From Correlation Function to Event-shape

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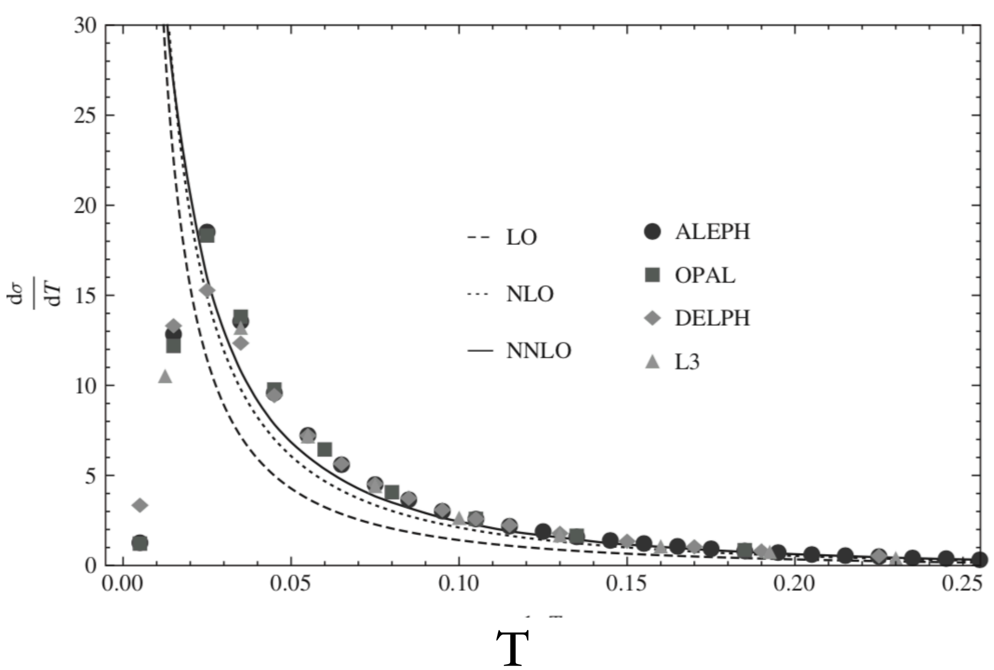
based on 2001.10806, 1903.05314 with Chicherin, Henn, Sokatchev, Zhiboedov

Event shape probes the structure of energy-momentum flow in the final state

- global geometry of hadronic final state, e.g. thrust, transverse thrust
- multi-prong jet substructure, e.g. N-subjettiness, D2
- angular distribution of energy/charge flow, e.g. EEC,QQC

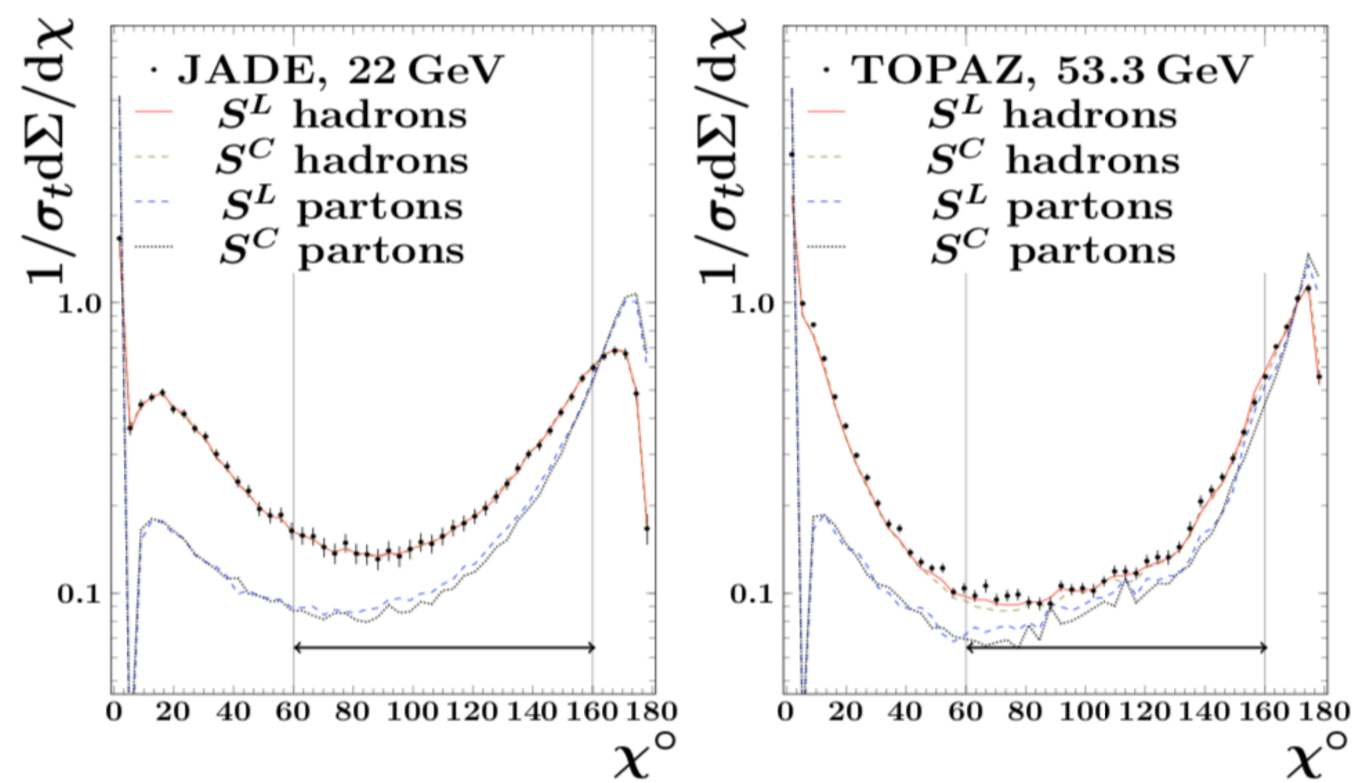
Thrust distribution

$$\int d\,\sigma_f\,\delta\Big(T\ -\max_{\vec{n}}\frac{\sum_{j\in f}|\vec{p}_j\cdot\vec{n}|}{\sum_{j\in f}|\vec{p}_j|}\Big)$$



