Renormalization group analysis of strongly anisotropic self-organized critical system subjected to isotropic turbulent flow

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9^{th} International Conference on New Frontiers in Physics

2 of October, 2020

Physics of Elementary Particles and Statistical Physics

In this talk we deal with two areas of physic: statistical physics and high energy physics:

- Hwa-Kardar equation describing the growth of the surface;
- stochastic description of the system;
- functional integration and calculation of Feynman graphs;
- renormalization group (RG).

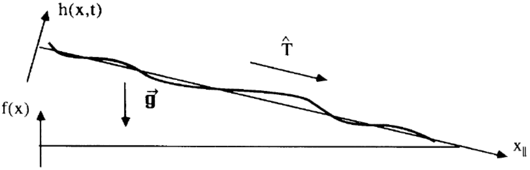
The problem under consideration is growth of the surface under turbulent motion of the environment and mutual influence of the isotropy of the system and anisotropy of the turbulent flow.

Plan of the talk

The main steps (general scheme) are following:

- Stochastic formulation of Hwa-Kardar model;
- quantum field action and Feynman diagrams;
- divergences of the diagrams;
- renormalization, RG, RG flow and fixed points;
- critical dimensions at different fixed points.

Hwa-Kardar model



T. Hwa and M. Kardar, Phys. Rev. Let., Vol.62, №16 (1989).

Stochastic equation (f is a random force)

$$\partial_t h(t,x) = \nu_{\|}\partial_{\|}^2 h(t,x) + \nu_{\perp}\partial_{\perp}^2 h(t,x) - \frac{1}{2}\partial_{\|} h^2(t,x) + f(t,x).$$

Anisotropy of the system

$$\mathbf{x} = \mathbf{x}_{\perp} + \mathbf{n}x_{\|}, \quad \mathbf{n}\perp\mathbf{x}_{\perp} \qquad (\hat{T} \equiv \mathbf{n}).$$

Hwa-Kardar model: turbulent mixing

Introducing of velocity field: anisotropic one

$$\partial_t \to \nabla_t \equiv \partial_t + v\partial_{\|};$$

$$\mathbf{v}(t,x) = \mathbf{n}v(t,x_\perp), \quad \partial_{\|} v = 0.$$

In the real life **v** obeys Navier-Stokes equation, but calculation should be very complicated. Simplified model: **v** is a Gaussian variable with given correlator function

$$\left\langle v(t,x)v(t',x')\right\rangle = \int \frac{dkd\omega}{(2\pi)^d} e^{ik(x-x')-i\omega(t-t')} B_v(\omega,k),$$

$$B_v(\omega,k) = 2\pi\delta(k_{\|}) B_0 \frac{k_\perp^{5-d-(\xi+\eta)}}{\omega^2 + [\alpha\nu_\perp k_\perp^{2-\eta}]^2}.$$

arXiv:2005.04756

Hwa-Kardar model: turbulent mixing

Introducing of velocity field: isotropic one

$$\langle v_i(t,\mathbf{x}) v_j(t',\mathbf{x}')\rangle = \delta(t-t') D_{ij}(\mathbf{x}-\mathbf{x}'),$$

$$D_{ij}(\mathbf{r}) = B_0 \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{k^{d+\xi}} P_{ij}(\mathbf{k}) \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}).$$

arXiv:2009.00302

Hwa-Kardar model: choise of the random force

Thermal noise

$$\left\langle f(x)f(x')\right\rangle = D_0\,\delta(t-t')\delta^{(d)}(\mathbf{x}-\mathbf{x}'), \quad D_0>0;$$

rapid correlations in both space and time.

Static noise (optional, do not consider here)

$$\left\langle f(x)f(x')\right\rangle = D_0\,\delta^{(d)}(\mathbf{x}-\mathbf{x}'), \quad D_0>0;$$

external average influence of the environment to the landscape is a constant.

