Turbulence as Statistical Field Theory

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Summary

The Turbulence in incompressible fluid is represented as a Field Theory in 3 dimensions. There is no time involved, so this is intended to describe stationary limit of the Hopf functional in a regime when viscosity $\nu \to 0$ at fixed energy flow $\mathcal{E}$.

The basic fields are Clebsch variables defined modulo gauge transformations (symplectomorphisms). Explicit formulas for gauge invariant Clebsch measure in space of steady flows compatible with finite energy flow are presented.

We introduce a concept of Clebsch confinement related to unbroken gauge invariance and study Clebsch instantons: singular vorticity sheets with nontrivial helicity related to winding numbers of Clebsch field. These singular solutions are involved in enhancing energy dissipation and creating exponential tails in velocity circulation PDF.

Summary

The turbulence which is a strong coupling phase of original field theory of fluctuating velocity field becomes a weak coupling phase of a dual string theory describing fluctuating surfaces. The effective expansion parameter is $\nu^{\frac{1}{5}}$ as opposed to $\frac{1}{\nu}$ expansion of original field theory. Our scaling laws are different from K41.

We computed the leading terms of WKB expansion around instanton solution with discontinuity at a minimal surface. In a turbulent limit the tangent components of vorticity around the surface grow as some negative power of viscosity and has Gaussian profile in normal direction with the width vanishing as another power of viscosity.

The distribution for PDF of velocity circulation has exponential tails which fit the numerical simulations within their systematic errors due to the finite lattice.

Fixed Point of the Hopf Equation

There are two alternative views of Turbulence:

- time -averaged distribution of the solution of stochastic differential equation over Gaussian distribution of random forces.
- ensemble-averaged distribution of steady state solution of the Hopf functional equation over initial or boundary conditions.

In principle, these approaches are equivalent, but at the technical level they are very different. They are like Newton Dynamics vs Gibbs Statistics.

Fixed Point of the Hopf Equation

The first approach is used in numerical simulations, perturbative expansion in $\frac{1}{\nu}$ and also in some non-perturbative calculations in the program "Instantons and Intermittency" started in the 90-ties . Unfortunately, this program was never implemented for Navier-Stokes equation, only for passive scalar (Falkovich et al. 1996) and for Burgers equation (Gurarie and Migdal 1996).

The second approach deals directly with observable single time statistical distribution, which is a significant advantage over unnecessary stochastic dynamics of the first approach.

This approach was modified in the 90-ties to produce Loop Equations (Migdal 1995) and recently (Migdal 2020c,Migdal 2020b,Migdal 2020a) we advanced that approach using Clebsch variables.

Fixed Point of the Hopf Equation

Navier-Stokes equation for vorticity

$$\dot{\omega}_\alpha = G_\alpha[\omega]$$
$$G_\alpha = \nu \partial^2 \omega_\alpha + \omega_\beta \partial_\beta v_\alpha - v_\beta \partial_\beta \omega_\alpha;$$

leads to the Hopf equation for generating functional

$$\mathcal{Z}[\vec{\lambda}, t] = \left\langle \exp\left(\imath \int_r \lambda_\alpha \omega_\alpha \right) \right\rangle;$$
$$\partial_t \mathcal{Z} = \imath \int_r \lambda_\alpha G_\alpha \left[-\imath \frac{\delta}{\delta \lambda} \right] \mathcal{Z}$$

with averaging over randomized initial conditions being implied.

Fixed Point of the Hopf Equation

The external potential random forces $f_\alpha(r,t) = -\partial_\alpha p^{ext}(\vec{r},t)$ drop from the Navier-Stokes equation for vorticity, therefore the only way these random forces would affect the vorticity distribution would be the boundary conditions and the energy flow constraint.

Instead of solving stochastic differential equation we would have to solve steady state equation with boundary conditions depending on realization of static random forces and corresponding to finite positive energy flow.

Then we would have to average the Hopf functional over ensemble of these random forces instead of averaging over time.

Fixed Point of the Hopf Equation

Let us consider a manifold $\mathcal{G}$ of locally steady solutions (generalized Beltrami flow, GBF)

$$\mathcal{G} : G_\alpha[\omega^\star, \vec{r}] = 0$$

Then an integral

$$\mathcal{Z} \propto \int_{\mathcal{G}} d\mu(\omega^\star) \exp\left(\imath \int_r \lambda_\alpha \omega^\star_\alpha\right);$$

with some invariant measure $d\mu(\omega^\star)$ on $\mathcal{G}$ would be a fixed point of the Hopf equation as one can check by direct substitution.

The random boundary conditions are hidden in the distribution $d\mu(\omega^\star)$ in this formula. As we shall discuss in detail later, in addition to the local variables parametrizing vorticity $\omega^\star$ there are some global parameters for the boundary forces.

