Coupled $\mathcal{N} = 2$ supersymmetric quantum systems: Symmetries and supervariable approach

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Introduction

Superspace approach description

- Start with a $\mathcal{N} = 2$ supervariable $X(t, \theta, \bar{\theta})$ on a $(1, 2)$-dimensional supermanifold.
- This supervariable expanded along the Grassmannian directions $(\theta, \bar{\theta})$

$$X(t, \theta, \bar{\theta}) = q(t) + i\,\theta\,\bar{\psi}(t) + i\,\bar{\theta}\,\psi(t) + \theta\,\bar{\theta}\,A(t),$$

- The two supercharges Q and $\bar{Q}$ are defined as

$$Q = \frac{\partial}{\partial \bar{\theta}} + i\,\theta\,\frac{\partial}{\partial t}, \qquad \bar{Q} = \frac{\partial}{\partial \theta} + i\,\bar{\theta}\,\frac{\partial}{\partial t},$$

`Cooper, Freedman, Ann.Phys.(1983)`

- The SUSY transformation (δ) on the supervariable

$$\begin{aligned} \delta X(t, \theta, \bar{\theta}) &= \delta q(t) + i\,\theta\,\delta\bar{\psi}(t) + i\,\theta\,\delta\psi(t) + \theta\,\bar{\theta}\,\delta A(t) \\ &\equiv (\bar{\epsilon}\,Q + \epsilon\,\bar{Q})\,X(t, \theta, \bar{\theta}), \end{aligned}$$

- The general Lagrangian for the $\mathcal{N}=2$ SUSY QM model, in $(1,2)$-dimensional supermanifold, can be given as

$$L = \int d\theta\, d\bar{\theta}\left[\frac{1}{2}\,\mathcal{D}X(t,\theta,\bar{\theta})\,\bar{\mathcal{D}}X(t,\theta,\bar{\theta}) - W\left(X(t,\theta,\bar{\theta})\right)\right],$$

Lahiri, Roy, Bagchi, IJMPA (1990)
Kumar, Malik, EPJC (2013)

- $\mathcal{D}$ and $\bar{\mathcal{D}}$ are supercovariant derivatives

$$\mathcal{D} = \frac{\partial}{\partial\bar{\theta}} - i\,\theta\,\frac{\partial}{\partial t}, \qquad \bar{\mathcal{D}} = \frac{\partial}{\partial\theta} - i\,\bar{\theta}\,\frac{\partial}{\partial t},$$

- $W(X)$ is the superpotential which is an arbitrary function of the supervariable $X(t,\theta,\bar{\theta})$.

Introduction

$\mathcal{N}=2$ SUSY QM model

- The superpotential $W(X(t,\theta,\bar{\theta}))$ can be Taylor expanded around ordinary space variable q and the proper Grassmannian integration yields (modulo a total derivative)

$$L = \frac{1}{2}\,\dot{q}^2 + i\,\bar{\psi}(t)\,\dot{\psi}(t) + W'(q)\,A(t) + \frac{1}{2}\,A^2(t) + W''(q)\,\bar{\psi}(t)\,\psi(t)$$

Kumar, Malik, EPJC (2013)

- Here $q(t),\ \psi(t),\ \bar{\psi}(t)$ are the basic dynamical variables and $A(t)$ is an auxiliary variable.

- The above Lagrangian remains invariant under *two* sets of off-shell nilpotent, fermionic symmetry transformations.

- Off-shell symmetries (s_1 & s_2)

$$
\begin{aligned}
s_1 x &= i\,\psi, & s_2 x &= i\,\bar{\psi},\\
s_1 \bar{\psi} &= -(\dot{x} + i\,A), & s_2 \psi &= -(\dot{x} - i\,A),\\
s_1 A &= -\dot{\psi}, & s_2 A &= \dot{\bar{\psi}},\\
s_1 \psi &= 0, & s_2 \bar{\psi} &= 0.
\end{aligned}
$$

  - Nilpotent of order two (i.e. $s_1^2 = 0, s_2^2 = 0$)
  - Continuous symmetries
  - Fermionic in nature

- The Lagrangian remains quasi-invariant

$$
s_1 L = \frac{d}{dt}\Big[-W'\,\psi\Big], \quad s_2 L = \frac{d}{dt}\Big[i\,\bar{\psi}\,(\dot{x} - i\,A) + \bar{\psi}\,W'\Big]
$$

Introduction

$\mathcal{N}=2$ SUSY QM model: Off-shell symmetries

- Bosonic symmetry (s_ω)

$$s_\omega \Phi \equiv \{s_1,\; s_2\}\,\Phi = -2\,i\,\dot{\Phi},$$
$$\Phi = x,\; \psi,\; \bar{\psi},\; A,\; W,\; W',\; W''$$

here Φ being any generic variable.

