Chiral Separation Effect in non-homogeneous systems

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- The Chiral Separation Effect (CSE) is one of the non - dissipative transport effects, which has been proposed by M.Metlitski and A.Zhitnitsky

$$J_5^k = -\frac{1}{4\pi^2}\epsilon^{ijk0}\mu F_{ij} \quad (1)$$

- It is related to the chiral anomaly, like the Chiral Magnetic Effect. The difference is that CSE can exist in equilibrium, while the CME vanishes, and can exist only out of the equilibrium.
- This is an axial current in the direction of external magnetic field.
- The possibility to observe it during the heavy ioins collisions was discussed by several authors - for example by Kharzeev
- The continuum QFT model was discussed, for example, by Miransky
- The homogenuous case, using the lattice regularization, was discribed by Buividovich, Zubkov and Khaidukov
- In the current workwe consider uniform external magnetic field and uniform chemical potential, but non-uniform fermionic action, using the Weyl-Wigner formalism.

Average of operator $\hat{A}$ with respect to quantum state Ψ, in quantum mechanics may be written as

$$\langle\Psi|\,\hat{A}\,|\Psi\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp A_W(x,p)\rho_W(x,p) \tag{2}$$

The Weyl symbol of operator $A_W(x,p)$ and Wigner distribution $W(x,p)$

$$A_W(x,p) \equiv \int_{-\infty}^{\infty} dy e^{-ipy}\,\langle x+\frac{y}{2}|\,\hat{A}\,|x-\frac{y}{2}\rangle = \int_{-\infty}^{\infty} dq e^{iqx}\,\langle p+\frac{q}{2}|\,\hat{A}\,|p-\frac{q}{2}\rangle \tag{3}$$

$$W(x,p) = \int_{-\infty}^{\infty} dy e^{ipy}\,\langle x-\frac{y}{2}|\Psi\rangle\,\langle\Psi|x+\frac{y}{2}\rangle = \int_{-\infty}^{\infty} dq e^{-iqx}\,\langle p-\frac{q}{2}|\Psi\rangle\,\langle\Psi|p+\frac{q}{2}\rangle \tag{4}$$

The Moyal product is defined as follows

$$(\hat{A}\hat{B})_W = \int_{-\infty}^{\infty} dq e^{iqx}\,\langle p+\frac{q}{2}|\,\hat{A}\hat{B}\,|p-\frac{q}{2}\rangle = A_W(x,p)\star B_W(x,p) = A_W(x,p)e^{\overleftrightarrow{\Delta}}B_W(x,p) \tag{5}$$

where $\overleftrightarrow{\Delta} \equiv \frac{i}{2}\left(\overleftarrow{\partial}_x\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_x\right)$

Relation between the Green function and Dirac operator in Weyl-Wigner formalism - Groenewold equation

The Green's funcion

$$(i\gamma_\mu \partial_x^\mu - m)G(x-y) = \delta(x-y) \tag{6}$$

which can be rewritten in the "operator" form

$$\langle x| \hat{D}\hat{G} |y\rangle = \langle x|y\rangle \tag{7}$$

where

$$\hat{D}(\partial_x) = i\gamma^\mu \partial_\mu - M \tag{8}$$

Applying Weyl-Wigner transformation, we obtain

$$(\hat{Q}\hat{G})_W = Q_W \star G_W = 1 \tag{9}$$

This is the Gronewold equation.

Approximate generalization of Weyl-Wigner fromalism from continuous space to lattice

The Weyl symbol of operator

$$[\hat{A}]_W(x_n, p) = \int_{\mathcal{M}} dq e^{iqx_n} \langle p + \frac{q}{2} | \hat{A} | p - \frac{q}{2} \rangle \tag{10}$$

Wigner distribution

$$[\hat{\rho}]_W(x_n, p) = W(x, p) = \int_{\mathcal{M}} dq e^{-iqx_n} \langle p - \frac{q}{2} | \hat{\rho} | p + \frac{q}{2} \rangle \tag{11}$$

$$\langle \Psi | \hat{A} | \Psi \rangle = \sum_{x_n} \int_{\mathcal{M}} \frac{dp}{\mathcal{M}} A_W(x_n, p) \rho_W(x_n, p) \tag{12}$$

Where $\mathcal{M}$ is the first Brillouin zone and x_n are the lattice points. We denote the trace of Weyl symbol as follows:

$$\mathrm{Tr} A_W = \sum_{x_n} \int_{\mathcal{M}} \frac{dp}{\mathcal{M}} A_W(x_n, p) \tag{13}$$

1. Star product identity

$$A_W(x,p) \star B_W(x,p) = (\hat{A}\hat{B})_W(x,p). \tag{14}$$

2. First trace identity

$$\mathrm{Tr}\, A_W = \mathrm{tr}\, \hat{A} \tag{15}$$

3. Second trace identity

$$\mathrm{Tr}[A_W(x,p) \star B_W(x,p)] = \mathrm{Tr}[A_W(x,p) B_W(x,p)]. \tag{16}$$

4. Weyl symbol of identity operator

$$(\hat{1})_W(x,p) = 1. \tag{17}$$

5. Star product

$$\star_{xp} \equiv e^{\frac{i}{2}\left(\overleftarrow{\partial_x}\,\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\,\overrightarrow{\partial_x}\right)} \tag{18}$$

Partition function

$$Z = \int D\bar{\psi}D\psi \; e^{S[\psi,\bar{\psi}]} \tag{19}$$

Action

$$S[\psi,\bar{\psi}] = \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p)\hat{Q}(i\partial_p, p)\psi(p) = \sum_{r_n} \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \operatorname{tr}\left[Q_W(r_n, p)W(r_n, p)\right] \tag{20}$$