Generalized Beltrami Flow

We parameterize the vorticity by unit vector $\vec{S} \in S_2$:

$$\omega_\alpha = \frac{1}{2} Z e_{ijk} e_{\alpha\beta\gamma} S_i \partial_\beta S_j \partial_\gamma S_k; \; S_i^2 = 1$$

where $Z =$ const is a global scale factor (see below). We redefine velocity and vorticity $\omega_\alpha \Rightarrow Z\omega_\alpha, v_\alpha \Rightarrow Z v_\alpha$ after which Z drops from Euler equations.

The Euler equations are then equivalent to passive convection of the Clebsch field by the velocity field:

$$\partial_t \vec{S} = -v_\alpha \partial_\alpha \vec{S}$$

It is convenient to use polar coordinates $\theta \in (0, \pi), \varphi \in (0, 2\pi)$ for the unit vector $S = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$:

$$\phi_1 = (1 - \cos\theta);$$

$$\phi_2 = \varphi \pmod{2\pi}$$

Generalized Beltrami Flow

The second variable ϕ_2 is multi-valued, but vorticity is finite and continuous everywhere. The helicity $\int d^3 r v_\alpha \omega_\alpha$ was ultimately related to winding number of that second Clebsch field [1].

The volume element on S_2

$$d^2\phi = d\cos\theta d\varphi$$

is equivalent to phase space volume element $d\phi_1 d\phi_2$ up to the scale factor Z. So, these ϕ_1, ϕ_2 are conventional Clebsch variables.

[1] To be more precise, it was Hopf invariant on a sphere S_3 instead of real space R_3 (see Kuznetsov and Mikhailov 1980 for details).

Generalized Beltrami Flow

There is some gauge invariance (canonical transformation in terms of Hamiltonian system, or area preserving diffeomorphisms geometrically)

$$\phi_a(r) \Rightarrow G_a(\phi(r))$$
$$\det \frac{\partial G_a}{\partial \phi_b} = \frac{\partial(G_1, G_2)}{\partial(\phi_1, \phi_2)} = 1.$$

These transformations manifestly preserve vorticity and therefore velocity.

These variables and their ambiquity were known for centuries but they were not utilized within hydrodynamics until pioneering work of Khalatnikov in 1952 and subsequent works of Kuznetzov and Mikhailov and Levich in early 80-ties.

Generalized Beltrami Flow

In terms of field theory, this is an exact gauge invariance, rather than the symmetry of observables, much like color gauge symmetry in QCD. This is why back in the early 90-ties I referred to Clebsch fields as "quarks of turbulence".

To be more precise, they are both quarks and gauge fields at the same time.

The choice of the target space S_2 for these Clebsch variables represents the gauge fixing.

The symplectomorphisms would change the metric while preserving its determinant. We verified in (**Mig20c**) that there are no ghosts needed for this gauge fixing.

The GBF Measure

We propose the following measure on GBF(with $I = (\vec{r}, \alpha)$, $\sum_I = \int d^3 r \sum_\alpha$ etc):

$$d\mu[\vec{S}] = \prod_r d^2 S_r d^3 U_r d^3 \Psi_r$$

$$\exp\left(\imath \sum_I U_I G^I + \frac{1}{2} [\Phi, \Phi] \right);$$

$$\Phi = \sum_I \Psi_I G^I;$$

$$G_\alpha = \frac{\nu}{Z} \partial^2 \omega_\alpha + \omega_\beta \partial_\beta v_\alpha - v_\beta \partial_\beta \omega_\alpha;$$

$$[A, B] = \int_r \frac{\delta A}{\delta S_j(\vec{r})} e_{ijk} S_i(\vec{r}) \frac{\delta B}{\delta S_k(\vec{r})}$$

We scaled the global normalization factor Z out of velocity and vorticity fields.

The GBF Measure

The Bose field U_I is Lagrange multiplier for the GBF condition, and Grassmann field Ψ_I is corresponding ghost. The Poisson brackets $[\Phi, \Phi]$ is not vanishing for the Grassmann functionals.

In the paper (Migdal 2020a) we have proven that this measure is uniform at the GBF. The proof involves so called SVD (Wikipedia 2020).

We can perform general linear transformation of both U_I, Ψ_I and Jacobian of this transformation will cancel between DU and $D\Psi$. Using such transformation one can eliminate the dependence of $d\mu(\omega*)$ from the point $\omega*$ at the GBFmanifold $\mathcal{G}$.

