauge invariance (up to an anomaly).

Equilibrium Partition Function

- Stress Tensor and $U(1)$ current ➞ under t-indep variations

$$\delta \log Z = \frac{1}{T_0} \int d^3x \sqrt{-g_3} \left(-\frac{1}{2} T_{\mu\nu} \delta g^{\mu\nu} + J^\mu \delta \mathcal{A}_\mu \right)$$

➞ $T_{\mu\nu} = -2 T_0 \frac{\delta \log Z}{\delta g^{\mu\nu}}$, $\qquad J^\mu = T_0 \frac{\delta \log Z}{\delta \mathcal{A}_\mu}$.

- In particular, for $\log Z = \mathcal{W}(e^\sigma, A_0, a_i, A_i, g^{ij}, T_0, \mu_0)$, where

$$A_0 = \mathcal{A}_0 \, , \qquad A_i = \mathcal{A}_i - a_i \mathcal{A}_0 \, ,$$

are KK invariant quantities, one gets the consistent currents

$$\langle J^i \rangle_{\rm cons} = \frac{T_0}{\sqrt{-G}} \frac{\delta \mathcal{W}}{\delta A_i} \, , \qquad \langle J_0 \rangle_{\rm cons} = -\frac{T_0 e^{2\sigma}}{\sqrt{-G}} \frac{\delta \mathcal{W}}{\delta A_0} \, ,$$

$$\langle T_0{}^i \rangle = \frac{T_0}{\sqrt{-G}} \left(\frac{\delta \mathcal{W}}{\delta a_i} - A_0 \frac{\delta \mathcal{W}}{\delta A_i} \right) \, , \qquad \langle T_{00} \rangle = -\frac{T_0 e^{2\sigma}}{\sqrt{-G}} \frac{\delta \mathcal{W}}{\delta \sigma} \, .$$

- ➞ $\mathcal{W}$ is a generating functional for the hydrodynamic constitutive relations.

Issues

Equilibrium Partition Function at 0th order

- Most general partition function up to 0th order in derivatives:

$$\log Z = \mathcal{W}_0 = \int d^3x \sqrt{g_3} \frac{e^{\sigma}}{T_0} \underbrace{P(T_0 e^{-\sigma}, e^{-\sigma} A_0)}_{\text{Arbitrary function of 2 variables}} \,, \qquad A_0 = \mathcal{A}_0 + \mu_0 \,.$$

- Constitutive relations ($a \equiv e^{-\sigma} T_0 \,,\; b \equiv e^{-\sigma} A_0$):

$$\langle T^{ij} \rangle = P g^{ij} \,, \qquad \langle T_{00} \rangle = e^{2\sigma}(P - a\partial_a P - b\partial_b P) \,, \qquad \langle T_0^i \rangle = 0 \,,$$
$$\langle J^0 \rangle = e^{-\sigma} \partial_b P \,, \qquad \langle J^i \rangle = 0 \,.$$

- By comparison with the hydrodynamic constitutive relations:

$$\langle T^{\mu\nu} \rangle = (\varepsilon + \mathcal{P}) u^\mu u^\nu + \mathcal{P} g^{\mu\nu} \,, \qquad \langle J^\mu \rangle = \rho u^\mu \,,$$

one gets

$$u^\mu = e^{-\sigma}(1, 0, \cdots, 0) \,,$$
$$\mathcal{P} = P \,, \qquad \varepsilon = -P + a\partial_a P + b\partial_b P \,, \qquad \rho = \partial_b P \,.$$

→ ε, $\mathcal{P}$ and ρ are not independent functions, but are determined in terms of a single *master* function.

Equilibrium Partition Function at first order

- Partition function at 1st order in derivative expansion **[Banerjee et al '12; EM, M.Valle '14]**:

$$\mathcal{W}^{(1)} = \int d^3x \sqrt{g_3} \left[\alpha_1(\sigma, A_0) \epsilon^{ijk} A_i A_{jk} + \alpha_2(\sigma, A_0) \epsilon^{ijk} A_i f_{jk} + \alpha_3(\sigma, A_0) \epsilon^{ijk} a_i f_{jk} \right]$$

where $A_{ij} = \partial_i A_j - \partial_j A_i$, $f_{ij} = \partial_i a_j - \partial_j a_i$.

