

Bekenstein bound from the Pauli principle

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Abstract

Assuming that the degrees of freedom of a black hole are finite in number and of fermionic nature, we naturally obtain, within a second-quantized toy model of the evaporation, that the Bekenstein bound is a consequence of the Pauli exclusion principle for these fundamental degrees of freedom. We show that entanglement, Bekenstein and thermodynamic entropies of the black hole all stem from the same approach, based on the entropy operator whose structure is the one typical of Takahashi and Umezawa's Thermofield Dynamics. We then evaluate the von Neumann black hole–environment entropy and noticeably obtain a Page-like evolution. We finally show that this is a consequence of a duality between our model and a quantum dissipative-like fermionic system.

Introduction

In Ref. [1], Bekenstein's argument that a black hole (BH) reaches the maximal entropy at disposal of a physical system [2] leads to two main proposals: i) the degrees of freedom (dof) responsible for the BH entropy have to take into account both matter and spacetime and hence must be of a new, more fundamental nature than the dof we know, here we call such dof "Xons" ii) the Hilbert space \mathcal{H} of the Xons of a given BH is necessarily finite dimensional [3].

Here we reverse that logic [4]. Namely, we start off by supposing that in a BH only free Xons exist, and we suppose that they are *finite* in number and *fermionic* in nature. This amounts to have a finite dimensional \mathcal{H} . With these assumptions, we show here that BH evaporation is a dynamical mechanism producing a maximal entanglement entropy, equal to the initial entropy of the BH.

Hilbert space of Xons and Bekenstein bound

We assume that the fundamental dof are fermionic. As a consequence, each quantum level can be filled by no more than one fermion. This assures that the Hilbert space \mathcal{H} of physical states with a finite number of levels is finite dimensional.

Now, say N is the total number of quantum levels available to the BH. The evaporation consists of the following, steady process: $N \rightarrow (N-1) \rightarrow (N-2) \rightarrow \dots$. That is, the number of *free* Xons steadily decreases, in favor of the Xons that, having evaporated, are arranged into quasi-particles and the spacetime they live in. One might think of a counter that only sees free Xons, hence keeps clicking in one direction as the BH evaporates, till its complete stop.

In this picture: i) there is no pre-existing *time*, because the natural evolution parameter is the average number of free Xons; and ii) there is no pre-existing *space* to define the regions inside and outside the BH, because a distinction of the total system into two systems, say *environment* (I) and *BH* (II), naturally emerges in the way just depicted. With this in mind, in what follows we shall nonetheless refer to the system I as *outside*, and to system II as *inside*. The Hilbert space of *physical states* is then built as a subspace of a larger tensor product (*kinematical*) Hilbert space

$$\mathcal{H} \subseteq \mathcal{H}_I \otimes \mathcal{H}_{II}. \quad (1)$$

We introduce fermionic ladder operators

$$\{\chi_{\tau n}, \chi_{\tau' n'}^\dagger\} = \delta_{\tau, \tau'} \delta_{nn'}, \quad (2)$$

with $n, n' = 1, \dots, N$, $\tau = I, II$, and all other anticommutators equal to zero. Then, we introduce the simplified notation

$$a_n \equiv \chi_{In}, \quad b_n \equiv \chi_{II n}. \quad (3)$$

We initialize the system in a pure state representing the BH at the beginning of the evaporation process, with all the slots occupied by free Xons. In the final state they will not be free Xons left, as they all recombined to form fields and spacetime.

A toy-model corresponding to such boundary conditions is defined by the entangled state

$$|\Psi(\sigma)\rangle = \prod_{i=1}^N \sum_{n_i=0,1} C_i(\sigma) (a_i^\dagger)^{n_i} (b_i^\dagger)^{1-n_i} |0\rangle_I \otimes |0\rangle_{II}, \quad (4)$$

which describes the state of the system, with $C_i = (\sin \sigma)^{n_i} (\cos \sigma)^{1-n_i}$. The parameter σ can be seen as an interpolating parameter, which describes the evolution of the system, from $\sigma = 0$, corresponding to the start of the evaporation of the BH, till $\sigma = \pi/2$, corresponding to its complete evaporation.

The Hilbert space of physical states has dimension

$$\Sigma \equiv \dim \mathcal{H} = 2^N. \quad (5)$$

In order to quantify entanglement at each stage, we define the entropy operator for environment modes as in Thermofield dynamics (TFD) [5]

$$S_I(\sigma) = - \sum_{n=1}^N \left(a_n^\dagger a_n \ln \sin^2 \sigma + a_n a_n^\dagger \ln \cos^2 \sigma \right). \quad (6)$$

We also define the entropy operator for BH modes, in a rather unconventional way

$$S_{II}(\sigma) = - \sum_{n=1}^N \left(b_n^\dagger b_n \ln \cos^2 \sigma + b_n b_n^\dagger \ln \sin^2 \sigma \right). \quad (7)$$