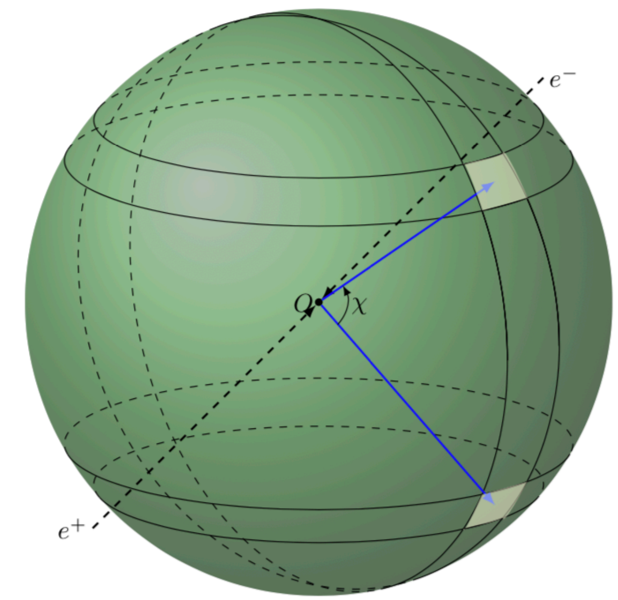
Energy-energy correlation

$$\int\sum_{i,j} d\,\Sigma_{ij}\,\delta\big(\cos\chi\ -\cos\theta_{ij}\big)\,,\qquad d\,\Sigma_{ij} = d\sigma_{ij+X}\frac{E_i E_j}{Q^2}$$



Energy-energy correlation (EEC) in e+e- annihilation

LO analytic result was available since the 70’s



$$\frac{1}{\sigma_0}\frac{d\Sigma}{d\cos\chi} = \frac{\alpha_s(\mu)}{2\pi}\, C_F\, \frac{3-2\zeta}{4(1-\zeta)\zeta^5}\left[3\zeta(2-3\zeta) + 2(2\zeta^2-6\zeta+3)\log(1-\zeta)\right]$$

χ : angle between detectors

Phys. Rev. Lett. 41, 1585 (1978).

$$\zeta := \frac{q^2(n\cdot n')}{2\,(q\cdot n)(q\cdot n')} = \frac{1-\cos\chi}{2} \quad (c.o.m)$$

$$EEC(\chi;q^2) = \int d\,\Pi_X\,(2\pi)^4\delta^4(q-P_x)\sum_{i,j\in X}\langle O\,|X\rangle\,\langle X|O\rangle\,\delta(\hat{n}-\Omega_i)\delta\big(\hat{n}'-\Omega_j\big)E_iE_j$$

O: electromagnetic currents

- admits representation in terms of a correlation function involving energy flow operators

$$\delta(\hat{n}-\Omega_i)E_i|k_i\rangle = \mathrm{E}(n)|k_i\rangle$$

$$EEC(\chi;q^2) = \int d^4x\, e^{i\,q\cdot x}\langle O(x)\,\mathrm{E}(n)\mathrm{E}(n')O(0)\rangle$$

E(n) energy flow (ANEC) operator, local operator on celestial sphere

- EEC lies at the intersection of collider physics and formal studies of conformal bootstrap and light-ray OPE

EEC is a central object in collider physics

- Precision determination of fundamental constants; precision jet substructure; 1804.09146, 1907.01435, 2004.11381.
- Small-angle and back-to-back limit: EFT 1905.01310,1801.02627 ; Light-cone operator product expansion 1905.01311,1905.01444
- Multi-point Energy Correlators: building block for computing physical cross section 1912.11050

new insights from supersymmetric theory (N=4 SYM)

- Novel methods for computation at higher-loop order: avoids infrared divergence Henn, Sokatchev, Yan, Zhiboedov 1903.05314.
- Direct application to event-shape in QCD Chicherin, Henn, Sokatchev, Yan, 2001.10806
- Function space of EEC in N=4 SYM shed light on QCD answer (e.g. via bootstrapping)
- End-point asymptotics: valuable CFT data; testing ground for understanding sub-leading power factorisation Moult, Vita, Yan 1912.02188

Novel methods : from correlation function to event shape

- **EEC in CFT can be built upon four-point conformal correlation function between the local operators**

$$EEC(\chi; q^2) = \int d^4x \, e^{i\, q\cdot x} \langle O(x)\, \mathrm{E}(n)\mathrm{E}(\mathrm{n}')O(0)\rangle$$

$$\mathrm{E}(n) \coloneqq \lim_{r\to\infty} r^2 \int d\, x_- \, n^j T_{0j}(t = x_- + r, r\,\vec{n})$$

$$\boxed{\sim \int dx_{2,-} dx_{3,-} \, \langle O\ T\ T\ O\rangle}$$

Detectors are sent to infinity and integrated over its working time

x_ : retarded time

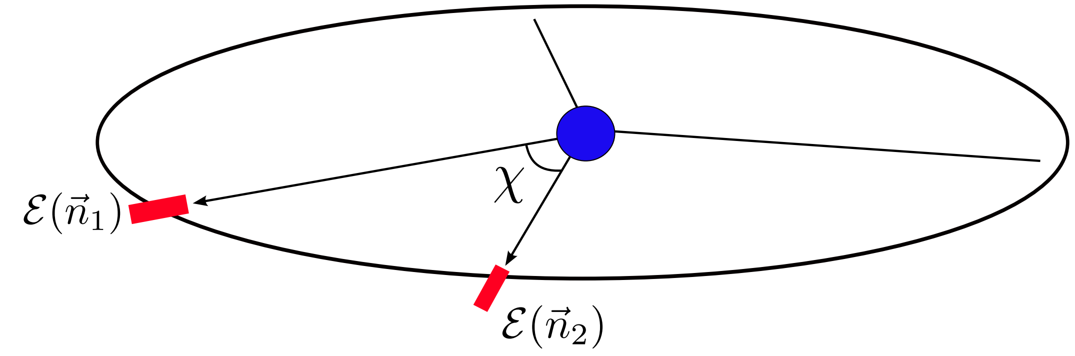
- **How we bypassed infrared divergence:**

phase-space integrals are replaced by off-shell conformal integrals to compute the correlation function + a two-fold, manifestly finite integral over detector time