Using *Peierls* substitution, in the presence of gauge field

$$Q_W(p) \to Q_W(p - A(i\partial_p)) \tag{21}$$

Propagator of fermions is defined as

$$\hat{G} = -\frac{1}{Z}\int D\bar{\psi}D\psi \, |\psi\rangle \langle\bar{\psi}| \exp\left(\int \frac{d^D p}{|\mathcal{M}|}\bar{\psi}(p)\hat{Q}(i\partial_p, p)\psi(p)\right) \tag{22}$$

$$\delta \log Z = \sum_{r_n} \int \frac{d^D p}{|\mathcal{M}|} \mathrm{tr}\left[\delta Q_W G_W\right] = \mathrm{tr}\left[\hat{G}\delta\hat{Q}\right] = \mathrm{Tr}[G_W \star \delta Q_W] = \mathrm{Tr}[G_W \delta Q_W] \tag{23}$$

Variation with respect to the gauge field $A \to A + \delta A$ gives

$$Q_W(x, p - (A + \delta A)) = Q_W(x, p - A) + \partial_{A_i} Q_W(x, p - A)\delta A_i \tag{24}$$

and

$$\delta Q_W = \partial_{A_i} Q_W \delta A_i = -\partial_{p_i} Q_W \delta A_i \tag{25}$$

Electric current is given by

$$j_i(x) = \frac{\delta \log Z}{\delta A_k(x)} = -\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \mathrm{tr}\left[G_W(x, p)\partial_{p_i} Q_W(x, p)\right] \tag{26}$$

Integrating (or summing on the lattice) local current density

$$J_i \equiv \int dx j_i(x) = -\int dx \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \operatorname{tr}\left[G_W(x,p)\partial_{p_i} Q_W(x,p)\right] = \\ = -\operatorname{Tr}\left[G_W(x,p)\partial_{p_i} Q_W(x,p)\right] = -\operatorname{Tr}\left[G_W(x,p)\star\partial_{p_i} Q_W(x,p)\right] \quad (27)$$

J is not the conventional current I, which is defined as an integral of current density over the cross section of a given sample. Relation between the two may be understood easily for the homogeneous system of rectangular form with length L at finite temperature $1/\beta$. Then $J = \beta L I$. For the variation of J we obtain

$$\delta J_i = \delta \operatorname{Tr}\left[G_W \star \partial_{p_i} Q_W\right] = \operatorname{Tr}\left[\delta G_W \star \partial_{p_i} Q_W + G_W \star \partial_{p_i} \delta Q_W\right] \quad (28)$$

Using the variation of the Groenewold equation

$$Q_W(p,x) \star G_W(p,x) = 1 \tag{29}$$

$$(Q_W + \delta Q_W) \star (G_W + \delta G_W) \approx Q_W \star G_W + \delta Q_W \star G_W + Q_W \star \delta G_W \tag{30}$$

which gives

$$\delta G_W = -G_W \star \delta Q_W \star G_W \tag{31}$$

and

$$G_W \star \partial_{p_i} \delta Q_W = \partial_{p_i} (G_W \star \delta Q_W) - \partial_{p_i} G_W \star \delta Q_W \tag{32}$$

as well as

$$\partial_{p_i} G_W = -G_W \star (\partial_{p_i} Q_W) \star G_W \tag{33}$$

and periodic boundary conditions in momentum space we get

$$\delta J_i = -\,\mathrm{Tr}\left[-G_W \star \delta Q_W \star G_W \star \partial_{p_i} Q_W + G_W \star (\partial_{p_i} Q_W) \star G_W \star \delta Q_W\right] \tag{34}$$

Current in Weyl-Wigner formalism - topological invariance

$$\delta J_i = -\operatorname{Tr}\left[-G_W \star \delta Q_W \star G_W \star \partial_{p_i} Q_W + G_W \star (\partial_{p_i} Q_W) \star G_W \star \delta Q_W\right] \tag{35}$$

The cyclic properties of the trace will give

$$\delta J_i = 0 \tag{36}$$

Hence, J_i is topological invariant in the presence of periodic spacial boundary conditions. Notice, that the above consideration fails in the presence of external electric field, when periodic boundary conditions cannot be imposed. Therefore, the appearance of non - vanishing response of J to external electric field does not contradict with the statement that J is topological invariant for the systems with periodic boundary conditions.

Let us consider the case when the external gauge field C is present

$$Q_W(p,x) = Q_W(p - C(x), x) \tag{37}$$

We assume here that Q_W has an additional space dependence to that coming from the gauge field. The gauge field itself is divided to the background one $B(x)$ and to that for which we are looking a linear response $A(x)$.

$$C(x) = A(x) + B(x) \tag{38}$$

Hence, the Dirac operator may be written as

$$Q_W(p,x) \approx Q_W^{(0)}(p,x) + \delta Q_W(p,x) = Q_W^{(0)}(p,x) - \partial_{p_k} Q_W^{(0)}(p,x) A_k(x) \tag{39}$$

where

$$Q_W^{(0)}(p,x) = Q_W^{(0)}\,(p - B(x), x) \tag{40}$$

The propagator may also be presented as a perturbation

$$G_W(p,x) \approx G_W^{(0)}(p,x) + \delta G_W(p,x) \tag{41}$$

substituting this variation back into Groenewold equation, we get

$$\delta G_W = -G_W^{(0)} \star \delta Q_W \star G_W^{(0)} = G_W^{(0)} \star \partial_{p_k} Q_W^{(0)}(p,x) A_k(x) \star G_W^{(0)} \tag{42}$$

which may be written up to linear term sin $A_{ij} = \partial_i A_j - \partial_j A_i$