Energy Flow Balance

As is well known, the energy is pumped into the turbulent flow from the largest scales (pipes, ships, etc.), and dissipated at the smallest scales due to viscosity effects. Let us see how that happens in some detail. Using Navier-Stokes equation

$$\dot{v_\alpha} = \nu \partial_\beta^2 v_\alpha - v_\beta \partial_\beta v_\alpha - \partial_\alpha p;\ \partial_\alpha v_\alpha = 0$$

we have

$$\partial_t \int d^3 r \frac{1}{2} v_\alpha^2 = \int d^3 r\, \nu v_\alpha \partial_\beta^2 v_\alpha - v_\alpha \left(v_\beta \partial_\beta v_\alpha + \partial_\alpha p \right) = 0$$

Energy Flow Balance

So, we have the balance of two terms cancelling each other in the time derivative of energy : dissipation and pumping. By Stokes theorem these terms can be reduced to the following

$$\mathcal{E} = \nu \int_V d^3 r \omega_\alpha(r)^2$$
$$= -\int_{\partial V} d\sigma_\beta \left(v_\beta \left(p + \frac{1}{2} v_\alpha^2 \right) + \nu v_\alpha (\partial_\beta v_\alpha - \partial_\alpha v_\beta) \right)$$

The first expression for $\mathcal{E}$ is the dissipation, concentrated in high vorticity regions, where the small viscosity is compensated by square of vorticity.

The second expression is the energy flow through the boundary.

Vorticity Cell

We are going to impose the boundary condition on the pressure, corresponding to fixed uniform force at the infinity

$$p\left(\vec{r} \to \infty\right) \to -r_\alpha f_\alpha$$

With this condition and decreasing velocity at infinity, the pressure term with $p = -r_\alpha f_\alpha$ is the only one contributing to the energy flow. Coming back to the volume integral

$$\mathcal{E} = f_\alpha Q_\alpha;$$
$$Q_\alpha = \int_V d^3 r v_\alpha$$

This is the limit of the conventional expression for energy flow in case of uniform force. Our derivation shows that in fact this is the energy flow through the infinite boundary.

Vorticity Cell

In our new normalization of velocity and vorticity

$$\mathcal{E} = Z f_\alpha Q_\alpha;$$

$$\mathcal{E} = Z^2 \nu \int_V d^3 r \omega_\alpha^2;$$

$$Q_\alpha = \int_V d^3 r v_\alpha$$

The global factor Z is determined from the energy balance (at fixed realization of the random force $\vec{f}$):

$$Z = \frac{f_\alpha Q_\alpha}{\nu \int_V d^3 r \omega_\alpha^2};$$

$$\mathcal{E} = \frac{(f_\alpha Q_\alpha)^2}{\nu \int_V d^3 r \omega_\alpha^2}$$

Master Equation

The loop equation in the minimal surface approximation was analyzed in detail and solved numerically by *Mathematica*® code in my work in 2019. It was assumed in that paper that vorticity was smooth and dominated by a region close to the minimal surface, where it was directed towards normal.

As we see now, with Clebsch instanton, this assumption is modified in a nontrivial way: the vorticity flux is still determined by smooth normal component of vorticity.

However, there is a tangential vorticity in an infinitely thin layer around the minimal surface $S : \vec{r} = \vec{X}(\xi), \xi = (\rho, \alpha); C : \rho = \rho(\alpha)$. Formally this tangential component comes as a delta function, related to the discontinuity of the Clebsch field.

Master Equation

Why minimal surface? The only boundary conditions for Clebsch field we can devise compatible with gauge invariance is the Neumann conditions

$$\partial_z \phi_a = R_{ab}(x,y)\phi_b$$

The gauge transformation $\phi_a \Rightarrow F_a(\phi_1, \phi_2)$

$$\frac{\partial F_a(\phi_1,\phi_2)}{\partial \phi_b}\partial_z \phi_b = 0$$

$$\det \frac{\partial F_a(\phi_1,\phi_2)}{\partial \phi_b} = 1$$

will preserve it only for $R_{ab} = 0$.

Master Equation

In linear vicinity of local tangent plane to the surface its equation reads (with K_1, K_2 being principal curvatures at this point)

$$z - \frac{K_1}{2}x^2 - \frac{K_2}{2}y^2 = 0$$

$$n_i = \frac{(-K_1 x, -K_2 y, 1)}{\sqrt{1 + K_1^2 x^2 + K_2^2 y^2}} = (0, 0, 1) + O(x, y)$$

$$\Omega = n_\alpha \omega_\alpha \to \frac{1}{2} e_{ij} e_{ab} \partial_i \phi_a \partial_j \phi_b + O(x, y)$$

$$n_\alpha \partial_\alpha \Omega(r) \to e_{ij} e_{ab} \partial_i \partial_z \phi_a \partial_j \phi_b + O(x, y)$$

The mixed derivatives $\partial_i \partial_z \phi_a$ vanish at $x = y = z = 0$ for our boundary conditions.

Master Equation

Simple algebra then yields

$$0 = \partial_\alpha \omega_\alpha = \Omega(r) \partial_\alpha n_\alpha = -\Omega(r)(K_1 + K_2) = 0$$

Therefore, with gauge invariant Neumann boundary conditions the Clebsch field is allowed to have discontinuity only across the minimal surface.