- Ideal gas of Dirac fermions ⟶ from a computation of $\langle T_0{}^i \rangle$ and $\langle J^i \rangle$, one gets

$$\alpha_1(\sigma, A_0) = \frac{C}{6T_0} A_0 \, , \quad \alpha_2(\sigma, A_0) = \frac{1}{2} \left(\frac{C}{6T_0} A_0^2 + C_2 T_0 \right) \, , \quad \alpha_3(\sigma, A_0) = 0 \, .$$

where: $C = -\frac{1}{4\pi^2}$ *(chiral anomaly)*: **[Son, Surowka '09], [Erdmenger et al '09], . . .**

$C_2 = \frac{1}{24}$ *(gauge-gravitational anomaly)*: **[Landsteiner, EM, Pena-Benitez '11]**

- Coefficients related to chiral magnetic $\sigma^{\mathcal{B}}$ and chiral vortical $\sigma^{\mathcal{V}}$ conductivities.

Issues

The chiral anomaly

- Under a general shift $\mathcal{A}_\mu^a \to \mathcal{A}_\mu^a + \delta\mathcal{A}_\mu^a$:

$$\delta\Gamma[\mathcal{A}] = \int d^D x \, \delta\mathcal{A}_\mu^a \, J_a^\mu(x)_{\rm cons} \, . \qquad (1)$$

- The anomaly is given by the *"failure of the effective action to be invariant under axial gauge transformations"*:

$$\mathcal{A}_\mu \longrightarrow U\mathcal{A}_\mu U^{-1} - i\partial_\mu U \, U^{-1} \, , \qquad U = \exp\left(i\Lambda_a^{\rm Axial} t_a\right) \, ,$$

$$\delta_{\rm gauge}\Gamma[\mathcal{A}] = -\int d^D x \, \Lambda^{{\rm Axial},\, a} \, G_a[\mathcal{A}] \, .$$

- Particularizing **(1)** to $\delta\mathcal{A}_\mu^a = (D_\mu\Lambda^{\rm Axial})^a$ ➞
➞ (non)-conservation law for the consistent current:

$$D_\mu J_a^\mu(x)_{\rm cons} = G_a[\mathcal{A}(x)] \, .$$

The non-abelian anomaly

- Bardeen form of the non-abelian anomaly for the symmetry group $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$ **[W.A. Bardeen, PR184, '69]**:

$$G_a[\mathcal{V}, \mathcal{A}] = \frac{iN_c}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}\Big\{ \lambda_a \Big[\mathcal{V}_{\mu\nu}\mathcal{V}_{\rho\sigma} + \tfrac{1}{3}\mathcal{A}_{\mu\nu}\mathcal{A}_{\rho\sigma} - \tfrac{32}{3}\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma + \tfrac{8}{3} i (\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{V}_{\rho\sigma} + \mathcal{A}_\mu\mathcal{V}_{\rho\sigma}\mathcal{A}_\nu + \mathcal{V}_{\rho\sigma}\mathcal{A}_\mu\mathcal{A}_\nu) \Big] \Big\} .$$

where

$$\mathcal{V}_{\mu\nu} = \partial_\mu \mathcal{V}_\nu - \partial_\nu \mathcal{V}_\mu - i[\mathcal{V}_\mu, \mathcal{V}_\nu] - i[\mathcal{A}_\mu, \mathcal{A}_\nu] ,$$
$$\mathcal{A}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i[\mathcal{V}_\mu, \mathcal{A}_\nu] - i[\mathcal{A}_\mu, \mathcal{V}_\nu] .$$

- This includes *triangle*, *square* and *pentagon* one-loop diagrams.
- The anomaly arises from the breaking of gauge invariance under *'axial'* gauge transformations of the effective action $\Gamma_0[\mathcal{V}, \mathcal{A}]$:

$$\mathscr{Y}_a(x)\Gamma_0[\mathcal{V}, \mathcal{A}] = 0 , \qquad \mathscr{X}_a(x)\Gamma_0[\mathcal{V}, \mathcal{A}] = G_a[\mathcal{V}, \mathcal{A}] . \qquad (2)$$

$(\mathscr{Y}_a(x), \mathscr{X}_a(x)) \equiv$ Local generator of (vector, axial) transform.