Action functional: General rules

Theorem: any stochastic equation of the type

$$\partial_t \phi(x) = U(x, \phi) + f(x), \quad \langle f(x) f(x') \rangle = D(x, x'),$$

where $\phi(x) = \phi(t, \mathbf{x})$ is a random field, $U(x, \phi)$ is a t-local functional depending on the fields and their derivatives, $f(x)$ is a random force, **is equivalent to quantum field model** of the double set of fields $\widetilde{\phi} = \{\phi, \phi'\}$ and action functional

$$S[\varphi] = \underbrace{\frac{1}{2}\varphi' D \varphi'}_{\text{noise term}} + \varphi' \underbrace{[-\partial_t \varphi + U]}_{\text{dynamics}},$$

integration over t and $\mathbf{x}$ implied.

Action functional: General rules

What does it mean:

- statistical average is equivalent to functional integration with weight $\exp S[\phi]$;
- classical random field $\rightarrow$ quantum field;
- we may use all techniques from quantum field theory: Feynman graphs, renormalization group, operator product expansion, *etc.*

Actions functional

Quantum field action:

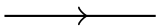
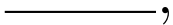
$$S(\Phi) = \frac{1}{2} h' D_0 h' + h'(-\nabla_t h + \nu_{\|_0} \partial_\|^2 h + \nu_{\perp_0} \partial_\perp^2 h - \frac{1}{2} \partial_\| h^2) + S_v.$$

All integrations are implied:

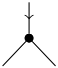
$$h' D_0 h' \equiv D_0 \int dt \int d^d x \, h'(t,x) h'(t,x).$$

Feynman rules and divergent functions

According to general rules both models contain three propagators

$\langle hh' \rangle_0 =$ , $\quad \langle hh \rangle_0 =$ , $\quad \langle vv \rangle_0 =$ 

and two vertices

$-\frac{1}{2} h' \partial_{\|} h^2 = h^2 \partial_{\|'} h' =$  ; $\quad -h'(v\partial_{\|})h = h(v\partial_{\|})h' =$ ,

The propagators are

$$\langle hh' \rangle_0 = \frac{1}{-i\omega + \epsilon(k)}, \quad \langle hh \rangle_0 = \frac{D_0}{\omega^2 + \epsilon^2(k)}; \quad \epsilon(k) \equiv \nu_{\|0} k_{\|}^2 + \nu_{\perp 0} k_{\perp}^2.$$

Logarithmic dimension is $d = 4$ and the only divergent Green function is $\langle h' h \rangle$.

Dimensions and scales

The key difference between anisotropic velocity ensemble

$$\mathbf{v}(t,x) = \mathbf{n} v(t, x_\perp)$$

and isotropic one (under consideration)

$$\langle v_i(t,\mathbf{x}) v_j(t',\mathbf{x}')\rangle = \delta(t-t') D_{ij}(\mathbf{x}-\mathbf{x}')$$

is possibility/impossibility to introduxe two independent momentum scales.

In first case any quantity F is described by three canonical dimensions

$$[F] \sim [T]^{-d_F^\omega} [L_{||}]^{-d_F^{||}} [L_\perp]^{-d_F^\perp},$$

in second case only by two:

$$[F] \sim [T]^{-d_F^\omega} [L]^{-d_F^k}.$$

Couplings and diagrams

This fact drastically changes RG analysis: we have one more dimensionless quantity, i.e., one more "coupling constant"

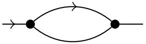
$$u_0 = \nu_{\|0}/\nu_{\perp 0}.$$

All together three couplings are

$$D_0 = g_0(\nu_{\|_0}\nu_{\perp_0})^{3/2}, \quad B_0 = w_0\nu_{\|_0}\nu_{\perp_0}^2, \quad u_0 = \nu_{\|0}/\nu_{\perp 0}.$$

Answers for the graphs are

$$= -\frac{3}{8}B_0 p^2\frac{m^{-\xi}}{\xi};$$



$$= -\frac{3}{16}\frac{D_0 p_{\|}^2}{\nu_{\|}^{\frac{1}{2}}\nu_{\perp}^{\frac{3}{2}}}\frac{m^{-\varepsilon}}{\varepsilon}.$$