- We have following symmetry transformation

$$s_\omega L = \big(s_1\, s_2 + s_2\, s_1\big)\, L \equiv \frac{d}{dt}\, L$$

- The Noether conserved charges corresponding to the continuous symmetries are Kumar, Malik, EPJC (2013)

$$\begin{aligned} Q &= (i\,\dot{x} - A)\,\psi, \qquad \bar{Q} = \bar{\psi}\,(i\,\dot{x} + A), \\ Q_\omega &= \frac{1}{2}\,p^2 - \frac{1}{2}\,A^2 - A\,W' - W''\,\bar{\psi}\,\psi = H, \end{aligned}$$

- *"Physically"* relevant discrete symmetry $(*)$

$$x \to -x, \qquad t \to -t, \qquad \psi \to +\bar{\psi}, \qquad \bar{\psi} \to -\psi,$$
$$A \to -A, \qquad W' \to -W', \qquad W'' \to +W''$$

- Algebraic structure: operator form

$$s_1^2 = 0, \quad s_2^2 = 0, \quad s_w = \{s_1,\ s_2\} = s_1\, s_2 + s_2\, s_1,$$
$$[s_w,\ s_1] = 0, \quad [s_w,\ s_2] = 0, \quad s_w = (s_1 + s_2)^2,$$
$$s_1\, \Phi = \pm * s_2 * \Phi, \quad * \text{ is the discrete symmetry}$$

- Obey exactly same algebraic structure as the algebra of de Rham cohomological operators (d, δ, Δ) of differential geometry.

Motivations

Proof of the conjucture

There might exist more than one set of discrete symmetry transformations which provide an analogue for the Hodge duality operator of differential geometry. `Kumar, Malik, EPJC (2013)`

Extension of the study

Generalized form of superpotential, in particular, superposition of 'polynomial-type' superpotentials.

(Anti-)chiral supervariable approach

On-shell nilpotent symmetries for the generalized superpotential.

Generalized superpotential

Superposition of superpotentials

- We take a generalized superpotential of the form

$$W' = \sum_{j=a}^{b} \beta_j q^j$$

  - a, b could be positive or negative integers in an 'appropriate order' (i.e. $a, a+1, a+2, ..., b = a+n$) with n being a positive integer.
  - β_j's are constant coefficients.

- If, we choose $a = -1,\ b = 1 \Longrightarrow W' \ = \ \dfrac{\lambda}{q} + \mu + \omega q$

  Bouquiaux, Dauby, Hussin, JMPA (1987)

- Various combinations of the constants ω, μ, λ yield different kind of superpotentials.

- The Lagrangian, with generalized superpotential, can be given as

$$L^{(g)} = \frac{\dot{q}^2}{2} - \frac{1}{2}\left(\sum_{j=a}^{b} \beta_j q^j\right)^2 + i\bar{\psi}\dot{\psi} - \left(\sum_{j=a}^{b} j\beta_j q^{j-1}\right)\bar{\psi}\psi.$$

- The fermionic symmetries $s_1^{(g)}$, $s_2^{(g)}$ associated with the above Lagrangian are

$$s_1^{(g)} q = -i\psi, \quad s_1^{(g)}\psi = 0, \quad s_1^{(g)}\bar{\psi} = \dot{q} + i\sum_{j=a}^{b} \beta_j q^j,$$

$$s_2^{(g)} q = i\bar{\psi}, \quad s_2^{(g)}\bar{\psi} = 0, \quad s_2^{(g)}\psi = -\dot{q} + i\sum_{j=a}^{b} \beta_j q^j.$$

Generalized superpotential

Bosonic and discrete symmetries

- We also have bosonic symmetry $s_w^{(g)} = \left\{ s_1^{(g)}, s_2^{(g)} \right\}$, as

$$s_w^{(g)} q = 2i\dot{q}, \;\; s_w^{(g)} \psi = i\dot{\psi} + \sum_{j=a}^{b} j\beta_j q^{j-1}\psi, \;\; s_w^{(g)} \bar{\psi} = i\dot{\bar{\psi}} - \sum_{j=a}^{b} j\beta_j q^{j-1}\bar{\psi}$$

- The Lagrangian remains quasi-invariant under $s_1^{(g)}$, $s_2^{(g)}$, and $s_w^{(g)}$.
- The Lagrangian is endowed with *eight* sets of discrete symmetries.
- Only following *two* sets are "physically" relevant (provide analog to the Hodge duality operator).

$$q \longrightarrow \pm q, \quad t \longrightarrow -t, \quad \psi \longrightarrow \pm\bar{\psi}, \quad \bar{\psi} \longrightarrow \mp\psi,$$

$\beta_{2n+1} \longrightarrow \beta_{2n+1}, \; \beta_{2n} \longrightarrow \pm\beta_{2n},$ where n is any integer.