$$\begin{aligned}
&\delta G_W(p,x) = \\
&\left[G_W^{(0)}(p,x) \star e^{-\frac{i}{2}\overleftarrow{\partial}_p \overrightarrow{\partial}_y} \partial_{p_k} Q_W^{(0)}(p,x) A_k(y) e^{\frac{i}{2}\overleftarrow{\partial}_y \overrightarrow{\partial}_p} \star G_W^{(0)}(p,x) \right]_{y=x} \approx \\
&\left[G_W^{(0)}(p,x) \star \left(1 - \frac{i}{2}\overleftarrow{\partial}_p \overrightarrow{\partial}_y \right) \partial_{p_k} Q_W^{(0)}(p,x) A_k(y) \left(1 + \frac{i}{2}\overleftarrow{\partial}_y \overrightarrow{\partial}_p \right) \star G_W^{(0)}(p,x) \right]_{y=x} \approx \\
&\left[G_W^{(0)}(p,x) \star \left(\partial_{p_k} Q_W^{(0)}(p,x) \right) \star G_W^{(0)}(p,x) \right] A_k(x) - \\
&\frac{i}{2}\left[\left(\partial_{p_i} G_W^{(0)}(p,x) \right) \star \left(\partial_{p_k} Q_W^{(0)}(p,x) \right) \star G_W^{(0)}(p,x) \right] \partial_{x_i} A_k(x) + \\
&\frac{i}{2}\left[G_W^{(0)}(p,x) \star \left(\partial_{p_k} Q_W^{(0)}(p,x) \right) \star \left(\partial_{p_i} G_W^{(0)}(p,x) \right) \right] \partial_{x_i} A_k(x) = \\
&\left[G_W^{(0)} \star \left(\partial_{p_k} Q_W^{(0)} \right) \star G_W^{(0)} \right] A_k + \frac{i}{2}\left[G_W^{(0)} \star \left(\partial_{p_i} Q_W^{(0)} \right) \star G_W^{(0)} \star \left(\partial_{p_j} Q_W^{(0)} \right) \star G_W^{(0)} \right] A_{ij}
\end{aligned} \tag{43}$$

Hence, we may represent δG_W as follows

$$\delta G_W(p,x) = G^{(1)}_{W(k)} A_k + G^{(2)}_{W(ij)} A_{ij} = \left[G^{(0)}_W \star \left(\partial_{p_k} Q^{(0)}_W \right) \star G^{(0)}_W \right] A_k + \frac{i}{2} \left[G^{(0)}_W \star \left(\partial_{p_i} Q^{(0)}_W \right) \star G^{(0)}_W \star \left(\partial_{p_j} Q^{(0)}_W \right) \star G^{(0)}_W \right] A_{ij} \tag{44}$$

$$G_W(p,x) \approx G^{(0)}_W + G^{(1)}_{W(k)} A_k + G^{(2)}_{W(ij)} A_{ij} \tag{45}$$

$$G^{(1)}_{W(i)} = \left[G^{(0)}_W \star \left(\partial_{p_i} Q^{(0)}_W \right) \star G^{(0)}_W \right] = -\partial_{p_i} G^{(0)}_W \tag{46}$$

$$G^{(2)}_{W(ij)} = \frac{i}{2} \left[G^{(0)}_W \star \left(\partial_{p_i} Q^{(0)}_W \right) \star G^{(0)}_W \star \left(\partial_{p_j} Q^{(0)}_W \right) \star G^{(0)}_W \right] \tag{47}$$

In case of sufficiently weak inhomogeneity

$$\delta \log Z = \sum_{x_n} \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \operatorname{tr} [\delta Q_W(x_n, p) G_W(x_n, p)] \approx \int dx \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \operatorname{tr} [\delta Q_W(x, p) G_W(x, p)] \tag{48}$$

As it was already mentioned above

$$j_k(x) = \frac{\delta \log Z}{\delta A_k(x)} = -\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \operatorname{tr} [G_W(x, p) \partial_{p_k} Q_W(x, p)] \tag{49}$$

Since

$$\partial_{p_i} Q_W = \partial_{p_i} Q_W^{(0)} - \left(\partial_{p_i} \partial_{p_j} Q_W^{(0)}\right) A_j \tag{50}$$

$$G_W \partial_{p_k} Q_W = \left(G_W^{(0)} + G_{W(l)}^{(1)} A_l + G_{W(mn)}^{(2)} A_{mn}\right) \left(\partial_{p_i} Q_W^{(0)} - \left(\partial_{p_i} \partial_{p_j} Q_W^{(0)}\right) A_j\right) \tag{51}$$

up to linear terms in A_i and A_{ij}, the current density may be written as follows

$$j_i(x) = j_i^{(0)}(x) + j_{i(k)}^{(1)}(x) A_k(x) + j_{i(mn)}^{(2)}(x) A_{mn}(x) \tag{52}$$

$$j_i(x) = j_i^{(0)}(x) + j_{i(k)}^{(1)}(x) A_k(x) + j_{i(mn)}^{(2)}(x) A_{mn}(x) \tag{53}$$

where

$$j_i^{(0)}(x) = -\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \,\mathrm{tr}\left[G_W^{(0)} \partial_{p_i} Q_W^{(0)} \right] = 0 \tag{54}$$

$$j_{i(k)}^{(1)}(x) = \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \,\mathrm{tr}\left[\partial_{p_k} \left(G_W^{(0)} \partial_{p_i} Q_W^{(0)} \right) \right] = 0 \quad \text{(in periodic BC)} \tag{55}$$

$$j_{i(mn)}^{(2)}(x) = -\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \,\mathrm{tr}\left[\frac{i}{2} \left[G_W^{(0)} \star \left(\partial_{p_m} Q_W^{(0)} \right) \star G_W^{(0)} \star \left(\partial_{p_n} Q_W^{(0)} \right) \star G_W^{(0)} \right] \partial_{p_i} Q_W^{(0)} \right] \tag{56}$$