- Computation of $\Gamma_0[\mathcal{V}, \mathcal{A}]$ from a solution of **(2)** by trial and error.

Issues

The anomalous functional

- Solution of the local functional $\Gamma_0[V, A, G]$ **[Mañes,EM,Valle,MAVZ'18]**:

$$\Gamma_0[V,A,G] = -\frac{N_c}{32\pi^2}\int dt\, d^3x \sqrt{g}\, \epsilon^{ijk}\, \mathrm{Tr}\left\{ \frac{32}{3} i\, V_0 A_i A_j A_k + \frac{4}{3}(A_0 A_i + A_i A_0) A_{jk} + 4(V_0 A_i + A_i V_0) V_{jk} + \frac{8}{3}(A_0^2 + 3V_0^2) A_i \partial_j a_k \right\} + C_2 T_0^2 \int dt\, d^3x \sqrt{g}\, \epsilon^{ijk} \mathrm{Tr} A_i\, \partial_j a_k$$

- V_μ and A_μ are KK inv fields: $A_0 = \mathcal{A}_0$, $A_i = \mathcal{A}_i - a_i \mathcal{A}_0$, etc.
- Γ_0 can be determined also from differential geometry methods: Chern-Simons effective action ⟶ dimensional reduction **[Jensen, Laganayagam, Yarom '14; Mañes, EM, Valle, MAVZ '18 '19]**.
- C_2 related to the mixed gauge-gravitational anomaly $\sim \mathrm{Tr}\{t_a\}\, \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\lambda\kappa} R_{\rho\sigma}{}^{\lambda\kappa}$ ⟶ One should take into account the Riemann tensor contribution in the anomaly polynomial $\sim \mathrm{Tr}\, \mathcal{F}_A R^\mu{}_\nu R^\nu{}_\mu$ **[Nair,Ray,Roy PRD86 '12]**.

Currents in equilibrium

- The covariant currents are defined by adding to the consistent currents the Bardeen-Zumino (BZ) polynomials:

$$J^\mu_{\rm cov} = J^\mu_{\rm cons} + J^\mu_{\rm BZ} \,,$$

with

$$J^\mu_{{\rm BZ}\,A} = \frac{N_c}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}{\rm Tr}\Big\{\lambda_a\left(\mathcal{A}_\nu\mathcal{A}_{\rho\sigma} + \mathcal{A}_{\rho\sigma}\mathcal{A}_\nu\right)\Big\},$$

$$J^\mu_{{\rm BZ}\,V} = \frac{N_c}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}{\rm Tr}\Big\{\lambda_a\big(\mathcal{A}_\nu\mathcal{V}_{\rho\sigma} + \mathcal{V}_{\rho\sigma}\mathcal{A}_\nu + \tfrac{8}{3}i\,\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\big)\Big\}.$$

- From $W_0 = i\Gamma_0$ the covariant currents and stress tensor are:

$$\langle J^i_{5a}\rangle_{\rm cov} = -\frac{N_c}{8\pi^2}e^\sigma\epsilon^{ijk}{\rm Tr}\Big\{t_a\big[(A_0A_{jk} + A_{jk}A_0) + (V_0V_{jk} + V_{jk}V_0) + 2(A_0^2 + V_0^2)\partial_j a_k\big]\Big\}, \qquad \langle J_{5a\,0}\rangle_{\rm cov} = 0\,, \qquad t_a = \frac{\lambda_a}{2}\,,$$

$$\langle T_0{}^i\rangle = \frac{N_c}{8\pi^2}e^{-\sigma}\epsilon^{ijk}\,{\rm Tr}\Big\{(A_0^2 + V_0^2)(\mathbb{F}_A)_{jk} + (V_0A_0 + V_0A_0)(\mathbb{F}_V)_{jk} + \left(\frac{2}{3}A_0^3 + 2A_0V_0^2\right)\partial_j a_k\Big\}, \qquad \langle T_{00}\rangle = \langle T^{ij}\rangle = 0\,.$$