Aditi, Anjali, Binu, Saurabh, CTP (2020)

Cohomological aspects

- Continuous symmetries satisfy the following algebra

$$s_1^{(g)2} = 0, \quad s_2^{(g)2} = 0, \quad s_w^{(g)} = \{s_1^{(g)}, s_2^{(g)}\} = s_1^{(g)} s_2^{(g)} + s_2^{(g)} s_1^{(g)},$$
$$s_w^{(g)} = (s_1^{(g)} + s_2^{(g)})^2, \quad [s_w^{(g)}, s_1^{(g)}] = 0, \quad [s_w^{(g)}, s_2^{(g)}] = 0.$$

- Algebra of de Rham cohomological operators

$$d^2 = 0, \quad \delta^2 = 0, \quad \Delta = \{d, \delta\} \equiv d\delta + \delta d \equiv (d + \delta)^2,$$

$$[\Delta, d] = 0, \quad [\Delta, \delta] = 0, \quad \delta = \pm * \ d \ * .$$

- For the *perfect* analogy, we should have

$$s_1^{(g)} \phi = \pm * s_2^{(g)} * \phi.$$

(Anti-)chiral supervariable approach

A brief synopsis

- Ordinary variables, in the theory, are generalized to their (anti-)chiral supervariables on the supermanifold parametrized by $x_\mu, \theta, \bar{\theta}$.
  - where $\theta, \bar{\theta}$ are Grassmannian variables (i.e. $\theta^2 = \bar{\theta}^2 = 0, \quad \theta\bar{\theta} + \bar{\theta}\theta = 0$).
- These supervariables are expanded along the Grassmannian directions, which contain the secondary variables.
- Finally, using SUSY invariant restrictions (SUSYIRs), the relationships between the basic and secondary variables can be determined.

Malik, *et al*, Ann.Phys.(2014)

Anti-chiral supervariable approach

On-shell nilpotent symmetries

- Ordinary variables $\longmapsto$ anti-chiral supervariables

$$\begin{aligned} \mathcal{Q}(t,\bar{\theta}) &= q(t) + \bar{\theta}\Lambda(t), \\ \Psi(t,\bar{\theta}) &= \psi(t) + i\bar{\theta}\Omega_1(t), \\ \bar{\Psi}(t,\bar{\theta}) &= \bar{\psi}(t) + i\bar{\theta}\Omega_2(t), \end{aligned}$$

Λ is a fermionic secondary variable and Ω_1, Ω_2 are bosonic secondary variables.

- SUSYIRs:

$$\begin{aligned} \Psi(t,\bar{\theta}) &= \psi(t), \\ \mathcal{Q}(t,\bar{\theta})\Psi(t,\bar{\theta}) &= q(t)\psi(t), \\ \dot{\mathcal{Q}}(t,\bar{\theta})\dot{\Psi}(t,\bar{\theta}) &= \dot{q}(t)\dot{\psi}(t). \end{aligned}$$

This yields: $\Omega_1(t) = 0, \quad \Lambda(t) = -i\psi(t)$.

- More SUSYIR:

$$\Gamma(t, \bar{\theta}) \;=\; \Upsilon(t),$$

with $\Upsilon(t) \;\equiv\; \frac{1}{2}\dot{q}^2(t) + i\bar{\psi}(t)\dot{\psi}(t) - \frac{1}{2}\Big(\sum_{j=a}^{b}\beta_j q^j(t)\Big)^2 -$

$$\Big(\sum_{j=a}^{b} j\beta_j q^{j-1}(t)\Big)\bar{\psi}(t)\psi(t) + i\Big(\sum_{j=a}^{b}\beta_j q^j(t)\Big)\dot{q}(t)$$

Aditi, Anjali, Binu, Saurabh, CTP (2020)

- $\Gamma(t, \bar{\theta})$ represents generalization of $\Upsilon(t)$ on the anti-chiral super-submanifold.
- This yields: $\Omega_2(t) \;=\; -i\Big(\dot{q}(t) + i\sum_{j=a}^{b}\beta_j q^j(t)\Big).$

Anti-chiral supervariable approach

On-shell nilpotent symmetries

- Expansion of supervariables in anti-chiral super-submanifold

$$\mathcal{Q}(t,\bar\theta) = q(t) + \bar\theta\big(-i\psi\big) \equiv q(t) + \bar\theta\big(s_1^{(g)} q\big),$$
$$\Psi(t,\bar\theta) = \psi(t) + \bar\theta\big(0\big) \equiv \psi(t) + \bar\theta\big(s_1^{(g)} \psi\big),$$
$$\bar\Psi(t,\bar\theta) = \bar\psi(t) + \bar\theta\Big(\dot q + i\sum_{j=a}^{b} \beta_j q^j\Big) \equiv \bar\psi(t) + \bar\theta\big(s_1^{(g)} \bar\psi\big).$$

- Translation along the Grassmannian direction $\bar\theta$

$$\Downarrow$$

as $s_1^{(g)}$ acting on the corresponding ordinary variable.