Master Equation

As we learn from analysis in (Migdal 2020c), the vorticity in vicinity of the surface $\vec{r} = \vec{X}(\xi) + z\vec{n}(\xi)$ has the structure

$$\vec{\omega}\,(\vec{r}) = \delta(z) 2\pi n \vec{\nabla}\Phi(\xi) \times \vec{n}(\xi) + \vec{n}(\xi)\Omega(\xi) + O(z^2)$$

$$\vec{n} = \frac{\partial_\rho \vec{X} \times \partial_\alpha \vec{X}}{\sqrt{\det G}}$$

$$\Omega(\xi) = \frac{\frac{\partial \Phi(\xi)}{\partial \rho}}{\sqrt{\det G}}$$

$$G_{ij} = \partial_i \vec{X}(\xi) \partial_j \vec{X}(\xi)$$

The delta term comes from the normal discontinuity of ϕ_2 while the other component is continuous

$$\phi_2\,(\vec{r}) = \alpha + 2\pi n \theta(z) + O(z^2);\ n \in \mathbb{Z};$$

$$\phi_1\,(\vec{r}) = \Phi(\xi) + O(z^2)$$

Master Equation

This delta term in vorticity is orthogonal to the normal vector and thus does not contribute to the flux through the minimal surface, so this flux is still determined by the second (regular) term and circulation is related to this $\Phi(\xi)$

$$\Gamma_C = Z \oint_C v_\alpha dr_\alpha =$$

$$Z \oint_C \phi_1 d\phi_2 = Z \int_0^{2\pi} \left(\Phi(\rho(\alpha), \alpha) - \Phi(\vec{0})\right) d\alpha$$

However, the Biot-Savart integral with this Clebsch instanton is dominated by the singular tangential component and is finite (though not continuous)

$$v_\beta(r) = 2\pi n \left(\delta_{\beta\gamma}\partial_\alpha - \delta_{\alpha\beta}\partial_\gamma\right)$$
$$\int_{S_C} d\sigma_\gamma(\xi)\partial_\alpha\Phi(\xi)\frac{1}{4\pi|\vec{X}(\xi) - \vec{r}|}$$

Master Equation

As we noted above, the GBF equations will be satisfied provided the Clebsch master equation

$$v_\alpha \partial_\alpha \phi_a = e_{ab} \frac{\partial h(\phi)}{\partial \phi_b}$$

with some gauge function $h(\phi)$.

The leading term in these equations near the minimal surface is still the normal flow restriction $v_n(r) = 0, r \in S$, which annihilates the $\delta(z)$ term in above equation. The next order terms will already involve the gauge function $h(\phi)$, which can in principle be non-zero.

These equations are quite different from those we deduced from the loop equations, because the singular terms in vorticity were missed there. Alas, trial and error is the only path we know in Physics.

Instanton On Flat Surface

The simplest case of our instanton is that of a flat loop in 3D space, which we assume to be in x, y plane. The minimal surface is a part D_C of x, y plane bounded by this flat loop.

The generic formula simplifies here (here $i, j = 1, 2$) :

$$v_i^{inst}(r_0) = 0,$$
$$v_z^{inst}(r_0) = \frac{n}{2} \int_{D(C)} d^2 r \partial_i \Phi(r) \partial_i \frac{1}{|r - r_0|}$$

The vanishing tangent velocity means that the regular part of master equation is satisfied identically with $h = 0$.

Instanton On Flat Surface

As for the singular part, proportional to $\delta(z)$ it requires $v_z(r) = 0$ at the minimal surface.

In fact, there is always extra contribution to the normal velocity from the smooth velocity $\vec{v}^s(r_0)$, related to background vorticity in the rest of space.

So, correct equation reads

$$v_z(r_0) = v_z^s(r_0) + \frac{n}{2}\int_{D(C)} d^2 r \partial_i \Phi(r) \partial_i \frac{1}{|r - r_0|} = 0$$

Minimization Problem

There is a way to reduce our master equation to a minimization of a quadratic functional.
Let us make the integral transformation

$$\Phi(\vec{r}) = \frac{\int_{D_C} d^2 r v_z^s(\vec{r}, z)}{n} \int_{D_C} d^2 r' \frac{H(\vec{r}')}{2\pi |r - r'|}$$

and we are arrive at universal equation

$$\frac{1}{4\pi^2} \int_{D_C} d^2 r' \partial_\alpha \frac{1}{|\vec{r}' - \vec{r}|} \int_{D_C} d^2 r'' H(\vec{r}'') \partial'_\alpha \frac{1}{|r'' - r'|} = R(\vec{r})$$

Here

$$R(\vec{r}) = \frac{v_z^s(\vec{r}, z)}{\int_{D_C} d^2 r v_z^s(\vec{r}, z)}$$

is normalized to unit integral over the domain.