Renormalization constants Z and β-functions

The renormalization constants Z are

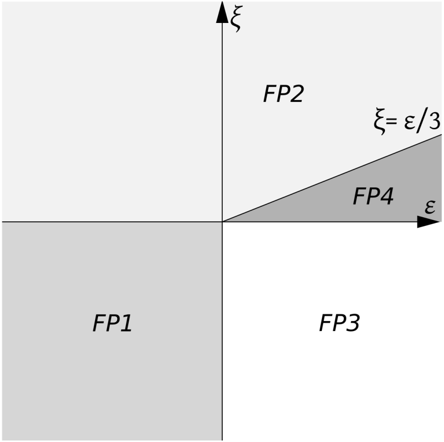
$$Z_{\nu_\parallel} = 1 - \frac{3}{8u}\frac{x}{\xi} - \frac{3}{16}\frac{g}{\varepsilon},$$
$$Z_{\nu_\perp} = 1 - \frac{3}{8}\frac{x}{\xi}, \quad \text{where} \quad x = uw.$$

The β-functions are

$$\beta_g = g\left(-\varepsilon + \frac{9}{32}g + \frac{9}{16}\frac{x}{u} + \frac{9}{16}x\right),$$
$$\beta_x = x\left(-\xi + \frac{3}{8}x\right),$$
$$\beta_u = u\left(-\frac{3}{16}g - \frac{3}{8}\frac{x}{u} + \frac{3}{8}x\right).$$

Fixed points and scaling regimes

Four fixed points depending on the parameters ξ and η



Critical dimensions: Fixed point FP2

The equation of critical scaling is a combination of equations of canonical scalilng with RG equation taken at fixed point.

$$\left(\mathcal{D}_{k_{\parallel}} + \mathcal{D}_{k_{\perp}} + \Delta_{\omega}\mathcal{D}_{\omega} - d_G^k - \Delta_{\omega} d_G^{\omega} - \gamma_G^*\right) G^R = 0,$$

where $\Delta_{\omega} = 2 - \gamma^*_{\nu_{\perp}}$. The critical dimension Δ_F of a quantity F reads

$$\Delta_F = d_F^k + \Delta_{\omega} d_F^{\omega} + \gamma_F^*.$$

Usually this step is very straightforward, and for the fixed point FP2 they are

$$\Delta_h = 1 - \xi, \quad \Delta_v = 1 - \xi, \quad \Delta_{h'} = 3 - \varepsilon + \xi, \quad \Delta_{\omega} = 2 - \xi.$$

Critical dimensions: Fixed points FP3 and FP4

The points FP3 and FP4 has the coordinate $\alpha^* = 1/u^* = 0$. Direct substitution of the coordinates g^*, x^*, α^* into $\gamma_{\nu_\perp}$ gives us trivial answer: $\gamma^*_{\nu_\perp} = 0$ and, consequently, $\Delta_F = d_F^k + 2d_F^\omega$ are simply canonical dimensions.

To obtain nontrivial corections we should expand β_α!

$$\left(\mathcal{D}_{k_\parallel} + \mathcal{D}_{k_\perp} + \Delta_\omega \mathcal{D}_\omega - \lambda^* \mathcal{D}_\alpha - d_G^k - \Delta_\omega d_G^\omega - \gamma_G^*\right) G^R = 0,$$

where $\lambda = \partial \beta_\alpha / \partial \alpha$ at $\alpha = 0$ and λ^* denotes $\lambda(g^*, x^*)$.

This trick allows us to determine nontrivial one-loop corrections to canonical dimensions at fixed points FP3 and FP4 and, moreover, reproduce well-known one-loop answers for pure Hwa-kardar model [Phys. Rev. Lett. 62, 1813 (1989); Phys. Rev. A 45, 7002 (1992)], which corresponds in our terminology to fixed point FP3.

Conclusion. Universe 6, 145 (2020).

We applied methods of **quantum field theory** (functional integration, calculation of Feynman graphs and renormalization group) to the Hwa-Kardar model which describes the profile of a surface.

- The key point is the possibility to reformulate initial stochastic problem into some quantum field theory.
- Feynman graphs are divergent. Renormalization group allows us to work with these objects and, moreover, provides critical dimensions of measurable quantities.
- We coupled isotropic velocity ensemble with anisotropic initial stochastic equation. As a consequence, one more coupling constant and corresponding β-function should be considered.
- As a result we should expand β-function to obtain nontrivial corrections to critical dimensions.

Thank you for your attention!