Constitutive relations

- Maximal number of chemical potentials to be consistently introduced = dimension of the Cartan subalgebra. Let us consider the background ($N_f = 2$):

$$V_\mu(x) = V_{0\,\mu}(x) t_0 + V_{3\,\mu}(x) t_3 \,, \qquad t_0 = \tfrac{1}{2} 1_{2\times 2} \,, \; t_3 = \tfrac{1}{2}\sigma_3$$
$$A_0 = A_{5\,0}\, t_0 \,, \qquad A_i = 0 \,, \qquad \mathcal{A}_i = a_i A_{5\,0}\, t_0 \,.$$

- Equilibrium velocity field and equilibrium baryonic, isospin and axial chemical potentials:

$$u_\mu = -e^\sigma (1, a_i) \,, \quad \mu_0 = \mathcal{V}_{0\,0}\, e^{-\sigma} \,, \quad \mu_3 = \mathcal{V}_{3\,0}\, e^{-\sigma} \,, \quad \mu_5 = \mathcal{A}_{5\,0}\, e^{-\sigma} \,.$$

  μ_5 controls chiral imbalance **[Gatto, Ruggieri '12; Planells et al, '13].**
- Distinguish between three currents:

$$J^\mu_{\text{electromagnetic}} = e\bar{\Psi}\gamma^\mu Q \Psi = \frac{e}{3}\bar{\Psi}\gamma^\mu t_0 \Psi + e\bar{\Psi}\gamma^\mu t_3 \Psi \,,$$
$$J^\mu_{\text{baryonic}} = \frac{2}{3} e\bar{\Psi}\gamma^\mu t_0 \Psi \,,$$
$$J^\mu_{\text{isospin}} = e\bar{\Psi}\gamma^\mu t_3 \Psi \,.$$

Constitutive relations

- Constitutive relations → Out-of-equilibrium expressions → Covariantization of the equilibrium currents:

$$J^\mu_{0\,\mathrm{cov}} = \frac{N_c}{16\pi^2}\mu_5\epsilon^{\mu\nu\lambda\rho}u_\nu\mathcal{V}_{0\,\lambda\rho}, \qquad J^\mu_{3\,\mathrm{cov}} = \frac{N_c}{16\pi^2}\mu_5\epsilon^{\mu\nu\lambda\rho}u_\nu\mathcal{V}_{3\,\lambda\rho},$$

$$J^\mu_{50\,\mathrm{cov}} = \frac{N_c}{16\pi^2}\epsilon^{\mu\nu\lambda\rho}u_\nu\left(\mu_0\mathcal{V}_{0\,\lambda\rho} + \mu_3\mathcal{V}_{3\,\lambda\rho} + (\mu_0^2+\mu_3^2-\mu_5^2)\partial_\lambda u_\rho\right),$$

$$J^\mu_{53\,\mathrm{cov}} = \frac{N_c}{16\pi^2}\epsilon^{\mu\nu\lambda\rho}\left(\mu_3 u_\nu\mathcal{V}_{0\,\lambda\rho} + \mu_0 u_\nu\mathcal{V}_{3\,\lambda\rho} + 2\mu_0\mu_3 u_\nu\partial_\lambda u_\rho\right).$$

- Covariant form of the stress tensor $T^{\mu\nu} = u^\mu q^\nu + u^\nu q^\mu$:

$$q^\mu = \frac{N_c}{16\pi^2}\epsilon^{\mu\nu\lambda\rho}u_\nu\mu_5\left[\mu_0\mathcal{V}_{0\,\lambda\rho} + \mu_3\mathcal{V}_{3\,\lambda\rho} + \left(\mu_0^2+\mu_3^2-\frac{1}{3}\mu_5^2\right)\partial_\lambda u_\rho\right].$$

- Chiral Magnetic and Chiral Vortical contributions.

