Induced Surface and Curvature Tensions approach for 3D and 2D multicomponent dense mixtures of hard spheres

Results for classical case and thoughts about its generalization for Lorentz contraction of excluded volume

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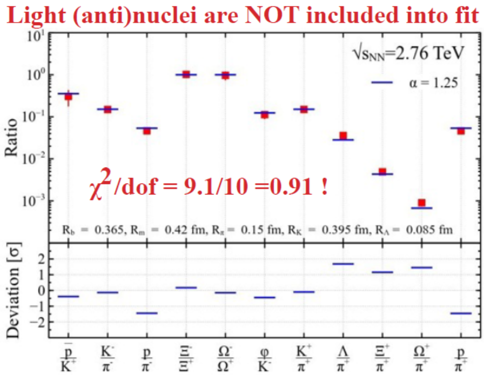
Plan

Hadron Resonance Gas Model (HRGM)

- HRGM is EoS with hard core repulsion written for all known hadrons and hadronic resonances
- For given temperature T, baryonic chem. potential, strange charge chem. potential, chem. potential of isospin 3-rd projection $\Rightarrow$ thermodynamic quantities $\Rightarrow$ all charge densities, to fit data.
- HRGM allows one to find **Chemical Freeze-out** - moment after which hadronic composition is fixed and only strong decays are possible. I.e. there are no inelastic reactions.
- Successfully describes hadron multiplicities measured in A+A collisions from $\sqrt{s_{NN}} \in [1\text{GeV}; 2.76\text{TeV}]$

Example of multicomponent hard-core repulsion



V. Sagun, K. Bugaev, L. Bravina, E. Zabrodin et al., Eur. Phys. J. A (2018) 54

- Radii are taken from the fit of AGS, SPS and RHIC data $\Rightarrow$ single parameter $T_{cfo} = 150 \pm 7$ MeV
- Combined fit of AGS, SPS, RHIC and LHC data gives $\chi^2_{tot}/dof \simeq 64.8/60 \simeq 1.08$. Best description achieved up to date.

Induced Surface Tension (IST) EoS

$$p = T\sum_{k=1}^{N} \phi_k \exp\left[\frac{\mu_k}{T} - v_k\frac{p}{T} - s_k\frac{\Sigma}{T}\right] \quad \rightarrow \text{Pressure}$$

$$\Sigma = T\sum_{k=1}^{N} L_k\phi_k \exp\left[\frac{\mu_k}{T} - v_k\frac{p}{T} - s_k\alpha\frac{\Sigma}{T}\right] \quad \rightarrow \text{Induced Surface Tension}$$

v_k and s_k are eigenvolume and eigensurface of hadron of sort k

$$\phi_k = g_k \int_{R^n} \frac{dp^n}{(2\pi\hbar)^n} e^{-\frac{\sqrt{p^2+m_k^2}}{T}} \rightarrow \text{thermal density}$$

Advantages:

- It allows us to go beyond the Van der Waals approximation since it reproduces 2-nd, 3-rd and 4-th virial coefficients of the gas of hard spheres.
- Number of equations does not depend on the number of different components.

Derivation

Mean hard core radius:

$\overline{R} = \frac{\sum\limits_l N_l R_l}{\sum\limits_l N_l} \to \frac{\sum\limits_l \langle N_l \rangle R_l}{\sum\limits_l \langle N_l \rangle}$ - assumption for the infinite system.

Introducing isobaric partition:

$$Z_{ISO}(T, \{\mu_k\}, \lambda) \equiv \int_0^{\infty} dV e^{-\lambda V} Z_{GCE}(T, \{\mu_k\}, V)$$

One can get equation for pressure: $p = T \sum\limits_{k=1}^{n} \phi_k e^{\frac{\mu_k}{T} - \frac{pV_k}{T} - \frac{p\overline{R}S_k}{T}}$

Introducing induced surface tension with extra α_k parameter:

$$p\overline{R} \to \Sigma S_k \to \Sigma S_k \alpha_k \text{ where } \alpha_k > 1$$

Induced Surface and Curvature Tension EoS

Introduction of curvature tension: $p\overline{R^2} \to c_k K$

Then replacing $c_k K \to \beta_k c_k K$ in eq. for K, where $\beta_k > 1$

$$p = T \sum_{k=1}^{N} \phi_k \exp\left[\frac{\mu_k}{T} - v_k \frac{p}{T} - s_k \frac{\Sigma}{T} - c_k \frac{K}{T}\right]$$