Then

$$S(\sigma) = \langle S_I(\sigma) \rangle_\sigma = -N \left(\sin^2 \sigma \ln \sin^2 \sigma + \cos^2 \sigma \ln \cos^2 \sigma \right) = \langle S_{II}(\sigma) \rangle_\sigma = S_{II}(\sigma),$$

where $\langle \dots \rangle_\sigma \equiv \langle \Psi(\sigma) | \dots | \Psi(\sigma) \rangle$. Therefore the averages of the operators coincide, as it must be for a bipartite system. This entropy is the von Neumann entropy quantifying the entanglement between environment and BH [4]. Remarkably, it has a behavior in many respect similar to that of the Page curve [6]. The maximum value is

$$S_{max} = N \ln 2 = \ln \Sigma, \quad (8)$$

so that

$$\Sigma = e^{S_{max}}. \quad (9)$$

Then, in our model $\dim \mathcal{H}$ is related to the maximal entanglement entropy of the environment with the BH. This happens exactly when the modes have *half* probability to be inside and *half* probability to be outside the

BH, and then a large amount of bits are necessary to describe the system. The system has thus an intrinsic way to know how big is the physical Hilbert space, hence how big is the BH at the beginning of the evaporation: when the maximal entanglement is reached, then that value of the entropy, S_{max} , tells how big was the original BH. Hence S_{max} must be some function of \mathcal{M}_0 , with \mathcal{M}_0 the original mass of the BH. This is the Bekenstein bound in this picture, obtained as a consequence of the finiteness of the fermionic fundamental dof, hence of a Pauli principle.

How the entropy of the BH, that should always decrease, and the entropy of the environment, that should always increase actually evolve in our model? It is important to stress that, in our formalism, the full kinematical Hilbert spaces associated to both sides have fixed dimension ($\dim \mathcal{H}_I = \dim \mathcal{H}_{II} = 2^N$), while only a subspace $\mathcal{H} \subseteq \mathcal{H}_I \otimes \mathcal{H}_{II}$ such that $\dim \mathcal{H} = 2^N$ is the one of physical states. Note that \mathcal{H} cannot be factorized and this is the origin of BH/environment entanglement.

Nonetheless, one could think that the physical Hilbert spaces of the two subsystems have to take into account only the number of modes truly occupied, at any given stage of the evaporation. Hence, the actual dimensions would be $2^{N_I(\sigma)}$, and $2^{N_{II}(\sigma)}$, where one easily finds that the mean number of modes on $|\Psi(\sigma)\rangle$ is

$$N_I(\sigma) = N \sin^2 \sigma, \quad N_{II}(\sigma) = N - N_I(\sigma) = N \cos^2 \sigma. \quad (10)$$

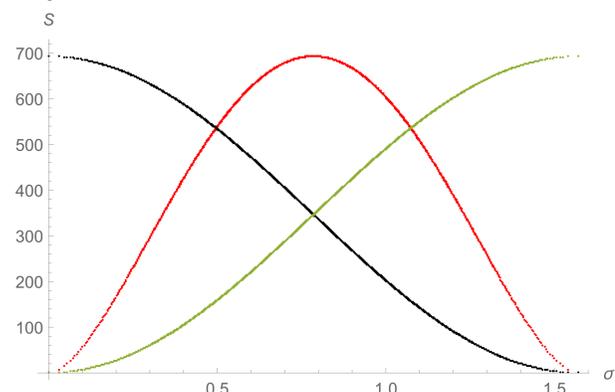
It becomes clear that σ is, in fact, a discrete parameter, essentially counting the diminishing number of free Xons. When we take this view it is natural to consider the partition

$$2^N = 2^{N_{II}(\sigma)} \times 2^{N_I(\sigma)} \equiv n \times m, \quad (11)$$

with $n = 2^N, 2^{N-1}, \dots, 1$, and $m = 1, \dots, 2^{N-1}, 2^N$, while σ runs in discrete steps in the interval $[0, \pi/2]$. It is then natural to define the Bekenstein and the environment entropy as

$$S_{BH} \equiv \ln n = N \ln 2 \cos^2 \sigma, \quad S_{env} \equiv \ln m = N \ln 2 \sin^2 \sigma. \quad (12)$$

It holds the inequality $S_I \leq S_{BH} + S_{env} = S_{max}$. In figure, the three entropies are plotted for $N = 1000$.



Number fluctuations, which make necessary to invoke the entire Hilbert space \mathcal{H} at each stage, represent a measure of entanglement as shown by a direct computation:

$$\Delta N_I(\sigma) = \Delta N_{II}(\sigma) = \frac{\sqrt{N} \sin(2\sigma)}{2}. \quad (13)$$

Duality with TFD

Let us perform the canonical transformation

$$A_n = a_n, \quad B_n = b_n^\dagger. \quad (14)$$

The vacua in the new representation satisfy $A_n |0\rangle_A = B_n |0\rangle_B = 0$. One can check that

$$|0\rangle_A = |0\rangle_I, \quad |0\rangle_B = |1_1 1_2 \dots 1_N\rangle_{II}. \quad (15)$$

It follows that

$$|\Psi(\sigma)\rangle = \prod_{n=1}^N \left(\cos \sigma + \sin \sigma A_n^\dagger B_n^\dagger \right) |0\rangle_A \otimes |0\rangle_B. \quad (16)$$

This has the form of the *thermal vacuum*, introduced in TFD [5]. The physical picture here is that, when the system evolves, a pair of A and B particles is created. The B -modes enter into the BH, annihilating BH modes, while the A -modes form the environment.

References

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