Minimization Problem

As we are interested in large size of domain D_C compared to the size of vorticity support in the thermostat, this $R(\vec{r})$ is concentrated inside a finite region near the center of D_C. Later we study this equation approximating $R(\vec{r})$ by a delta function. Now we proceed for a general $R(\vec{r})$.

We observe that this problem is equivalent to minimization of positive quadratic functional

$$Q[H] = -\int_{D_C} d^2 r H(r) R(\vec{r}) + \frac{1}{2} \int_{D_C} d^2 r F_\alpha^2[H, \vec{r}];$$

$$F_\alpha[H, \vec{r}] = \frac{1}{2\pi} \int_{D_C} d^2 r' H(\vec{r}') \partial'_\alpha \frac{1}{|\vec{r} - \vec{r}'|}$$

As we shall see later, the position of the origin drops from asymptotic formulas at large area.

Minimization Problem

This $F_\alpha[H, \vec{r}]$ is proportional to $\partial_\alpha \Phi(\vec{r})$. Thus, the quadratic part of our target functional is just a kinetic energy of a free scalar field, but it is the linear term which forces us to use $H(\vec{r})$ as an unknown.

In order for $\Phi(\vec{r})$ and its gradients to remain finite at the boundary C the new field H should satisfy Dirichlet boundary condition

$$H(C) = 0$$

Minimization Problem

In order for vorticity to remain finite at the origin we have to have

$$F_\alpha[H, \vec{0}] = 0$$

Coulomb poles disappeared from this problem, being replaced by weaker, logarithmic singularities.

The circulation integral

$$\Gamma[C] = Z \oint d\theta \left(\Phi\left(R\vec{f}(\theta) \right) - \Phi(\vec{0}) \right)$$

with $C : \vec{r} = L\vec{f}(\theta)$ being the equation for the contour C in polar coordinates on the plane.

Smeared Vorticity and Dissipation in Navier-Stokes Equations

The square of delta function entering the dissipation from the instanton has to be smeared at viscous scales.
The GBF equation for velocity field (with our new normalization)

$$0 = \frac{\nu}{Z}\partial^2 v_\alpha - v_\beta \partial_\beta v_\alpha + \partial_\alpha p;$$
$$\partial^2 p = \partial_\alpha v_\beta \partial_\beta v_\alpha$$

Before we substitute the singular instanton solution into above GBF equation, we need to smear the theta function.

$$\theta_h(z) = \int_{-\infty}^{z} dz' \delta_h(z'),$$

where $\delta_h(z)$ is some approximation to the delta function with width $h \to 0$. The shape of smeared delta function will follow from the Navier-Stokes equations.

Smeared Vorticity and Dissipation in Navier-Stokes Equations

The Clebsch representation

$$v_\alpha = -\phi_2 \partial_\alpha \phi_1 + \partial_\alpha \tilde{\phi}_3;$$
$$\tilde{\phi}_3 = \phi_3 + \phi_1 \phi_2$$

allows us to single out the singular terms in local tangent x, y frame, with z being the normal distance.

$$v_i(x,y,z) = -2\pi n \theta_h(z) \partial_i \Phi(x,y) + \dots;$$
$$v_z(x,y,z) = z v'_z(x,y) + \dots;$$
$$\partial^2 p \to 2 \partial_z v_i \partial_i v_z =$$
$$-4\pi n \left(\delta_h(z) \partial_i \Phi(x,y)\right) \partial_i \left(z v'_z(x,y)\right) + \dots;$$
$$p \to z \delta_h(z) P(x,y) + \dots;$$
$$\partial_i^2 P = -4\pi n \partial_i \Phi(x,y) \partial_i v'_z(x,y)$$

where $\dots$ stand for a regular parts at $z \to 0$.

Smeared Vorticity and Dissipation in Navier-Stokes Equations

Let us collect singular terms, proportional to $z\delta_h(z), \delta'_h(z)$ with coefficients depending only of x, y:

$$2\pi n \left(\frac{\nu}{Z} \delta'_h(z) + v'_z z \delta_h(z) \right) \partial_i \Phi + \partial_i P z \delta_h(z) = 0;$$
$$\partial_i^2 P = -4\pi n \partial_i \Phi \partial_i v'_z$$

Eliminating P we find

$$\partial_i P(x, y) = \partial_i v'_z(x, y) = 0$$

Smeared Vorticity and Dissipation in Navier-Stokes Equations

This leads to the Gaussian for normalized distribution $\delta_h(z)$

$$\delta_h(z) = \frac{1}{h\sqrt{2\pi}} \exp\left(-\frac{z^2}{2h^2}\right);$$
$$v'_z(x,y) = \frac{\nu}{Zh^2}$$

This is viscosity anomaly we were talking about: the singular term $\propto z\delta(z)$ in the Euler equation is balanced by the singular contribution $\propto \delta'(z)$ from dissipation term. Matching these terms leads to the Gaussian smearing of the delta function.