EEC in N=4 supersymmetric theory

source: half BPS operator
(protected operator analog to EM current in QCD)

$$\mathcal{O}^{IJ} = \mathrm{Tr}\left(\Phi^I\Phi^J - \frac{1}{6}\delta^{IJ}\Phi^K\Phi^K\right)$$



N=4 superconformal symmetry relates
energy flow to scalar flow

$$\mathrm{EEC}(\zeta) \sim \int d^4x\, e^{iq\cdot x} \int_{-\infty}^{\infty} dx_{2-} dx_{3-} \lim_{x_{2+,3+}\to\infty} x_{2+}^2 x_{3+}^2 \langle 0|O^\dagger(x)\mathcal{O}(x_2)\mathcal{O}(x_3)O(0)|0\rangle$$

This correlation function is given by off-shell
conformal integrals defined in 4d-Euclidean space

4-point correlation function of identical scalars.

Given explicitly as polylogarithmic function $G(u, v)$

$$u = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}.$$

Detector-time integration

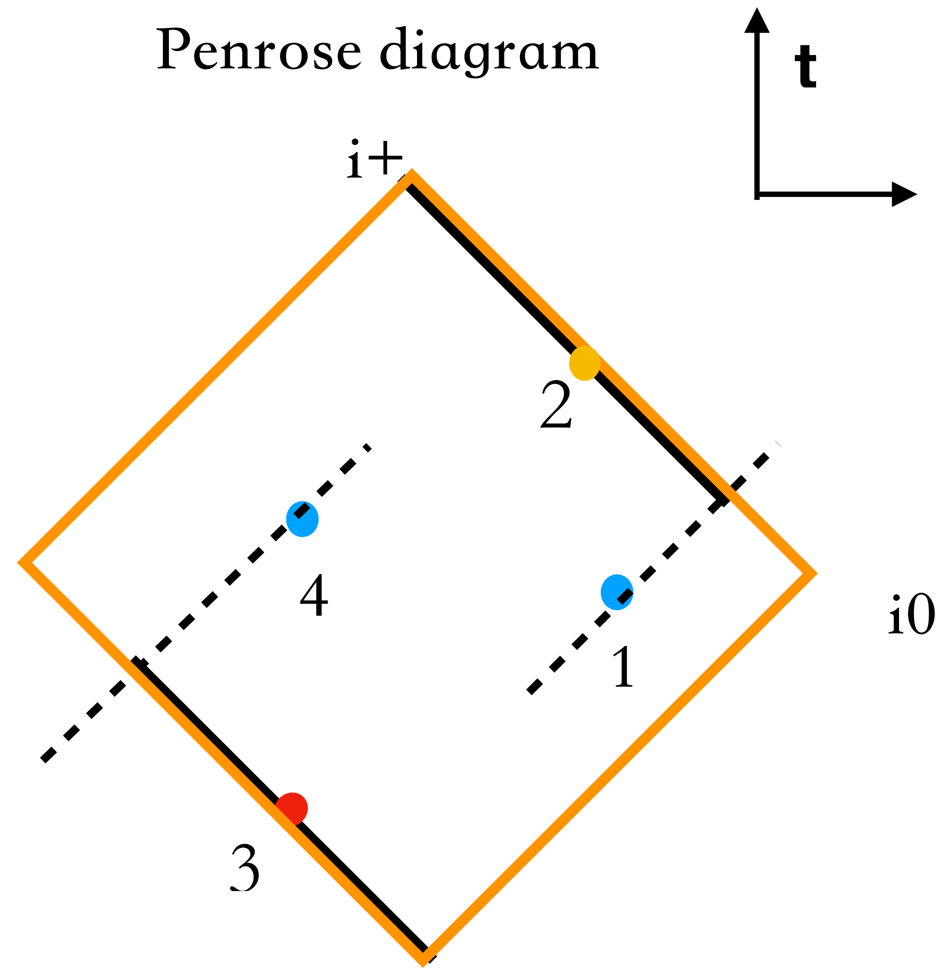
Integrated correlation function:

$$G_{EEC}(\gamma) := \frac{(n\cdot\bar{n})(n'\cdot\overline{n'})}{x^2\,(n\cdot n')}\int d\,x_{2,-}\,dx_{3,-}\,G_W(u,v)$$

$$\gamma := \frac{2(n\cdot x)(n'\cdot x)}{x^2(n\cdot n')}$$

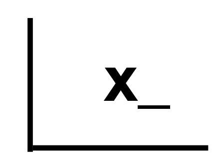
$\mathcal{G}_W(u,v)$:analytically continued 4-pt correlator; multi-valued function in time-like separated configuarations

Penrose diagram

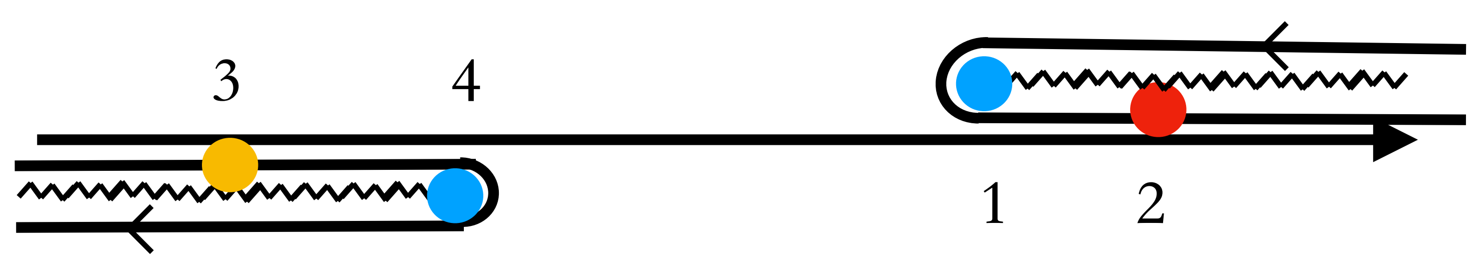


$$\int_{C_2} dx_{2-}\int_{C_3} dx'_{3-}\,\boxed{\mathrm{disc}_{x_{2-}=x_-}\,\mathrm{disc}_{x'_{3-}=0}[\mathcal{G}(u,v)]}$$

Integrate detectors (2,3) along a null line.