The $j_{i(mn)}^{(2)}(x)$ is the local electric conductivity tensor since it is a coefficient in front of electromagnetic tensor.
The total integrated current

$$\begin{aligned} J_k = \int dx j_k(x) = -\int dx \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \,\mathrm{tr}\left[G_W(x,p) \partial_{p_k} Q_W(x,p) \right] = \\ = -\,\mathrm{Tr}\left[G_W(x,p) \partial_{p_k} Q_W(x,p) \right] = -\,\mathrm{Tr}\left[G_W(x,p) \star \partial_{p_k} Q_W(x,p) \right] \end{aligned} \tag{57}$$

The average electric conductivity (we assume $A_{ij} = const$) is to be obtained from:

$$\bar{J}_i^{(2)} = J_i^{(2)}/V^{(D)} \equiv \frac{1}{V^{(4)}} A_{mn} \int d^D x j^{(2)}_{i(mn)}(x) = \frac{1}{\beta \mathrm{V}} A_{mn} \int dx j^{(2)}_{i(mn)}(x)$$
$$= -\frac{iA_{mn}}{2\beta \mathrm{V}} \mathrm{Tr} \left[\left(G_W^{(0)} \star \left(\partial_{p_m} Q_W^{(0)} \right) \star G_W^{(0)} \star \left(\partial_{p_n} Q_W^{(0)} \right) \star G_W^{(0)} \right) \partial_{p_i} Q_W^{(0)} \right] = \mathcal{W}_{mni} A_{mn} \tag{58}$$

$$\mathcal{W}_{mni} \equiv -\frac{i}{2\beta \mathrm{V}} \mathrm{Tr} \left[\left(G_W^{(0)} \star \left(\partial_{p_m} Q_W^{(0)} \right) \star G_W^{(0)} \star \left(\partial_{p_n} Q_W^{(0)} \right) \star G_W^{(0)} \right) \partial_{p_i} Q_W^{(0)} \right] \tag{59}$$

The conductivity tensor

$$J_i = \sigma_{ij} E_j \tag{60}$$

may be obtained by converting the Euclidean fields back to Minkowski ones

$$J_i = \mathcal{W}_{4ji} F^E_{4j} = \frac{\mathcal{W}_{4ji}}{i} E_j \tag{61}$$

where, the Hall conductivity is

$$\sigma_{kj} \equiv \frac{\mathcal{W}_{4[kj]}}{i} = \frac{1}{i} \left(\mathcal{W}_{4kj} - \mathcal{W}_{4jk} \right) = \frac{1}{2\pi^2} \epsilon^{kjl4} \mathcal{N}_l \tag{62}$$

$$\mathcal{N}_l = -\frac{T \epsilon_{ijkl}}{\mathrm{V}\, 3!\, 8\pi^2} \int d^4x d^4p \, \mathrm{Tr} \left[G_W(p,x) \star \frac{\partial Q_W(p,x)}{\partial p_i} \star \frac{\partial G_W(p,x)}{\partial p_j} \star \frac{\partial Q_W(p,x)}{\partial p_k} \right] \tag{63}$$

The local axial current density may be defined as

$$j_k^5(x) = -\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \operatorname{tr}\left[\gamma^5 G_W(x,p)\partial_{p_k} Q_W(x,p)\right] \tag{64}$$

Repeating all steps of the previous section

$$j_k^5(x) = -\frac{i}{2}\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \operatorname{tr}\left[\gamma^5 \left[G_W^{(0)} \star \left(\partial_{p_i} Q_W^{(0)}\right) \star G_W^{(0)} \star \left(\partial_{p_j} Q_W^{(0)}\right) \star G_W^{(0)}\right] \partial_{p_k} Q_W^{(0)}\right] F_{ij} \tag{65}$$

Integrating (or summing on the lattice) the local current

$$J_i^5 \equiv \int d^D x j_i^5(x) = -\int d^D x \int_{\mathcal{M}} \frac{d^D p}{\mathrm{v}|\mathcal{M}|} \operatorname{tr}\left[\gamma^5 G_W(x,p)\partial_{p_i} Q_W(x,p)\right] \tag{66}$$

Dividing by the total 4 - volume we obtain the average axial current

$$\begin{aligned} \bar{J_k}^5 = \frac{J_k^5}{\beta \mathrm{V}} = -\frac{i}{2}\frac{1}{\beta \mathrm{V}}\int d^D x \int_{\mathcal{M}} \frac{d^D p}{(2\pi)^D} \\ \operatorname{tr}\left[\gamma^5 \left[G_W^{(0)} \star \left(\partial_{p_i} Q_W^{(0)}\right) \star G_W^{(0)} \star \left(\partial_{p_j} Q_W^{(0)}\right) \star G_W^{(0)}\right] \partial_{p_k} Q_W^{(0)}\right] F_{ij} \end{aligned} \tag{67}$$

Chiral Separation Effect - (The absence of) topological invariance for the total axial current

The total integrated axial current

$$J_i^5 \equiv -\int d^D x \int_{\mathcal{M}} \frac{d^D p}{(2\pi)^D} \,\mathrm{tr}\left[\gamma^5 G_W(x,p)\partial_{p_i} Q_W(x,p)\right] = -\,\mathrm{Tr}\left[\gamma^5 G_W(x,p) \star \partial_{p_i} Q_W(x,p)\right] \tag{68}$$

The variation

$$\delta J_i^5 = -\delta\,\mathrm{Tr}\left[\gamma^5 G_W \star \partial_{p_i} Q_W\right] = -\,\mathrm{Tr}\left[\gamma^5 \delta G_W \star \partial_{p_i} Q_W + \gamma^5 G_W \star \partial_{p_i} \delta Q_W\right] \tag{69}$$