See also [Neiman, Oz, JHEP03 '11].

Issues

Wess-Zumino-Witten partition function

[Kaiser ’01], [Son, Stephanov ’08], [Fukushima, Mameda ’12], [Brauner, Kadam ’17], . . .

- Effects of the anomaly when symmetry is spontaneously broken.
- The WZW action describes the interactions of the Goldstone bosons among them and with the background fields.
- Application to QCD in the confined phase → Hadronic fluids → Fluid of pions, kaons, . . . interacting with external EM fields.
- WZW action:

$$\Gamma^{WZW}[\xi, A] = \Gamma_0[A] - \Gamma_0[A_{-\xi}]$$

  $\Gamma_0 \equiv$ anomalous functional in absence of symmetry breaking (computed above).
  $A_{-\xi} \equiv$ gauge field transformed with gauge parameters $\Lambda_a = -\xi_a$.
  [Wess,Zumino ’71; Witten ’83; Mañes ’85; Chu, Ho, Zumino ’96].
- $\xi_a \equiv$ Goldstone bosons

$$U(\xi) = \exp\left(2i\sum_a \xi_a t_a\right) .$$

Goldstone bosons

- Spontaneous breaking of axial symmetry:

$$\mathrm{U}(2)_L \times \mathrm{U}(2)_R \to \mathrm{U}(2)_V .$$

- The matrix of Goldstone bosons (GB)

$$U(\xi) = \exp\left(2i \sum_{a=1}^{3} \xi_a t_a\right) ,$$

includes three GBs from the broken $\mathrm{SU}(2)_A$ symmetry. The fourth GB ξ_0 is absent, as the $\mathrm{U}(1)_A$ sym is violated by non-pert effects.

- In terms of conventionally normalized GB fields $\pi^0, \pi^\pm$:

$$2 \sum_a \xi_a t_a = \frac{\sqrt{2}}{f_\pi} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 & \pi^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 \end{pmatrix} ,$$

where $f_\pi \approx 92\,\mathrm{MeV}$ is the pion decay constant.

- Lagrangian to zeroth order

$$\mathcal{L} = -\frac{f_\pi^2}{4} G^{\mu\nu} \mathrm{Tr}\left\{ D_\mu U (D_\nu U)^\dagger \right\} , \qquad \xi_\mu := D_\mu U = \partial_\mu U - i[\mathcal{V}_{3\,\mu} t_3, U] .$$

The lowest order description

- The lowest order description is

$$T_0 W = \int d^3x \sqrt{g} e^{\sigma} \left(P_0(\mu_0, \mu_3, T) + \mathcal{L} \right) ,$$

with

$$\mathcal{L} = \frac{f_\pi^2}{4} \left(e^{-2\sigma} \mathrm{Tr}(\xi_0 \xi_0^\dagger) - g^{ij} \mathrm{Tr}(\Phi_i \Phi_j^\dagger) \right) ,$$

and

$$\xi_0 = -i \mathcal{V}_{3\,0}[t_3, U] , \quad \text{and} \quad \Phi_i = \partial_i U - i\, V_{3\,i}[t_3, U] .$$

- Constitutive relations at the lowest order:

$$(J_{0\,\mu})_{\text{perfect}} = n_0\, u_\mu ,$$

$$(J_{3\,\mu})_{\text{perfect}} = n_3\, u_\mu + i \frac{f_\pi^2}{4} \mathrm{Tr} \left\{ [t_3, U] \partial_\mu U^\dagger + [t_3, U^\dagger] \partial_\mu U \right\} + \frac{f_\pi^2}{2} \mathcal{V}_{3\,\mu} \mathrm{Tr} \left\{ [t_3, U][t_3, U^\dagger] \right\}$$

$$T^{\mu\nu}_{\text{perfect}} = (\varepsilon + P) u^\mu u^\nu + P G^{\mu\nu} + \frac{f_\pi^2}{2} G^{\mu\rho} G^{\nu\lambda} \mathrm{Tr} \left\{ D_{(\rho} U (D_{\lambda)} U)^\dagger \right\} .$$

- $u^\mu = e^{-\sigma}(1,0,0,0)$ ➞ Equilibrium currents.