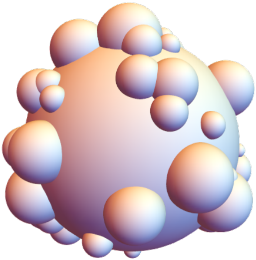
$$\Sigma = AT \sum_{k=1}^{N} L_k \phi_k \exp\left[\frac{\mu_k}{T} - v_k \frac{p}{T} - s_k \alpha_k \frac{\Sigma}{T} - c_k \frac{K}{T}\right]$$

$$K = BT \sum_{k=1}^{N} L_k^2 \phi_k \exp\left[\frac{\mu_k}{T} - v_k \frac{p}{T} - s_k \alpha_k \frac{\Sigma}{T} - c_k \beta_k \frac{K}{T}\right]$$

Here $c_k = \frac{s_k}{L_k}$ - curvature of particle of k-th sort.

N. Yakovenko, K. Bugaev, L. Bravina, E. Zabrodin arXiv:1910.04889 (to appear in EPJ A)

Hills and Dales model



Generalization with parameters α_k and β_k is an attempt to effectively include surface deformations of large Clusters with hard-core repulsion between constituents.

- *K. A. Bugaev, L. Phair and J. B. Elliott, Surface Partition of Large Clusters, Phys. Rev. E 72 (2005) 047106-1–047106-4*;
- *K. A. Bugaev and J. B. Elliott, Exactly Soluble Models for Surface Partition, Ukr. J. Phys. 52 (2007) 301-308*;

Hard Spheres

Compressibility

$$Z = \frac{p}{\rho T}, \qquad \rho = \left(\frac{\partial p}{\partial \mu}\right)_T$$

of gas of hard spheres very well described by Carnahan-Starling equation:

$$Z_{CS} = \frac{1 + \eta + \eta^2 - \eta^3}{(1 - \eta)^3}, \qquad \eta = \rho v$$

Norman F. Carnahan and Kenneth E. Starling J. Chem. Phys. 51, 635 (1969);

Monte Carlo calculation for HS (van Rensburg, 1993):

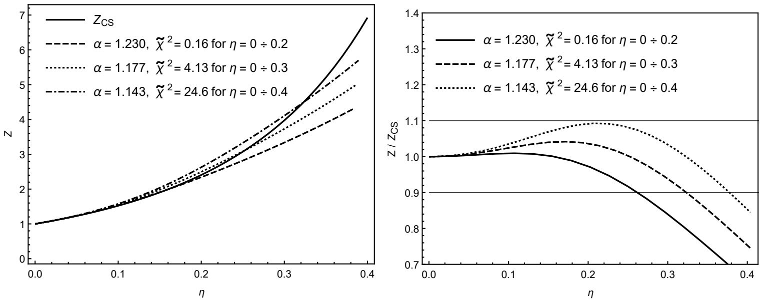
$$Z = 1 + 4\eta + 10\eta^2 + 18.36\eta^3 + 28.23\eta^4 + 39.74\eta^5 + 53.5\eta^6 + 70.8\eta^7$$

Carnahan-Starling eq. virial expansion:

$$Z = 1 + 4\eta + 10\eta^2 + 18\eta^3 + 28\eta^4 + 40\eta^5 + 54\eta^6 + 70\eta^7$$

One-component IST EoS

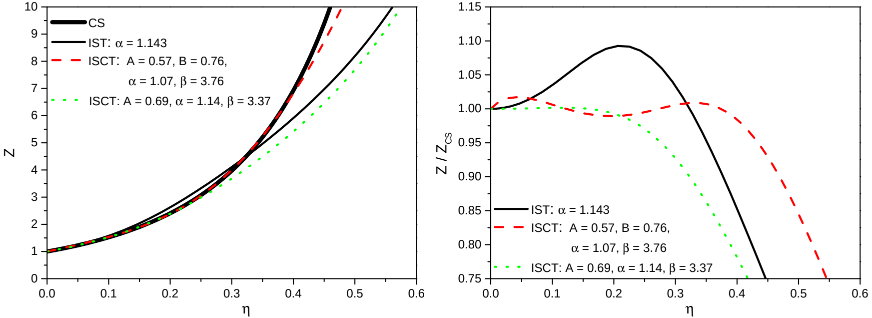
Compressibility



Left: Comparison of the compressibility factors of the IST and CS EoS. $\tilde{\chi}^2$ is the mean deviation squared per the interval of fit. **Right:** The same as in left but for the ratio of the compressibility factors of IST and CS EoS.