Using the identities (like in the case of electric current)

$$\gamma^5 \delta G_W \star \partial_{p_i} Q_W = -\gamma^5 G_W \star \delta Q_W \star G_W \star \partial_{p_i} Q_W \tag{70}$$

$$\gamma^5 G_W \star \partial_{p_i} \delta Q_W = \gamma^5 \partial_{p_i} \left(G_W \star \delta Q_W\right) - \gamma^5 \partial_{p_i} G_W \star \delta Q_W \tag{71}$$

$$\partial_{p_i} G_W = -G_W \star (\partial_{p_i} Q_W) \star G_W \tag{72}$$

we obtain (for the case of periodic boundary conditions) that under the trace we may substitute

$$\gamma^5 G_W \star \partial_{p_i} \delta Q_W = \gamma^5 G_W \star (\partial_{p_i} Q_W) \star G_W \star \delta Q_W \tag{73}$$

Chiral Separation Effect - (The absence of) topological invariance for the total axial current

As a result we come to

$$\delta J_i = -\,\mathrm{Tr}\left[-\gamma^5 G_W \star \delta Q_W \star G_W \star \partial_{p_i} Q_W + \gamma^5 G_W \star (\partial_{p_i} Q_W) \star G_W \star \delta Q_W\right] \qquad (74)$$

If γ^5 commutes (or anti - commutes) with G and Q then

$$\begin{aligned}\mathrm{Tr}\left[\gamma^5 G_W \star \delta Q_W \star G_W \star \partial_{p_i} Q_W\right] &= \mathrm{Tr}\left[\partial_{p_i} Q_W \star \gamma^5 G_W \star \delta Q_W \star G_W\right] = \\ \mathrm{Tr}\left[G_W \star \partial_{p_i} Q_W \star \gamma^5 G_W \star \delta Q_W\right] &= \mathrm{Tr}\left[\gamma^5 G_W \star \partial_{p_i} Q_W \star G_W \star \delta Q_W\right]\end{aligned} \qquad (75)$$

Under this (very restrictive) condition we obtain

$$\delta J_i^5 = 0 \qquad (76)$$

We implied that there are no singularities of the Green function. That means that the fermions are gapped, and the Fermi energy is within the gap. We conclude that for the systems with gapped fermions in the presence of precise chiral symmetry the total integrated axial current J_i^5 would be a topological invariant. In practise the corresponding requirements are too restrictive. For lattice Dirac fermions the presence of a gap (mass) excludes chiral symmetry.

Chiral Separation Effect - Axial current for gapless fermions at finite temperature

We are going to regularize the theory by finite (but small) temperature in order to deal with gapless fermions. Matsubara frequencies are

$$p_4 = \omega_n = \frac{2\pi\left(n+\frac{1}{2}\right)}{\beta} \tag{77}$$

Here inverse temperature $\beta = 1/T$ is taken in lattice units: $N_t \equiv \frac{1}{T}$, and the values of p_4 are

$$p_4 = \frac{2\pi\left(n_4+\frac{1}{2}\right)}{N_t} \qquad n_4 = -\frac{N_t}{2}, .., \frac{N_t}{2} - 1 \tag{78}$$

The boundary values are

$$\omega_{n=-\frac{N_t}{2}} = \frac{2\pi\left(-\frac{N_t}{2}+\frac{1}{2}\right)}{N_t} = -\pi + \frac{\pi}{N_t} \qquad \omega_{n=\frac{N_t}{2}-1} = \frac{2\pi\left(\frac{N_t}{2}-\frac{1}{2}\right)}{N_t} = \pi - \frac{\pi}{N_t} \tag{79}$$

The Matsubara frequencies most close to zero are: $\omega_{n=0} = \frac{\pi}{N_t}$ and $\omega_{n=-1} = -\frac{\pi}{N_t}$. One can see that ω_n never equals to zero. Therefore, even for the system of massless/gapless fermions the propagator never has poles in momentum space.

As a result the expression for the axial current is well - defined

$$\bar{J}_k^5 = -\frac{i}{2}\frac{1}{\beta V}\sum_{n=-\frac{N_t}{2}}^{\frac{N_t}{2}-1}\int d^3x \int_{\mathcal{M}_3}\frac{d^3p}{(2\pi)^3}$$
$$\mathrm{tr}\left[\gamma^5\left[G_W^{(0)}\star\left(\partial_{p_i}Q_W^{(0)}\right)\star G_W^{(0)}\star\left(\partial_{p_j}Q_W^{(0)}\right)\star G_W^{(0)}\right]\partial_{p_k}Q_W^{(0)}\right]F_{ij} \tag{80}$$

Introducing the chemical potential $\omega_n \to \omega_n - i\mu$ we obtain

$$\bar{J}_k^5 = -\frac{1}{2V\beta}\sum_{n=-\frac{N_t}{2}}^{\frac{N_t}{2}-1}\int d^3x \int_{\mathcal{M}_3}\frac{d^3p}{(2\pi)^3}$$
$$\partial_{\omega_n}\mathrm{tr}\left[\gamma^5\left[G_W^{(0)}\star\left(\partial_{p_i}Q_W^{(0)}\right)\star G_W^{(0)}\star\left(\partial_{p_j}Q_W^{(0)}\right)\star G_W^{(0)}\right]\partial_{p_k}Q_W^{(0)}\right]F_{ij}\mu \tag{81}$$

Chiral Separation Effect - Axial current for gapless fermions at finite temperature