- In other words: $\dfrac{\partial}{\partial\bar\theta}\Phi(t,\bar\theta) = s_1^{(g)}\phi(t)$

Chiral supervariable approach

On-shell nilpotent symmetries

- The other fermionic continuous symmetry $s_2^{(g)}$ can be deduced using chiral super-submanifold characterized by (t, θ).
- SUSYIRs provide a relation among basic and secondary variables of the theory.
- Expansion of supervariables in chiral super-submanifold

$$\mathcal{Q}(t,\theta) = q(t) + \theta\big(i\bar{\psi}\big) \equiv q(t) + \theta\big(s_2^{(g)} q\big),$$

$$\Psi(t,\theta) = \psi(t) + \theta\Big(-\dot{q} + i\sum_{j=a}^{b} \beta_j q^j\Big) \equiv \psi(t) + \theta\big(s_2^{(g)} \psi\big),$$

$$\bar{\Psi}(t,\theta) = \bar{\psi}(t) + \theta(0) \equiv \bar{\psi}(t) + \theta\big(s_2^{(g)} \bar{\psi}\big).$$

- We can identify: $\dfrac{\partial}{\partial\theta}\Phi(t,\theta) = s_2^{(g)}\phi(t)$

Invariance of the Lagrangian

- Starting Lagrangian can be generalized onto the $(1,1)$-dimensional (anti-)chiral super-submanifold

$$\begin{aligned}\tilde{L}_0^{(ac)} &= \frac{1}{2}\dot{\mathcal{Q}}^{(1)}(t,\bar{\theta})\dot{\mathcal{Q}}^{(1)}(t,\bar{\theta}) + i\bar{\Psi}^{(1)}(t,\bar{\theta})\dot{\Psi}^{(1)}(t,\bar{\theta}) \\ &- \frac{1}{2}\tilde{W}'(\mathcal{Q}^{(1)})\tilde{W}'(\mathcal{Q}^{(1)}) - \tilde{W}''(\mathcal{Q}^{(1)})\bar{\Psi}^{(1)}(t,\bar{\theta})\Psi^{(1)}(t,\bar{\theta}),\end{aligned}$$

$$\begin{aligned}\tilde{L}_0^{(c)} &= \frac{1}{2}\dot{\mathcal{Q}}^{(2)}(t,\theta)\dot{\mathcal{Q}}^{(2)}(t,\theta) + i\bar{\Psi}^{(2)}(t,\theta)\dot{\Psi}^{(2)}(t,\theta) \\ &- \frac{1}{2}\tilde{W}'(\mathcal{Q}^{(2)})\tilde{W}'(\mathcal{Q}^{(2)}) - \tilde{W}''(\mathcal{Q}^{(2)})\bar{\Psi}^{(2)}(t,\theta)\Psi^{(2)}(t,\theta).\end{aligned}$$

- The superscripts $(1)2$ denote the expansion of the supervariables in the (anti-)chiral directions.

Invariance of the Lagrangian

Supervariable approach

- The translational derivative of $\tilde{L}_0^{(ac)}$ and $\tilde{L}_0^{(c)}$ along the $\bar{\theta}, \theta$ direction, respectively, provides

$$\frac{\partial}{\partial\bar{\theta}}\tilde{L}_0^{(ac)} \;=\; -\frac{d}{dt}\Big(\sum_{j=a}^{b}\beta_j q^j \psi\Big), \quad \frac{\partial}{\partial\theta}\tilde{L}_0^{(c)} \;=\; \frac{d}{dt}\big(i\dot{q}\bar{\psi}\big).$$

- Translation along the (anti-)chiral direction $(\bar{\theta})\theta$ on superspace Lagrangian

$$\Downarrow$$

same as $(s_1^{(g)})s_2^{(g)}$ acting on the SUSY Lagrangian.

- Provide a proof for the nilpotency of fermionic symmetries in a straightforward manner.

Conclusions

- Superpostion of superpotentials – Fermionic, Bosonic and Discrete symmetries.
- A proof of the conjecture – Two sets of discrete symmetries corresponding to Hodge duality operation.
- On-shell nilpotent fermionic symmetries of $\mathcal{N} = 2$ SUSY QM model – Supervariable approach.
- Nilpotency of fermionic symmetries – Caputred with the aid of translational generators.

Thank You!!