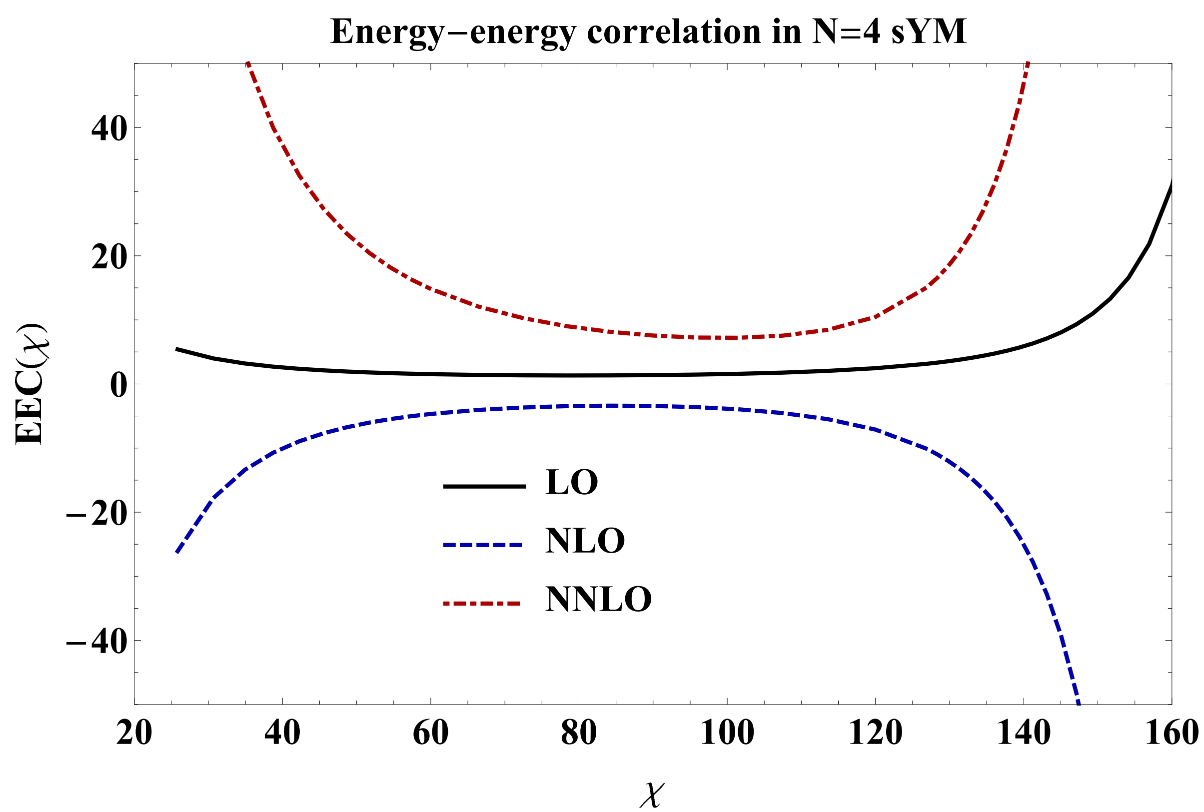
Two-fold contour integral — integrated double discontinuity



First calculation of EEC at NNLO :

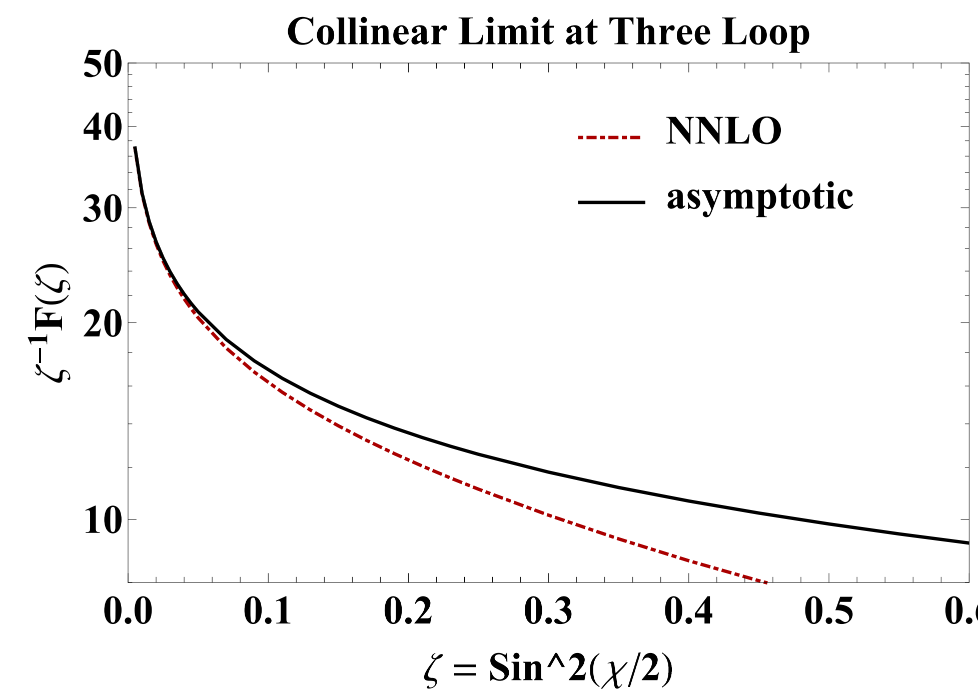
Henn, Sokatchev, Yan, Zhiboedov 1903.05314.