One-component ISCT EoS

Compressibility



Comparison of the compressibility factors Z of the CS EoS 0with the one-component IST EoS and ISCT EoS with best-fit parameters on the interval of packing fraction $\eta \in [0.; 0.4]$ and the ISCT EoS which exactly reproduces the five virial coefficients of the CS EoS (dotted curve).

Hard discs

Compressibility of gas of hard discs (2D case):

$$Z_S = \frac{1 + \frac{\eta^2}{8} - \frac{\eta^4}{10}}{(1-\eta)^2}$$

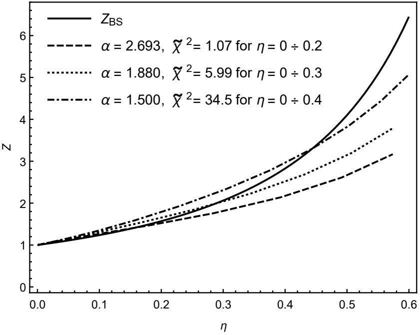
Numeric simulations for hard discs

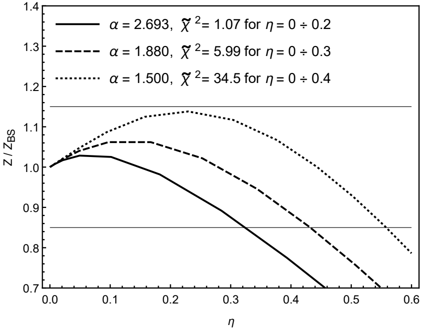
$$Z_{VE} = 1 + 2\eta + 3.128\eta^2 + 4.25786\eta^3 + 5.3369\eta^4 + 6.36303\eta^5 + 7.35208\eta^6 + 8.31867\eta^7 + 9.27236\eta^8 + 10.2161\eta^9$$

C. Barrio and J.R. Solana, Phys. Rev. E 63, 011201 (2001)

One-component IST EoS

Compressibility





Comparison of the compressibility factors of the IST and BS EoS for hard discs.

Hard Spheres

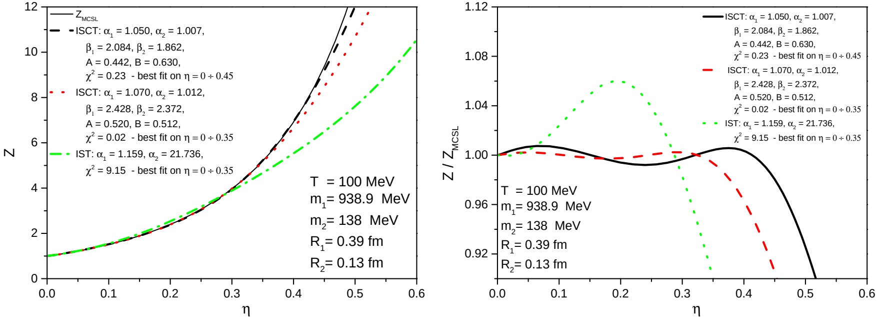
Mansoori-Carnahan-Starling-Leland equation for multicomponent system:

$$p_{MCSL} = \frac{6T}{\pi}\left(\frac{\xi_0}{1-\xi_3} + \frac{3\xi_1\xi_2}{(1-\xi_3)^2} + \frac{3\xi_2^3 - \xi_3\xi_2^3}{(1-\xi_3)^3}\right),$$

where $\xi_n = \frac{\pi}{6}\sum\limits_{k=1}^{N} \rho_k (2L_k)^n$

IST EoS:

$$\begin{cases} p = T\sum\limits_{k=1}^{N} \phi_k \exp\left[\frac{\mu_k}{T} - v_k\frac{p}{T} - s_k\frac{\Sigma}{T}\right] \\ \Sigma = T\sum\limits_{k=1}^{N} L_k\phi_k \exp\left[\frac{\mu_k}{T} - v_k\frac{p}{T} - s_k\alpha\frac{\Sigma}{T}\right] \end{cases}$$



Left: Comparison of the compressibility factors with the best-fit parameters obtained with IST and ISCT EoS for hard spheres gas with multicomponent CS EoS Z_{MCSL}. **Right:** Similar to the left, but for the ratios of the compressibility factors Z/Z_{MCSL}