Smeared Vorticity and Dissipation in Navier-Stokes Equations

Now we have to assume some scaling law in the turbulent limit

$$h \propto \nu^{\alpha}$$

The index α will be determined from the energy balance equation.

Smeared Vorticity and Dissipation in Navier-Stokes Equations

With Gaussian regularization of the delta function we have

$$\int_r \nu \omega_\alpha^2 \to \nu \int_r \delta_h(z)^2 \left(2\pi n \partial_i \Phi\right)^2$$

$$\to \Lambda \int_S d^2 r \left(2\pi n \partial_i \Phi\right)^2;$$

$$\Lambda = \frac{\nu}{h}\sqrt{\frac{1}{4\pi}};$$

$$Z = \frac{Q_{\alpha\beta} f_\alpha f_\beta}{\Lambda A};$$

$$A = \int_S d^2 r \left(2\pi n \partial_i \Phi\right)^2$$

Smeared Vorticity and Dissipation in Navier-Stokes Equations

From the last relation we finally find the estimate of the random force variance σ and pancake width h in the turbulent limit

$$\sigma \sim \sqrt{\mathcal{E}}\nu^{\frac{1}{2}(1-\alpha)};$$
$$Z \sim \frac{h\sigma}{\nu} \sim \sqrt{\mathcal{E}}\nu^{-\frac{1}{2}(1-\alpha)};$$
$$h \sim \nu^{\alpha};$$
$$v'_z(x,y) = \frac{\nu}{Zh^2} \sim \nu^{\frac{3-5\alpha}{2}}$$

The self-consistency requires

$$\alpha = \frac{3}{5}$$

in which case the anomaly contributes to the Navier-Stokes equations in the Turbulent limit.

$$\sigma \sim \sqrt{\mathcal{E}}\nu^{\frac{1}{5}};$$
$$h \sim \nu^{\frac{3}{5}};$$
$$Z \sim \sqrt{\mathcal{E}}\nu^{-\frac{1}{5}}$$

As expected, both the variance and the width go to zero in the turbulent limit. One can estimate the next corrections to the saddle point equation, coming from the Z dependence of vorticity by means of the viscous term in GBF equation. Differentiating equations by Z and estimating the corrections to the energy flow we find that these corrections are smaller than the leading terms in the turbulent limit.

Smeared Vorticity and Dissipation in Navier-Stokes Equations

As for the Zeldovich pancake, it is filled with coiled vortex lines coming and exiting in the normal direction and making n coils within the thickness h of the pancake

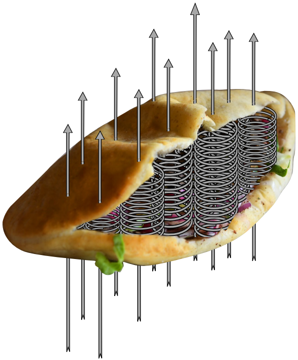

Figure: The vortex lines coiling inside the Zeldovich pancake in our Instanton solution.

Circulation PDF

In this section we are going to finally derive predictions for the circulation PDF.

$$\Gamma[C] \propto \frac{1}{n} Z \int_0^{2\pi} d\theta \int_{D_C} d^2 r$$

$$\frac{H(\vec{r})}{\bar{H}} \left(\frac{1}{|\vec{r} - L\vec{f}(\theta)|} - \frac{1}{|\vec{r}|} \right);$$

$$\bar{H} = \frac{\int_{D_C} d^2 r H(\vec{r}) R(\vec{r})}{\int_{D_C} d^2 r R(\vec{r})}$$

We remind that the origin is placed at geometric center of the domain D_C.
The integral $\int_{D_C} d^2 r H(r) R(\vec{r})$ in $\bar{H}$ is concentrated on finite scales $\vec{r} \sim 1$ due to decrease of $R(\vec{r})$, so this $\bar{H}$ scales as $H(\vec{0})$, same as $H(\vec{r})$ in the integral in the numerator.

Circulation PDF

Collecting scales of the remaining factors we see that $\Gamma[C] = LF[C/L]$ in agreement with the loop equation arguments (Migdal 2019).

Taylor expansion of $\vec{Q}(\vec{f})$ would be justified if, just like in a critical phenomena in statistical physics, the susceptibility would grow to infinity to compensate small value of external force.

This is what happens in a ferromagnet near the Curie point, when infinitesimal external magnetic field is enhanced by large susceptibility, resulting in a spontaneous magnetization.