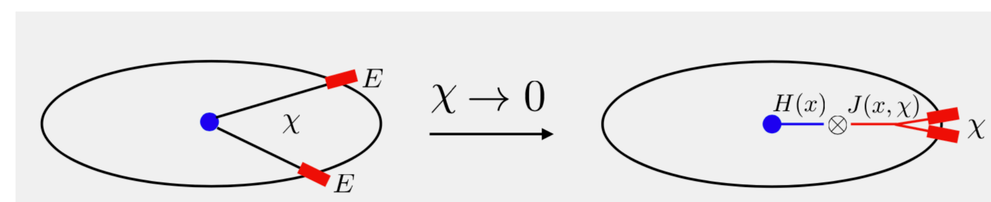
Fully analytic result : HPL + explicit two-fold finite elliptic integral



Fixed-order perturbative corrections displayed separately by loop order

$\chi \to 0$





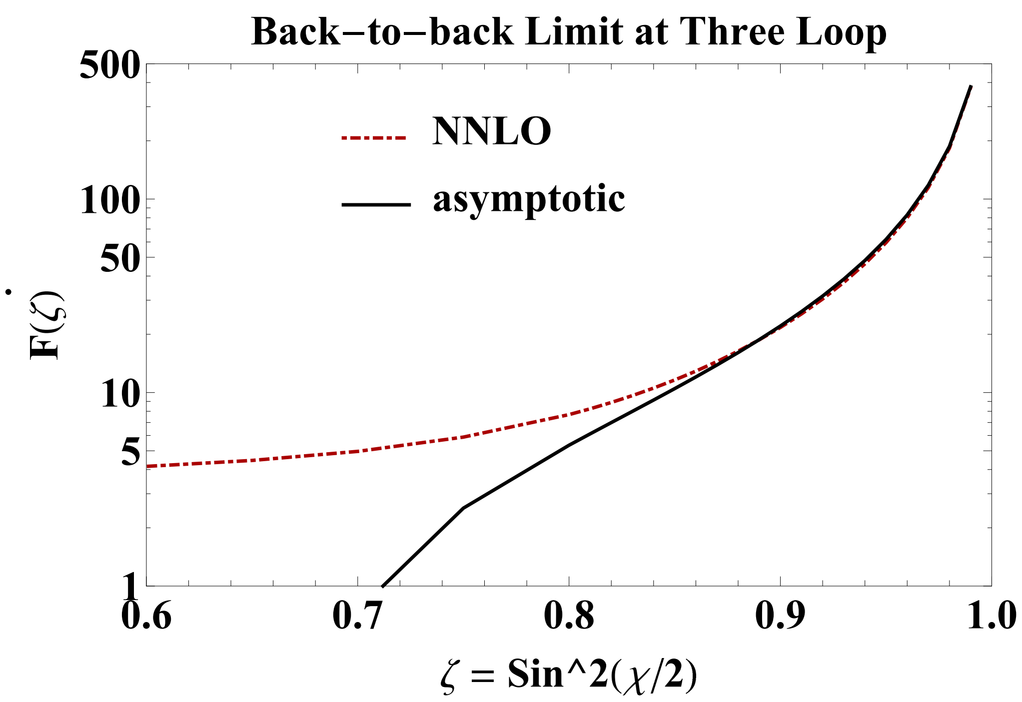
$$J(zQ^2,\mu) = C_J(\alpha_s)\left(\frac{zQ^2}{\mu^2}\right)^{\boxed{\gamma_J^{\mathcal{N}=4}(\alpha_s)}}$$

$\chi \to \pi$

Leading power: sudakov factorisation

$$\frac{1}{2}\boxed{H(a)}\int_0^\infty db\, bJ_0(b)\exp\left\{-\frac{1}{2}\Gamma_{\text{cusp}}(a)L^2 - \boxed{\Gamma(a)}L\right\}\,.$$

Extract hard function and collinear anomalous dimension for N^3LL resummation



Sub-leading power:

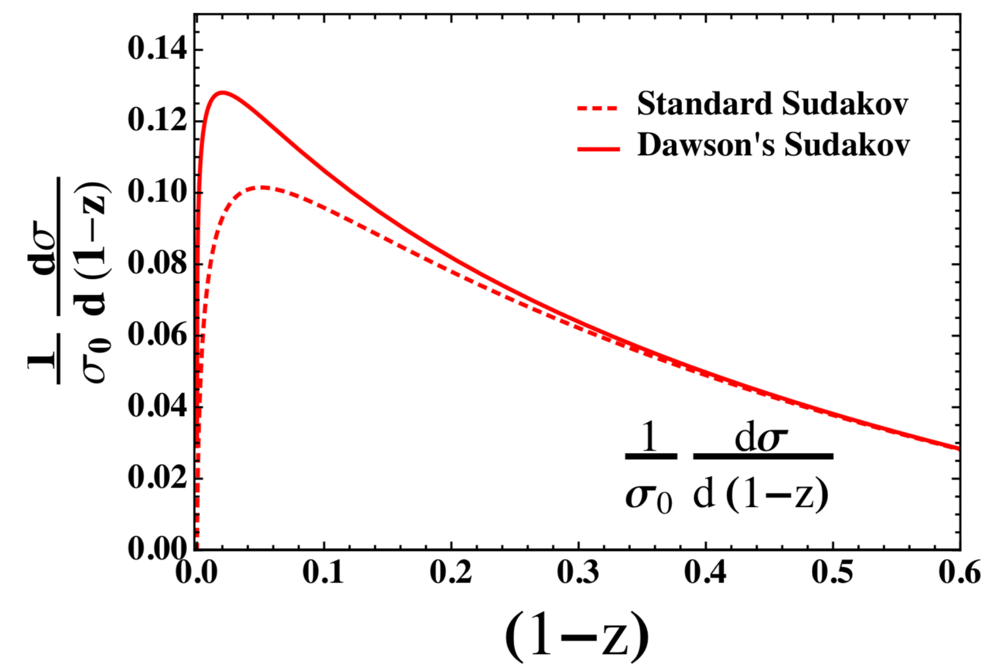
The LL series at NLP is a double factorial sum.