Hard Discs

Multicomponent Santos equation for hard discs mixture:

$$Z_{SHDM} = (1-\xi)\frac{1}{1-\eta} + \xi Z_S(\eta),$$

where

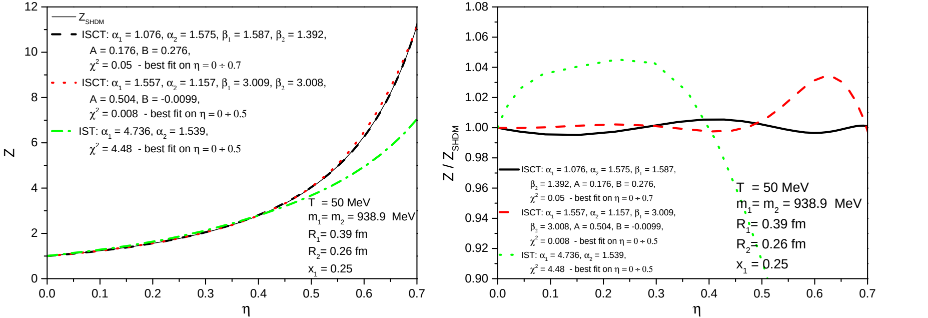
$$Z_S(\eta) = \frac{1+3\eta/\eta_0}{1-\eta/\eta_0} + \sum_{n=2}^{6}(b^n \eta_0^{n-1} - 4)\left(\frac{\eta}{\eta_0}\right)^{n-1}$$

- Woodcock's EoS, and $\xi = \overline{\sigma}^2/\overline{\sigma^2}$

Santos et al., Phys. Rev. E 66, 031202 (2002)

ISCT EoS

Compressibility of Hard Discs



Left: Comparison of the compressibility factors with the best-fit parameters obtained with IST and ISCT EoS for hard discs mixture with multicomponent SHDM EoS Z_{SHDM}. **Right:** Same as in the upper panel, but for the ratios of the compressibility factors Z/Z_{SHDM}.

Lorentz contracted excluded volume

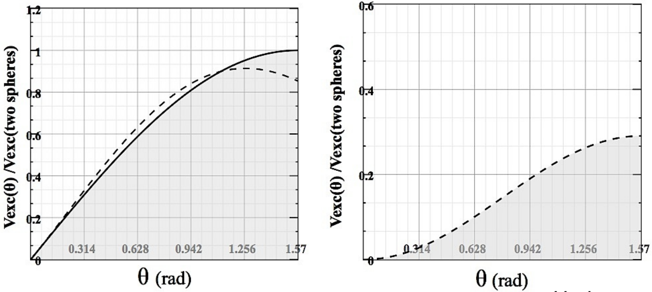
The approximate expression for excluded volume of Lorentz contracted hard spheres as a function of azimuthal angle Θ_v between the 3-momenta of particles k and l:

$$2v_{kl}^{Urel}(\Theta_v) = \frac{4}{3}\pi\frac{R_k}{\gamma_k}\left(R_k + R_l\cos^2\left(\frac{\Theta_v}{2}\right)\right)^2 + \frac{4}{3}\pi\frac{R_l}{\gamma_l}\left(R_l + R_k\cos^2\left(\frac{\Theta_v}{2}\right)\right)^2 + 2\pi a R_k R_l (R_k + R_l)\left|\sin\left(\Theta_v\right)\right| ,$$

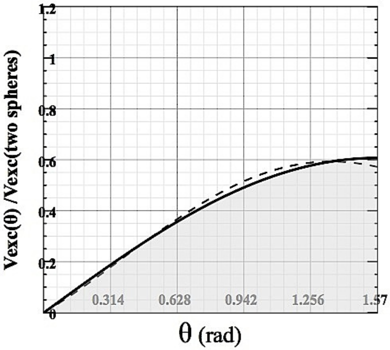
where $\hat{v}_k \equiv \dfrac{v_k}{\gamma_k}$.

It was obtained with the parameter value $a = 1$ in the ultrarelativistic limit, i.e. for two thin disks. The introduction of the coefficient $a \simeq 0.775$ allows one to reproduce the limit of two nonrelativistic spheres.

Bugaev K. A. The Van-der-Waals Gas EoS for the Lorentz Contracted Spheres. Nucl. Phys. A, 807:251-268, 2008.



The relation of Lorentz contracted excluded volumes $2v_{12}^{Urel}(\Theta_v)\sin(\Theta_v)$ of two hard core spheres to their nonrelativistic volumes $2v_{12}^{Nrel}$ with radiuses $R_1 = R_2 = 0.39$ fm as a function of the angle between the 3-momentum vectors of the particles. **The left panel** shows the nonrelativistic limit $\gamma_1 = \gamma_2 = 1$ for two spheres. The solid line is the exact result and the dashed line is obtained from equation for $2v_{kl}^{Urel}(\Theta_v)$. **The right panel** shows the ultrarelativistic limit $\gamma_1 = \gamma_2 = 1000$ for two thin disks.