We represent the above expression as

$$J_k^5(x) = \sigma_{ijk} F_{ij} \mu \tag{82}$$

where

$$\begin{gathered}\sigma_{ijk} = -\frac{1}{2V\beta} \sum_{n=-\frac{N_t}{2}}^{\frac{N_t}{2}-1} \int d^3x \int_{\mathcal{M}_3} \frac{d^3p}{(2\pi)^3} \\ \partial_{\omega_n} \mathrm{tr} \left[\gamma^5 \left[G_W^{(0)} \star \left(\partial_{p_{[i}} Q_W^{(0)} \right) \star G_W^{(0)} \star \left(\partial_{p_{j]}} Q_W^{(0)} \right) \star G_W^{(0)} \right] \partial_{p_k} Q_W^{(0)} \right]\end{gathered} \tag{83}$$

has the meaning of the CSE conductivity when external field strength corresponds to a constant magnetic field H: $F_{ij} = -\epsilon_{ijk} H_k$. Then

$$J_k^5(x) = -\sigma_{ijk} \epsilon_{ijk'} H_{k'} \mu$$

We will see below that for the wide range of systems $-\epsilon_{ijk}\sigma_{ijk'} = \delta^{kk'} \sigma_{CSE}$.

the expression for the CSE conductivity may be written as

$$\sigma_{ijk} = \sum_{n=-\frac{N_t}{2}}^{\frac{N_t}{2}-1} \partial_{\omega_n} \sigma^{(3)}_{ijk} \tag{84}$$

where

$$\sigma^{(3)}_{ijk} = -\frac{1}{2\mathrm{V}} \int d^3x \int_{\mathcal{M}_3} \frac{d^3p}{(2\pi)^3} \mathrm{tr}\left[\gamma^5 \left[G^{(0)}_W \star \left(\partial_{p_{[i}} Q^{(0)}_W\right) \star G^{(0)}_W \star \left(\partial_{p_{j]}} Q^{(0)}_W\right) \star G^{(0)}_W\right] \partial_{p_k} Q^{(0)}_W\right] \tag{85}$$

The limit of small temperature $T \to 0$, $N_t \to \infty$, $\frac{\pi}{N_t} = \epsilon \to 0$ allows to replace the sum by an integral. However, the point $\omega = 0$ is excluded from this integral

$$\sum_{n=-\frac{N_t}{2}}^{\frac{N_t}{2}-1} \quad \to \quad \frac{\beta}{2\pi} \int_{-\pi+\epsilon}^{0-\epsilon} d\omega + \frac{\beta}{2\pi} \int_{0+\epsilon}^{\pi-\epsilon} d\omega \tag{86}$$

The CSE conductivity becomes

$$\begin{aligned}\sigma_{ijk} &= \lim_{\epsilon\to 0}\int_{-\pi+\epsilon}^{0-\epsilon} d\omega\partial_\omega\sigma_{ijk}^{(3)} + \int_{0+\epsilon}^{\pi-\epsilon} d\omega\partial_\omega\sigma_{ijk}^{(3)} \\ &= \lim_{\epsilon\to 0}\left[\sigma_{ijk}^{(3)}(-\pi+\epsilon) - \sigma_{ijk}^{(3)}(0-\epsilon) + \sigma_{ijk}^{(3)}(0+\epsilon) - \sigma_{ijk}^{(3)}(\pi-\epsilon)\right]\end{aligned} \tag{87}$$

using periodicity, $\sigma_{ijk}^{(3)}(-\pi) = \sigma_{ijk}^{(3)}(\pi)$, we obtain

$$\sigma_{ijk} = \lim_{\epsilon\to 0}\left[\sigma_{ijk}^{(3)}(0+\epsilon) + \left(-\sigma_{ijk}^{(3)}(0-\epsilon)\right)\right] \tag{88}$$

where

$$\begin{aligned}&\sigma_{ijk}^{(3)}(\omega = 0\pm\epsilon) = \\ &= -\frac{1}{2V}\int d^3x \int_{\mathcal{M}_3}\frac{d^3p}{(2\pi)^4}\,\mathrm{tr}\left[\gamma^5\left[G_W^{(0)}\star\left(\partial_{p_i}Q_W^{(0)}\right)\star G_W^{(0)}\star\left(\partial_{p_j}Q_W^{(0)}\right)\star G_W^{(0)}\right]\partial_{p_k}Q_W^{(0)}\right]\Bigg|_{\omega=0\pm\epsilon} \\ &= -\frac{1}{2V}\int_{\mathcal{M}_3}\frac{d^3p}{(2\pi)^4}\int d^3x\,\mathrm{tr}\left[\gamma^5\left[G_W^{(0)}\star\left(\partial_{p_i}Q_W^{(0)}\right)\star G_W^{(0)}\star\left(\partial_{p_j}Q_W^{(0)}\right)\star G_W^{(0)}\right]\partial_{p_k}Q_W^{(0)}\right]\Bigg|_{\omega=0\pm\epsilon}\end{aligned} \tag{89}$$

The integrals in the CSE conductivity expression

$$\sigma_{ijk} = \lim_{\epsilon \to 0} \left[\sigma^{(3)}_{ijk}(0+\epsilon) + \left(-\sigma^{(3)}_{ijk}(0-\epsilon) \right) \right] \quad (90)$$

cancel each other except those in the small vicinities of the mentioned above singularities.

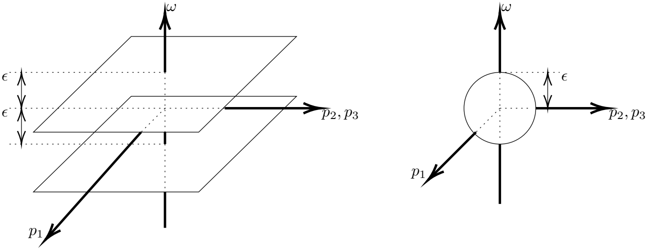

Figure: Deformation of the surface surrounding singularities

Hence, we may restrict the integrations by the small regions of the Brillouin zone above/below the singularities. In this region we assume the presence of precise chiral symmetry, which means that the effective low energy theory of our lattice model is chiral invariant (if the chiral anomaly is ignored).