$$\text{EEC}^{(2)} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!!}a_s^{n+1}\log((1-z)^2)^{2n+1}$$

$$\boxed{\text{EEC}^{(2)} = -\sqrt{2a_s}\; D\left[\sqrt{\frac{\Gamma^{\text{cusp}}}{2}}\log(1-z)\right]\,.}$$

$$D(x) = \frac{1}{2}\sqrt{\pi}e^{-x^2}\text{erfi}(x)$$

Dawson ’ s function



Moult, Vita, Yan 1912.02188

Generalisation to event-shape in QCD

Proof of principle: one-loop charge-charge correlation in QCD

Chicherin, Henn, Sokatchev, Yan
2001.10806

$$\langle \mathcal{Q}(\vec{n})\mathcal{Q}(\vec{n}')\rangle = \sigma_{\text{tot}}^{-1}\int d^4x_1 e^{iqx_{14}}\int dx_{2-}dx_{3-}$$
$$\lim_{x_{2+},x_{3+}\to\infty} x_{2+}^2 x_{3+}^2 \underline{\underline{\langle J_{\alpha\dot\alpha}(x_1)\, J_{\beta\dot\beta}(x_2)\, J_{\gamma\dot\gamma}(x_3)\, J_{\delta\dot\delta}(x_4)\rangle}}\, \bar{n}^{\dot\beta\beta}\bar{n}^{\dot\gamma\gamma}$$

QQC is built upon the correlation function of four identical vector currents.

A toy example to study event shape in QCD at the level where it can still be treated as conformal.

- compute four point correlation function in QCD
- implement detector time integration

Validity of the approach: IR and UV finiteness

In the one-loop approximation in our setup, QQC is infrared safe. Moreover, the complications due to UV renormalisation can be avoided.

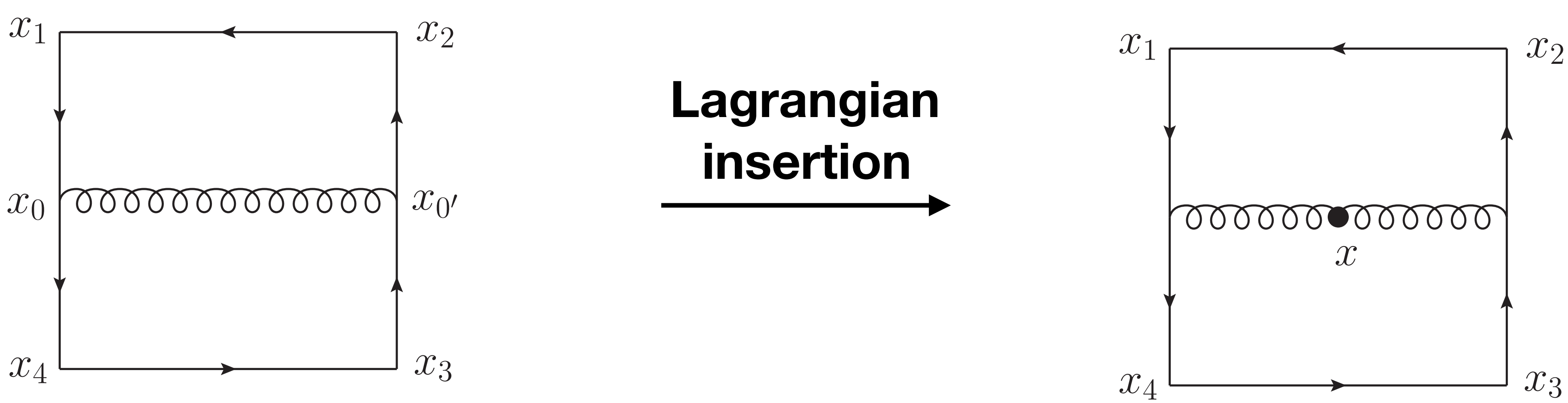
- Electromagnetic vector current is conserved.
- The interaction vertex renormalisation effects play no role since $\beta(g) = O(g^3)$.
- The field strength renormalisation is finite in our calculation scheme at one-loop level.

The correlation functions of vector currents in the one-loop approximation is finite four-dimensional objects which do not require any regularisation.

$$\langle J_\mu(x_1)\dots J_\nu(x_n)\rangle = G^{\text{tree}}_{\mu\dots\nu}(x_1,\dots,x_n) + g^2\, G^{\text{1-loop}}_{\mu\dots\nu}(x_1,\dots,x_n) + \mathcal{O}(g^4)$$

Tools developed the study of conformal correlators in N=4 SYM can be implemented here.

Feynman diagram calculation in coordinate space



$$A_\mu \to \tfrac{1}{g} A_\mu \qquad L = \frac{1}{g^2} L_{\rm YM} + i\psi^\alpha \mathcal{D}^+_{\alpha\dot\alpha} \bar\psi^{\dot\alpha} + i\chi^\alpha \mathcal{D}^-_{\alpha\dot\alpha} \bar\chi^{\dot\alpha} \qquad L_{\rm YM} \equiv -\frac{1}{2} {\rm tr}\, F_{\mu\nu} F^{\mu\nu}$$