Wess-Zumino-Witten partition function

- Spontaneous breaking of axial symmetry:

$$\mathrm{U}(2)_L \times \mathrm{U}(2)_R \to \mathrm{U}(2)_V .$$

Suitable gauge transformation is an axial transformation with parameters $\Lambda_a^{\mathrm{Axial}}(x) = -\xi_a(x)$.

- WZW partition function:

$$\begin{aligned} T_0 W^{WZW} = & \frac{N_c}{8\pi^2} \int d^3x \sqrt{g}\, V_{0\,0}\, \epsilon^{ijk} \left[-\tfrac{1}{2} \mathrm{Tr}\{\partial_i (R_j + L_j) t_3\}\, V_{3\,k} + \tfrac{1}{6} i\, \mathrm{Tr}\{L_i L_j L_k\} \right] \\ & + \frac{N_c}{16\pi^2} \int d^3x \sqrt{g}\, \epsilon^{ijk} \mathrm{Tr}\{(R_i + L_i) t_3\} \\ & \times \left(V_{0\,0}\, \partial_j V_{3\,k} + V_{3\,0}\, \partial_j V_{0\,k} + V_{0\,0} V_{3\,0}\, \partial_j a_k \right) \\ & + \frac{N_c}{48\pi^2} A_{5\,0} \int d^3x \sqrt{g}\, \epsilon^{ijk} \left[\mathrm{Tr}\{(L_i - R_i) t_3\} + 2\mathrm{Tr}\{t_3^2 - U t_3 U^{-1} t_3\} V_{3\,i} \right] \\ & \times \left(\partial_j V_{3\,k} + V_{3\,0} \partial_j a_k \right) \end{aligned}$$

where $\quad L_j = i\partial_j U\, U^{-1}, \quad R_j = iU^{-1}\partial_j U.$

Constitutive relations in hydrododynamics

- Contribution to the constitutive relations:

$$T^{\mu\nu} = T^{\mu\nu}_{\rm perfect} + \underbrace{\pi^{\mu\nu}}_{\text{Dissipative \& Anomalous}} , \qquad J^{\mu} = J^{\mu}_{\rm perfect} + \underbrace{\delta J^{\mu}}_{\text{Dissipative \& Anomalous}} .$$

- Consistent expansion ➞

$$T^{\mu\nu}_{\rm perfect}(\mu_{a0}+\delta\mu_a, T_0+\delta T, \dots) , \quad J^{\mu}_{\rm perfect}(\mu_{a0}+\delta\mu_a, T_0+\delta T, \dots).$$

- Choose the frame in which the first derivative corrections to the fluid quantities vanish, i.e. $\delta\mu_a^{(1)} = 0$, $\delta T^{(1)} = 0$ and $\delta u^{\mu\,(1)} = 0$
- In this frame

$$\pi^{\mu\nu} = 0 ,$$

$$\delta J^{\mu} = -\underbrace{(\delta J^{\nu} u_{\nu})}_{\text{Longitudinal}} u^{\mu} + \underbrace{P^{\mu}{}_{\nu}\delta J^{\nu}}_{\text{Transverse}} ,$$

where $P^{\mu\nu} = G^{\mu\nu} + u^{\mu}u^{\nu}$.

Constitutive relations at first order in derivatives

- Written in terms of 6 pseudo-scalars:

$$S_1 = u_\lambda \epsilon^{\lambda\nu\rho\sigma} \mathcal{I}_\nu \mathcal{V}_{3\,\rho\sigma}\,, \qquad S_2 = u_\lambda \epsilon^{\lambda\nu\rho\sigma} \mathcal{I}_\nu \partial_\rho u_\sigma\,,$$

$$S_3 = u_\lambda \epsilon^{\lambda\nu\rho\sigma}\left[-\frac{1}{2}\partial_\nu\left(\mathrm{Tr}\{(L_\rho + R_\rho)t_3\}\right)\mathcal{V}_{3\,\sigma} + \frac{i}{6}\mathrm{Tr}\{L_\nu L_\rho L_\sigma\}\right]$$

$$S_{4(a)} = u_\lambda \epsilon^{\lambda\nu\rho\sigma} \mathcal{T}_\nu \mathcal{V}_{a\,\rho\sigma}\,, \quad a = 0,3, \qquad S_5 = u_\lambda \epsilon^{\lambda\nu\rho\sigma} \mathcal{T}_\nu \partial_\rho u_\sigma\,,$$