Circulation PDF

In our theory this happens because the pancake thickness $h \propto \nu^{\frac{3}{5}}$ becomes small in a turbulent limit together with variance of external force $\sigma \propto \nu^{\frac{1}{5}}$. The resulting factor $\frac{h}{\nu} \sim \nu^{-\frac{2}{5}}$ enhances the leading term $(Q_{\alpha\beta} f_\alpha f_\beta)^2 \sim \sigma^2$ so that the nonlinear terms of expansion would be negligible. In other words, singularities of the instanton are the origin of the critical phenomena in our theory.

The transformation of the Gaussian distribution to an exponential one, happens because of the $\vec{Q}(\vec{f})$ factor multiplying the Gaussian force in the Z factor in the circulation.

Resulting square of Gaussian variable transforms the Gaussian distribution to the exponential one.

Circulation PDF

Also, we observe that the sign of Γ is proportional to the sign of the winding number n.

Clearly, in addition to instanton with winding number n there are always an anti-instanton with $-n$.

The probability for this GBF solution in our functional integral is exactly the same as for the positive n, so the contributions from these flows must be added.

This provides the negative branch of circulation PDF.

Circulation PDF

Summing up contribution from both signs we obtain an explicit formula for a Wilson loop

$$\langle \exp\left(\imath\,\gamma\Gamma_C\right)\rangle_n = \frac{1}{2}\left(W\left(\frac{\gamma}{n}\right) + W\left(-\frac{\gamma}{n}\right)\right)$$
$$W(\gamma) = \frac{1}{\sqrt{\prod_{i=1}^{3}\left(1 - \imath\,\gamma\mu_i\Sigma[C]\right)}}$$

where $\mu_i \propto \nu^{\frac{1}{5}}$ are three positive eigenvalues of the matrix (in decreasing order)

Circulation PDF

$$\mu_{\alpha\beta} = \frac{\sigma Q_{\alpha\beta}}{\Lambda}$$

$$\Sigma[C] = \int_0^{2\pi} d\theta \int_{D_C} d^2 r \frac{H(\vec{r})}{\bar{H}} \left(\frac{1}{|\vec{r} - L\vec{f}(\theta)|} - \frac{1}{|\vec{r}|} \right)$$

This corresponds to asymptotic law

$$P(\Gamma) \propto \sqrt{\left|\frac{n}{\Sigma[C]\Gamma}\right|} \exp\left(-\left|\frac{n\Gamma}{\mu_1 \Sigma[C]}\right|\right)$$

Discussion. Do we have a theory yet?

We identified the instanton mechanism of enhancement of infinitesimal random force in Euler equation and demonstrated how this enhancement takes place at small viscosity in Navier-Stokes equations.

The required random force needed to create the energy flow and asymptotic exponential distribution of circulation, has the variance $\sigma \sim \sqrt{\mathcal{E}}\nu^{\frac{1}{5}}$. This small force is enhanced by large susceptibility.

This large susceptibility can be traced back to the singular behavior of the vorticity field near the minimal surface in the turbulent limit of Navier-Stokes equations: $\omega \sim \frac{1}{h}$ in a layer $|z| \sim h \propto \nu^{\frac{3}{5}}$. The profile of amplitude of tangent vorticity is Gaussian of normal direction z and the Clebsch vector $\vec{S}$ is rapidly rotating while crossing minimal surface.

Discussion. Do we have a theory yet?

We presented an explicit solution for the shape of circulation PDF generated by instanton. We claim it is realized in high Reynolds flows for the large loops and large circulations, not as a model, but rather as an exact asymptotic law.

The instanton is topologically stable and is related to a quadratic minimization problem.

We confirmed the dependence $|\Gamma| \propto \sqrt{A_C}$ predicted earlier (Migdal 2019) based on the Loop equations. The raw data from (Iyer, Sreenivasan, and Yeung 2019) were compared with this prediction. We took the ratio of the moments $M_p = \langle \Gamma^p \rangle$ at largest available p and defined the circulation scale as $S = \sqrt{\frac{M_8}{M_6}}$.

Discussion. Do we have a theory yet?

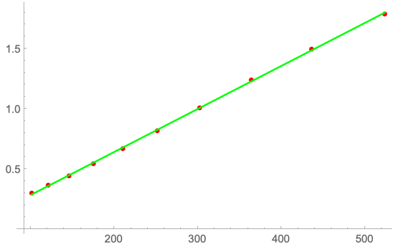

Figure: Linear fit of the circulation scale $S = \sqrt{\frac{M_8}{M_6}}$ (with $M_p = \langle \Gamma^p \rangle$) as a function of the $R = a/\eta$ for inertial range $100 \leq R \leq 500$. Here a is the side of the square loop C and η is a Kolmogorov scale . The linear fit $S = -0.073404 + 0.00357739R$ is almost perfect: adjusted $R^2 = 0.999609$

Discussion. Do we have a theory yet?