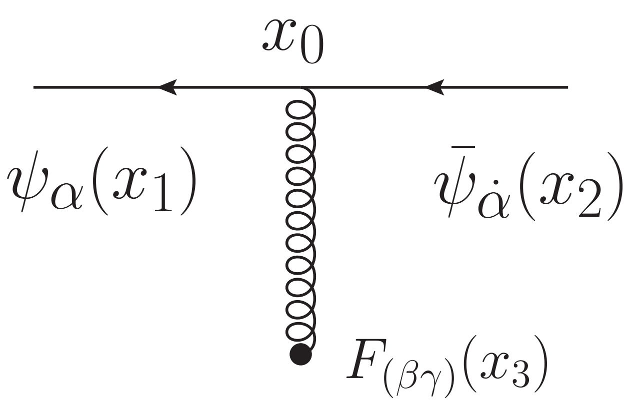
taking derivative of the path integral w.r.t to the coupling,

$$g^2 \frac{\partial}{\partial g^2} \langle J_\mu(x_1) \ldots J_\nu(x_n) \rangle \quad = -\frac{i}{g^2} \int D\Phi\, e^{i\int d^4x L(x)} \int d^4x\, L_{\rm YM}(x) J_\mu(x_1) \ldots J_\nu(x_n)$$

one-loop correction of the 4-pt correlation function is equal to the integrated 5-pt correlator with one additional Lagrangian point

$$\boxed{g^2 G^{\text{1-loop}}_{\mu\ldots\nu}(x_1, \ldots, x_n) = -i \int d^4x\, \langle L_{\rm YM}(x) J_\mu(x_1) \ldots J_\nu(x_n) \rangle_{\rm Born}}$$

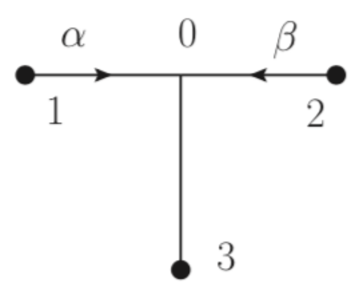
Diagrammatic building block



$$\langle \psi_\alpha(x_1)\bar\psi_{\dot\alpha}(x_2)F_a^{\mu\nu}(x_3)\rangle_g = -\frac{ig\,T_a}{(2\pi)^6}\int d^4x_0\,\partial_\alpha^{\dot\beta}\frac{1}{x_{10}^2}\partial_{\dot\alpha}^{\beta}\frac{1}{x_{20}^2}\sigma^{[\nu}_{\beta\dot\beta}\partial^{\mu]}\frac{1}{x_{30}^2}$$

$$= -\frac{ig\,T_a}{(2\pi)^4}\left\{\frac{4(x_{12})_{\alpha\dot\alpha}}{x_{12}^4}\frac{x_{31}^{[\mu}x_{32}^{\nu]}}{x_{13}^2x_{23}^2} - \frac{(x_{12}\tilde x_{23}\sigma^{[\mu}\tilde\sigma^{\nu]}x_{32})_{\alpha\dot\alpha}}{2x_{12}^2x_{13}^2x_{23}^4} - \frac{(\tilde x_{21}x_{13}\tilde\sigma^{[\mu}\sigma^{\nu]}\tilde x_{31})_{\alpha\dot\alpha}}{2x_{12}^2x_{13}^4x_{23}^2}\right\}$$

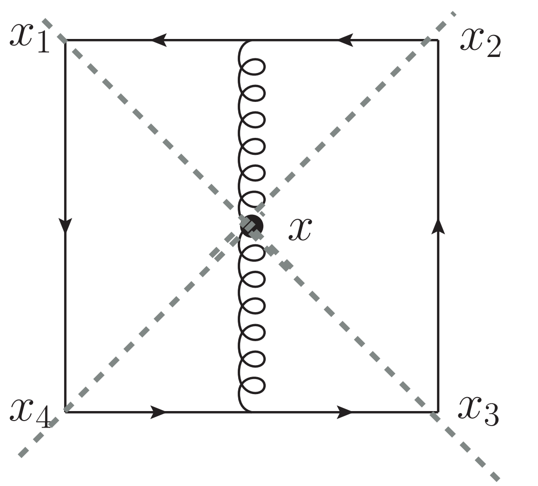
can be related to a conformal integral: answer is rational function completely fixed by conformal property

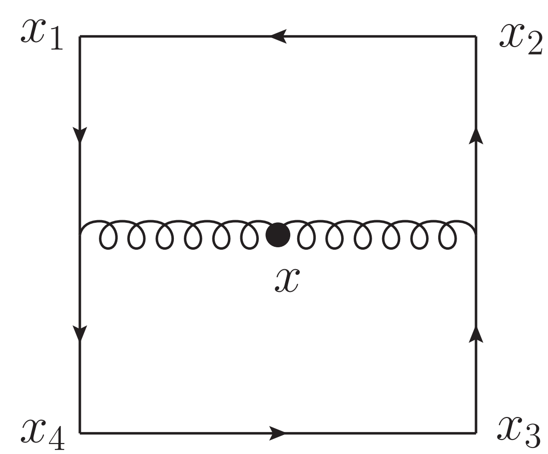


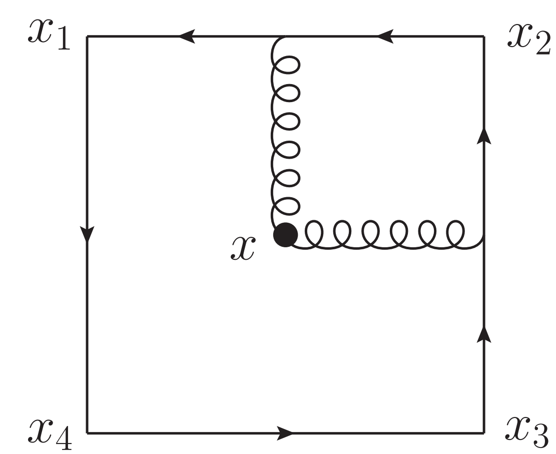
$$-\frac{1}{\pi^2}\int d^4x_0\;x_{30}^{-2}\,\partial_{1\,\alpha\dot\alpha}x_{10}^{-2}\,\tilde\partial_2^{\dot\alpha\beta}x_{20}^{-2} = \frac{(x_{13}\tilde x_{32})_\alpha{}^\beta}{x_{12}^2x_{13}^2x_{23}^2}$$

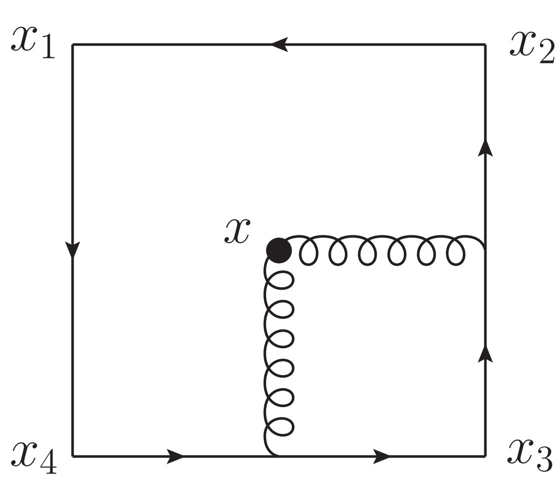
conformal symmetry of the integral is easily shown by inversion.
The rhs is the unique conformal covariant with the required weights, made from the three vectors x_i.