- and 8 pseudo-vectors:

$$P_1^\mu = \epsilon^{\mu\nu\lambda\sigma} u_\nu \mathcal{I}_\lambda \partial_\sigma\left(\frac{\mu_3}{T}\right), \qquad P_2^\mu = \epsilon^{\mu\nu\lambda\sigma} u_\nu \partial_\lambda \mathcal{I}_\sigma\,,$$

$$P_{3(a)}^\mu = \epsilon^{\mu\nu\lambda\sigma} u_\nu \mathcal{T}_\lambda \partial_\sigma\left(\frac{\mu_a}{T}\right), \qquad P_4^\mu = \epsilon^{\mu\nu\lambda\sigma} u_\nu \partial_\lambda \mathcal{T}_\sigma\,,$$

$$P_{5(a)}^\mu = \epsilon^{\mu\nu\lambda\sigma} u_\nu \mathcal{V}_{a\,\lambda\sigma}\,, \qquad P_6^\mu = \epsilon^{\mu\nu\lambda\sigma} u_\nu \partial_\lambda u_\sigma\,.$$

- We define:

$$\mathcal{H} = \mathrm{Tr}\left\{t_3^2 - U t_3 U^{-1} t_3\right\}, \qquad \mathcal{I}_\mu \equiv \mathrm{Tr}\left\{(L_\mu + R_\mu)t_3\right\},$$

$$\mathcal{T}_\mu \equiv \mathrm{Tr}\left\{(L_\mu - R_\mu)t_3 + 2(t_3^2 - U t_3 U^{-1} t_3)\mathcal{V}_{3\,\mu}\right\}.$$

Constitutive relations at first order in derivatives

- Baryonic currents [In color the BZ contributions: CME and CVE]:

$$u^\mu \delta J_{0\,\mu\,\mathrm{cov}} = \frac{N_c}{16\pi^2}\left[S_1 + 2S_3\right] ,$$

$$P^\mu{}_\nu \delta J^\nu_{0\,\mathrm{cov}} = \frac{N_c}{16\pi^2}\left[-2TP_1^\mu + \mu_3 P_2^\mu\right] - \frac{N_c}{16\pi^2}\mu_5 P^\mu_{5(0)} ,$$

$$u^\mu \delta J_{50\,\mu\,\mathrm{cov}} = \frac{N_c}{96\pi^2} S_{4(3)} ,$$

$$P^\mu{}_\nu \delta J^\nu_{50\,\mathrm{cov}} = -\frac{N_c}{48\pi^2}\left[TP^\mu_{3(3)} + \mu_3 \mathcal{H} P^\mu_{5(3)}\right] + \frac{N_c}{24\pi^2}\mu_5^2 P_6^\mu .$$

- Isospin currents:

$$u^\mu \delta J_{3\,\mu\,\mathrm{cov}} = -\frac{N_c}{48\pi^2}\mu_5 S_5 ,$$

$$P^\mu{}_\nu \delta J^\nu_{3\,\mathrm{cov}} = \frac{N_c}{48\pi^2}\mu_5\left[P_4^\mu + (\mathcal{H} - 3)\, P^\mu_{5(3)} + 2\mu_3 \mathcal{H} P_6^\mu\right] .$$

$$u^\mu \delta J_{53\,\mu\,\mathrm{cov}} = \frac{N_c}{32\pi^2} S_{4(0)} - \frac{N_c}{48\pi^2}\mu_5 S_2 ,$$

$$P^\mu{}_\nu \delta J^\nu_{53\,\mathrm{cov}} = -\frac{N_c}{16\pi^2}\left[TP^\mu_{3(0)} + \mu_3 \mathcal{H} P^\mu_{5(0)}\right] + \frac{N_c}{48\pi^2}\mu_5 P_2^\mu .$$