The relation of Lorentz contracted and nonrelativistic excluded volumes with radius $R_1 = R_2 = 0.39$ fm as a function of the angle between the 3-vectors of the momenta of the particles. Plots are shown in the limits of the sphere $\gamma_1 = 1$ and the thin disk $\gamma_2 = 1000$. The solid curve is the exact result, and the dashed curve obtained using approximate formula $2v_{kl}^{Urel}(\Theta_v)$ for $a = 0.775$

Relativistic ISCT EoS

$$\frac{p}{T} = \sum_k \hat{\phi}_k \exp\left[\frac{\mu_k}{T} - \hat{v}_k \frac{p}{T} - \hat{s}_k \frac{\tilde{\Sigma}}{T} - \hat{c}_k \frac{\tilde{K}}{T}\right],$$

$$\frac{\Sigma}{T} = \alpha_p \sum_k R_k \hat{\phi}_k \exp\left[\frac{\mu_k}{T} - \hat{v}_k \frac{p}{T} - \hat{s}_k \alpha \frac{\tilde{\Sigma}}{T} - \hat{c}_k \frac{\tilde{K}}{T}\right],$$

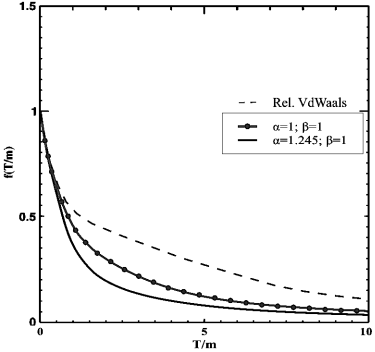
$$\frac{K}{T} = \beta_p \sum_k R_k^2 \hat{\phi}_k \exp\left[\frac{\mu_k}{T} - \hat{v}_k \frac{p}{T} - \hat{s}_k \alpha \frac{\tilde{\Sigma}}{T} - \hat{c}_k \beta \frac{\tilde{K}}{T}\right].$$

where

$$\hat{\phi}_k(T) = 4\pi g_k \int \frac{dk\, k^2}{(2\pi\hbar)^3} e^{-\frac{\sqrt{m_k^2+k^2}}{T}} \int_0^{\frac{\pi}{2}} d\Theta_v \sin(\Theta_v).$$

The hat over the thermal density means that the double integration over momentum k and angle Θ_v acts like an operator on any function with the same hat appearing to the right of the function $\hat{\phi}_k(T)$.

Temperature dependence of relativistic excluded volume



Averaged excluded volumes of gas of Lorentz contracted and nonrelativistic rigid spheres of radius $R_1 = 0.39$ fm as a function of the temperature of the system T. The dashed curve was obtained in the VdW gas model, the solid curves correspond to the model with induced surface and curvature tensions discussed here.

Conclusions

- ISCT EoS consists of only 3 equations for any number of components
- Greatly describes CS, Barrio & Solana and Santos equations just with few parameters
- It is the first approach which is able to describe the gaseous phase of both 2- and 3-dimensional spheres
- Gives the best description of hadron multiplicities measured by the ALICE in Pb+Pb collisions at the $\sqrt{s_{NN}} = 2.76$ TeV
- Possible generalization for even more precise description of relativistic and quantum systems
- General form allows its applications for various systems and not only mixtures of hadrons.

Appendix

$$\hat{s}_k(\Theta_v) \equiv s_k \left[\frac{1}{2\gamma_k} \left(1 + \frac{\Delta_{c1}}{2} \right) + \frac{a}{2} \left| \sin\left(\Theta_v\right) \right| \right] ,$$

$$\hat{c}_k(\Theta_v) \equiv c_k \frac{1}{2\gamma_k} \left(1 + \frac{\Delta_{c1}}{2} + \frac{\Delta_{c2}}{6} \right) ,$$

$$\Delta_{c1} \equiv \cos(\Theta_v) - 1 \, , \quad \Delta_{c2} \equiv [\cos(\Theta_v)]^2 - 1 \, ,$$

$$v_j = \frac{4}{3}\pi R_j^3 \, , \quad s_j = 4\pi R_j^2 \, , \quad c_j = 4\pi R_j \, .$$