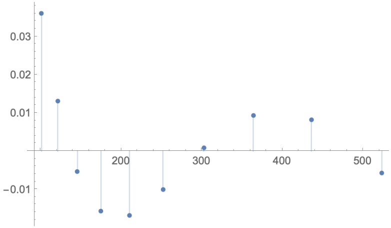

Figure: The relative residuals $\frac{\delta S}{S}$ of the linear fit are shown as a function of the side of the square. The smooth harmonic wave suggests that these errors are affected by harmonic wave forcing on a $16K$ lattice rather than genuine oscillations in infinite isotropic system.

Discussion. Do we have a theory yet?

The PDF is given by sum over positive integer winding numbers n

$$P(\Gamma) = \int_{-\infty}^{\infty} \frac{d\gamma}{2\pi} e^{-\imath\gamma\Gamma} \left\langle \exp\left(\imath\gamma \oint_C dr_\alpha v_\alpha\right)\right\rangle$$

$$\propto \frac{1}{\sqrt{|\Gamma|\bar{\mu}\Sigma[C]}} \sum_{n=1}^{\infty} \exp\left(-n\frac{|\Gamma|}{\bar{\mu}\Sigma[C]}\right)\sqrt{n}$$

Negative winding numbers are responsible for another branch of the PDF, so that resulting PDF is an even function of circulation at large $|\Gamma|$. This sum reduces to so called integral logarithm $\mathrm{Li}_{-\frac{1}{2}}$.

Obviously, at large circulation only the $n = 1$ term remains, matching numerical experiments. We found that our formula fits the latest data by Kartik Iyer within error bars of DNS with adjusted $R^2 = 0.9999$.

Discussion. Do we have a theory yet?

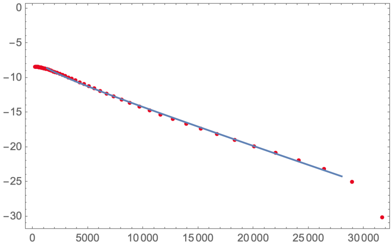

Figure: $\log P(x)$ (red dots) together with fitted line $\log P \approx -0.000526724x - 4.3711 - 0.5\log(x) \pm 0.116469,\ 1300 < x < 28000$. Here $x = \frac{|\Gamma|}{\nu}$. Last two points have low statistics in DNS and were discarded from fit. Remaining data match the theoretical formula within statistical errors of DNS. Adjusted $R^2 = 0.999929$

Discussion. Do we have a theory yet?

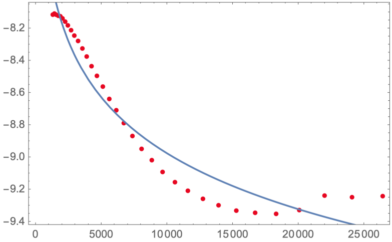

Figure: Subtracting the slope. $0.000526724x + \log P(x)$ (red dots) together with fitted line $-4.3711 - 0.5\log(x),\ 1300 < x < 28000$. Here $x = \frac{|\Gamma|}{\nu}$. We see that the pre-exponential factor $1/\sqrt{|\Gamma|}$ fits the data, though with less accuracy after subtracting the leading term.

Discussion. Do we have a theory yet?

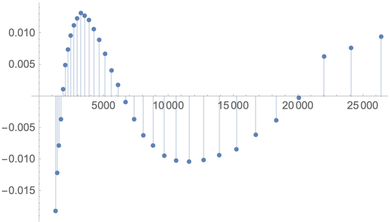

Figure: Relative residuals of the log fit of PDF. The harmonic wave behavior suggests that these are artefacts of harmonic random forcing on a $16K^3$ cubic lattice rather than genuine oscillations in infinite isotropic system. Such residuals do not imply contradictions with the theory.

Discussion. Do we have a theory yet?

The sum over integers emerges here by the same mechanism as in Planck's distribution in quantum physics.

There we had to sum over all occupation numbers in Bose statistics. Here we sum over all winding numbers of the Clebsch field across the minimal surface in physical space.

In Bose statistics the discreteness of quantum numbers is related to the compactness of the domain for the corresponding degree of freedom.

In our case this also follows from compactness of the domain for the Clebsch fields, varying on a sphere. The quantum numbers are counting covering of that sphere when vortex lines coil inside the viscous layer around minimal surface.

Discussion. Do we have a theory yet?

The physical reason why the multi-valued Clebsch fields are acceptable in a real world is the unbroken gauge invariance, or Clebsch confinement. Clebsch fields are unobservable, just like quarks or gluons.

So, do we have a theory of turbulence? Not yet IMHO, but we may be getting there. There are still some issues to be clarified and some computations to be made and some limits to be proven to exist. And maybe some errors to be corrected.

Once again I am appealing to young mathematical physicists and string theorists: come and help me! Do not wait until the turbulence experts will finally endorse this theory, they will take forever. The gauge-string duality is in play here and you know it better than anyone. You can develop this approach into a Theory of Turbulence.