We restrict integrations to the region, where γ^5 commutes/anti - commutes with Q and G. As a result the sum of the integrals represents a topological invariant, which does not depend on the form of the surface in $4D$ momentum space surrounding the singularities.

We may deform this surface arbitrarily in such a way that it remains surrounding the singularities. This way instead of the two pieces of the infinitely close planes (situated above and below the singularities) we may integrate over the sphere in momentum space.

$$\sigma_{ijk} = - \frac{1}{2\mathrm{V}} \int_{\Sigma_3} \frac{d^3p}{(2\pi)^4} \int d^3x \, \mathrm{tr} \left[\gamma^5 \left[G_W^{(0)} \star \left(\partial_{p_{[i}} Q_W^{(0)} \right) \star G_W^{(0)} \star \left(\partial_{p_{j]}} Q_W^{(0)} \right) \star G_W^{(0)} \right] \partial_{p_k} Q_W^{(0)} \right] \quad (91)$$

Here the integral is over Σ_3, which is the 3D hypersurface in 4D momentum space. It consists of the two infinitely close pieces of the planes situated above and below the singularities of expression standing in the integral.
Since γ^5 commutes/anti - commutes with G and Q in this region, we may rewrite this expression as

$$\sigma_{ijk} = \epsilon_{ijk}\sigma_H/2 \quad (92)$$

where $\sigma_H = \frac{\mathcal{N}}{2\pi^2}$, and

$$\mathcal{N} = -\frac{\epsilon_{ijk}}{48\pi^2\mathrm{V}} \int_{\Sigma_3} d^3p \int d^3x \, \mathrm{tr} \left[\gamma^5 \left[G_W^{(0)} \star \left(\partial_{p_i} Q_W^{(0)} \right) \star G_W^{(0)} \star \left(\partial_{p_j} Q_W^{(0)} \right) \star G_W^{(0)} \right] \partial_{p_k} Q_W^{(0)} \right] \quad (93)$$

When space inhomogeneities are sufficiently weak we are able to omit the star product in the above expression

$$\mathcal{N} = -\frac{\epsilon_{ijk}}{48\pi^2 \mathrm{V}} \int_{\Sigma_3} d^3p \int d^3x \,\mathrm{tr}\left[\gamma^5 \left[G_W^{(0)} \left(\partial_{p_i} Q_W^{(0)}\right) G_W^{(0)} \left(\partial_{p_j} Q_W^{(0)}\right) G_W^{(0)}\right] \partial_{p_k} Q_W^{(0)}\right] \tag{94}$$

Let us discuss for the definiteness example of the system with Wilson fermions in the presence of weakly varying external field. We assume that this model is used for the description of the continuous field theory with one massless fermion. This means that parameter $m^{(0)}$ is set to zero. In the absence of electric potential the model has one Fermi point at $p = 0$. The calculation in this case gives $\mathcal{N} = 1$.

Conclusions and discussions

In the present work we discussed the Chiral Separation Effect in essentially nonhomogeneous systems of chiral fermions.

The integral, in the CSE conductivity expression, is taken along the closed surface Σ_3 that surrounds positions $\Xi(x)$ of all singularities of expression standing inside the integral.

We assumed that along $\Xi(x)$, for each x, γ^5 commutes/anti - commutes with Q_W, G_W.

Under these conditions the CSE conductivity is a topological invariant, and, it is robust to the smooth deformations of the system as long as Σ_3 does not cross $\Xi(x)$ for any x. This property of $\mathcal{N}$ allows us to make a strong statement about the CSE conductivity of nonhomogeneous system.

It remains non-sensitive to the particular form of lattice Dirac operator, which means that we may start our consideration from the system with Wilson fermions in the presence of weak external electric potential, for which the value of $\mathcal{N}$ may be calculated, and then, deform the system smoothly to a more complex system. As long as the conditions mentioned above are fulfilled, the $\mathcal{N}$ will remain the same, and it equals the number of effective chiral Dirac fermions.

Wilson fermions

Lattice Dirac operator of Wilson fermions is defined as

$$Q_{\alpha\beta}(p) \equiv \tilde{K}_{\alpha\beta}(p) = \left[\sum_{k=1,2,3,4} i\gamma_k g_k(p) + m(p)\right]_{\alpha\beta} = i\left[\sum_{k=1,2,3,4} \gamma_k g_k(p) - im(p)\right]_{\alpha\beta} \quad (95)$$

where

$$g_k(p) = \sin(p_k) \qquad m(p) = m^{(0)} + \sum_{\nu=1}^{4}(1 - \cos(p_\nu)) \quad (96)$$

The two-point function

$$G_{\alpha\beta}(n,m) \equiv K^{-1}_{\alpha\beta}(n,m) = \frac{\left[\sum_q -i\gamma_q g_q(p) + m(p)\right]_{\alpha\beta}}{\sum_k g_k^2(p) + m^2(p)} \quad (97)$$

In the presence of external field

$$Q_{\alpha\beta}(\hat{p} - A(\hat{x})) = \left[\sum_k i\gamma_k g_k(\hat{p} - A(\hat{x})) + m(\hat{p} - A(\hat{x}))\right]_{\alpha\beta} \quad (98)$$

$$G_{\alpha\beta}(\hat{p} - A(\hat{x}))) \approx \frac{\left[\sum_q -i\gamma_q g_q(\hat{p} - A(\hat{x})) + m(\hat{p} - A(\hat{x}))\right]_{\alpha\beta}}{\sum_k g_k^2(\hat{p} - A(\hat{x})) + m^2(\hat{p} - A(\hat{x}))} \quad (99)$$