Constitutive relations at first order in derivatives

Properties:

- The only dependence in the background is through the physical electromagnetic fields $\mathcal{V}_{a\,\mu}$, so that the dependence in a_i has disappeared ➞ Constitutive relations should be expressed in terms of quantities of the fluid: fluid velocity, external fields, etc.
- Chiral electric effect in vector current **[Neiman, Oz, JHEP09 '11]**

$$P^\mu{}_\nu \delta J^\nu_{0\,\mathrm{cov}} = -\frac{N_c}{8\pi^2} T\epsilon^{\mu\nu\lambda\sigma} u_\nu \mathrm{Tr}\{(L_\lambda + R_\lambda)t_3\}\partial_\sigma\left(\frac{\mu_3}{T}\right) + \cdots .$$

- Chiral electric effect (CEE) in axial currents:

$$P^\mu{}_\nu \delta J^\nu_{50\,\mathrm{cov}} = -\frac{N_c}{48\pi^2} T\epsilon^{\mu\nu\lambda\sigma} u_\nu \mathrm{Tr}\{(L_\lambda - R_\lambda)t_3 + 2(t_3^2 - Ut_3U^{-1}t_3)\mathcal{V}_{3\,\lambda}\}\partial_\sigma\left(\frac{\mu_3}{T}\right)$$

- We provide explicit values for the corresponding transport coefficients of the CEE. Derived from an equilibrium partition function ➞ CEE cannot lead to entropy production ➞ CEE is non-dissipative.

Remark: Covariant corrents from BZ currents

[Mañes, EM, Valle, MAVZ JHEP '18 and '19]. See also **[Jensen, Laganayagam, Yarom '14].**

- In presence of Spontaneous Symmetry Breaking (SSB), one can compute the covariant currents directly from the BZ currents:

$$J^{\mu}_{\mathrm{cov}\,L} = W\left(\mathscr{U}\, J^{\mu}_{\mathrm{BZ}\,L}\right) W^{-1} \,, \qquad J^{\mu}_{\mathrm{cov}\,R} = W^{-1}\left(\mathscr{U}\, J^{\mu}_{\mathrm{BZ}\,R}\right) W \,,$$

  where $W^{-1}W^{-1} = U(\xi)$ and $\mathscr{U}[-\xi] \equiv$ operator that performs a finite gauge transformation with parameters $\Lambda_a = -\xi_a$.
- The covariant currents follow directly from the BZ currents ⟶ One does not need to compute the effective action.
- We have checked our results with this procedure as well.
- Energy-momentum tensor with SSB: $\langle T^{\mu\nu}\rangle = 0$ ⟶ This is consistent with $T^{\mu\nu}_{\mathrm{BZ}} = 0$.

Conclusions

- We have studied non-dissipative transport effects up to 1st order in the hydrodynamic expansion in non-abelian theories.
- Effects are induced by *external electromagnetic fields*, *vortices* and curvature in a relativistic fluid $\Longrightarrow$ Anomalous Transport.
- Equilibrium partition function method can only account for non-dissipative effects: *time reversal properties*.
- Dissipative effects (shear viscosity, electric conductivity, ...) ➞ other methods: Kubo formulae, Fluid/gravity correspondence, ...
- Anomalous effects in presence of Spontaneous Breaking of non-abelian gauge symmetries:
  - Interactions of Goldstone bosons among them and with external electromagnetic fields.
  - Application to QCD in confined phase: Fluid of pions, kaons, etc.
- Future directions:
  - Application to other sectors of the SM ➞ ElectroWeak sector.
  - Application to condensed matter systems with triangle anomalies ➞ Weyl semi-metals **[Basar, Kharzeev, Yee '14], [Landsteiner '14]**.

Thank You!