Calculation of CSE topological invariant for Wilson fermions model in the presence of slowly varying external field

$$m(\hat{p}) = m^{(0)} + \sum_{k=1}^{4}(1 - \cos(\hat{p}_k)) = m^{(0)} + \sum_{k=1}^{4} 2\sin^2\left(\frac{\hat{p}}{2}\right) = m^{(0)} + \sum_{k=1}^{4} 2g_k^2\left(\frac{\hat{p}}{2}\right) \tag{100}$$

The Green's function poles are given by the solutions of equation

$$\sum_k g_k^2(\hat{p} - A(\hat{x})) + m^2(\hat{p} - A(\hat{x})) = 0 \tag{101}$$

For the massless fermions, when $m^{(0)} = 0$,

$$Q(\hat{p} - A(\hat{x})) = \sum_k \left[i\gamma_k g_k(\hat{p} - A(\hat{x})) + 2g_k^2\left(\frac{\hat{p} - A(\hat{x})}{2}\right)\right] \tag{102}$$

and

$$\hat{G}(\hat{p} - A(\hat{x})) = \frac{\left[\sum_q -i\gamma_q g_q(\hat{p} - A(\hat{x})) + 2g_q^2\left(\frac{\hat{p}-A(\hat{x})}{2}\right)\right]}{\sum_k g_k^2(\hat{p} - A(\hat{x})) + 4\left(\sum_j g_j^2\left(\frac{\hat{p}-A(\hat{x})}{2}\right)\right)^2} \equiv \frac{R(p-A)}{U(p-A)} \tag{103}$$

Calculation of CSE topological invariant for Wilson fermions model in the presence of slowly varying external field

$$R = \sum_{k=1}^{3}\left[-i\gamma_k \sin(p_k - A_k) + 2\sin^2\left(\frac{p_k - A_k}{2}\right)\right] - \\ - i\gamma_4 \sin(p_4 - A_4) + \sin^2\left(\frac{p_4 - A_4}{2}\right) \tag{104}$$

and

$$U = \sum_{k=1}^{3}\sin^2(p_k - A_k) + \sin^2(p_4 - A_4) + \\ + 4\left[\sum_{k=1}^{3}\sin^2\left(\frac{p_k - A_k}{2}\right) + \sin^2\left(\frac{p_4 - A_4}{2}\right)\right]^2 \tag{105}$$

The poles are space dependent and for $A_4 = i\phi = 0$ they correspond to the single points

$$p_i = A_i(x) \tag{106}$$

In the neighborhood of these points

$$p_i - A_i(x) = \xi_i \to 0 \qquad \to \qquad g_i(\xi) \approx \xi_i \tag{107}$$

Calculation of CSE topological invariant for Wilson fermions model in the presence of slowly varying external field

In the case of nonzero $A_4(x) = i\phi(x) \to 0$ instead of the singularities concentrated at a point in momentum space for any given x we have singularities concentrated along the closed surfaces in momentum space. The form of these surfaces depends on x. More explicitly, we have spheres with the center at $p = A(x)$ and radius $|\phi_0(x)|$. The Dirac operator becomes

$$Q(\hat{p} - A(\hat{x})) = \sum_k \left[i\gamma_k \xi_k + \frac{1}{2}\xi_k^2 \right] \tag{108}$$

while

$$\partial_{p_i} Q = \partial_{\xi_i} Q = i\gamma_i + \xi_i \tag{109}$$

For the Green function we have

$$\hat{G}(\hat{p} - A(\hat{x})) = \frac{\left[\sum_q -i\gamma_q \xi_q + \frac{1}{2}\xi_q^2\right]}{\sum_k \xi_k^2 + \frac{1}{4}\left(\sum_j \xi_j^2\right)^2} \tag{110}$$

$$\partial_{p_i} G = \partial_{\xi_i} G = -i\frac{\gamma_i - 2\xi^i(\gamma\xi)/\xi^2}{\xi^2} \tag{111}$$

Calculation of CSE topological invariant for Wilson fermions model in the presence of slowly varying external field

Here surface Σ_3 surrounds all singularities of the Green functions at all values of x. The key point for the calculation of $\mathcal{N}$ is that we may deform the system smoothly removing the fields A, ϕ at all. This will bring us to a homogeneous system with $A = \phi = 0$. Then we chose Σ_3 of the form of the 3 sphere that surrounds point $p = 0$.

As a result integral over x is irrelevant, and we have an integral over the surface of sphere ($d\sigma^i$ is a vector orthogonal to the surface of the sphere, its absolute value is equal to the area element):

$$\begin{aligned} \mathcal{N} &= \frac{1}{48\pi^2} \int_{\Sigma_3} \mathrm{tr} \left[\gamma^5 G \, (dQ) \wedge dG \wedge (dQ) \right] \\ &= \frac{\epsilon_{ijkl}}{48\pi^2} \int_{\Sigma_3} d\sigma^i \, \mathrm{tr} \left[\gamma^5 G \partial^j Q \partial^k G \partial^l Q \right] \\ &= \frac{\epsilon_{ijkl}}{48\pi^2} \int_{\Sigma_3} \frac{d\sigma^i}{\xi^4} \, \mathrm{tr} \left[\gamma^5 \gamma^a \xi^a \gamma^j \gamma^k \gamma^l \right] \\ &= \frac{4\epsilon_{ijkl}}{48\pi^2} \int_{\Sigma_3} \frac{d\sigma^i \xi^a}{\xi^4} \epsilon^{ajkl} \\ &= \frac{24\delta_{ia}}{48\pi^2} \int_{\Sigma_3} \frac{d\sigma^i \xi^a}{\xi^4} = 1 \end{aligned} \tag{112}$$