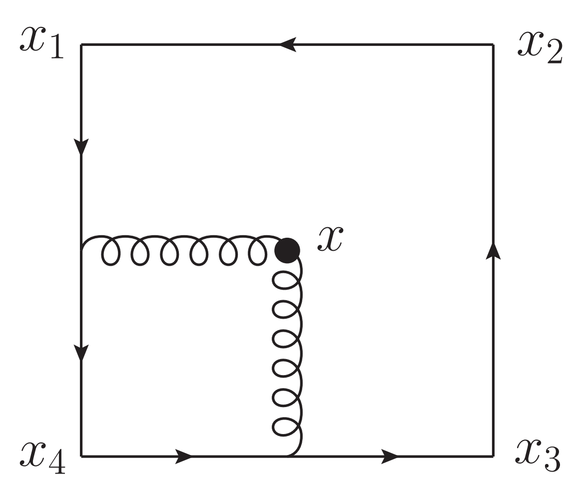
Class of diagrams built out of T-blocks and fermion propagators

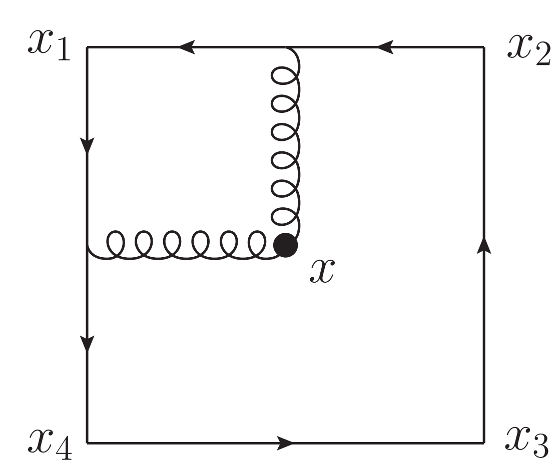


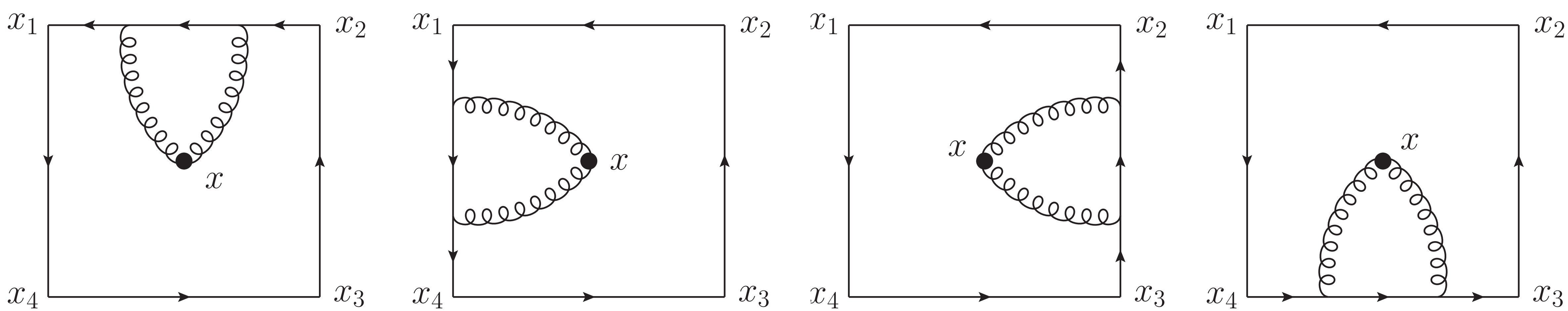


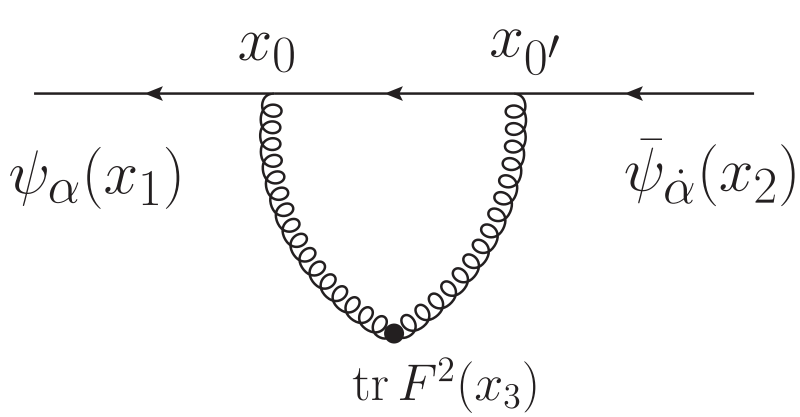












Our approach (via Lagrangian insertion) defines a renormalisation scheme, where one loop propagator correction is finite and rational.

$$= \frac{g^2 C_2}{2\pi^2}\frac{(x_{12})_{\alpha\dot\alpha}}{x_{12}^4 x_{13}^4 x_{23}^4}[x_{12}^4 + x_{13}^4 + x_{23}^4 - 2x_{12}^2 x_{13}^2 - 2x_{12}^2 x_{23}^2 - 2x_{13}^2 x_{23}^2]$$

Putting together all diagrams contributing to the 5-pt correlator, the obtained integrand is a finite rational function living in four space-time dimensions.

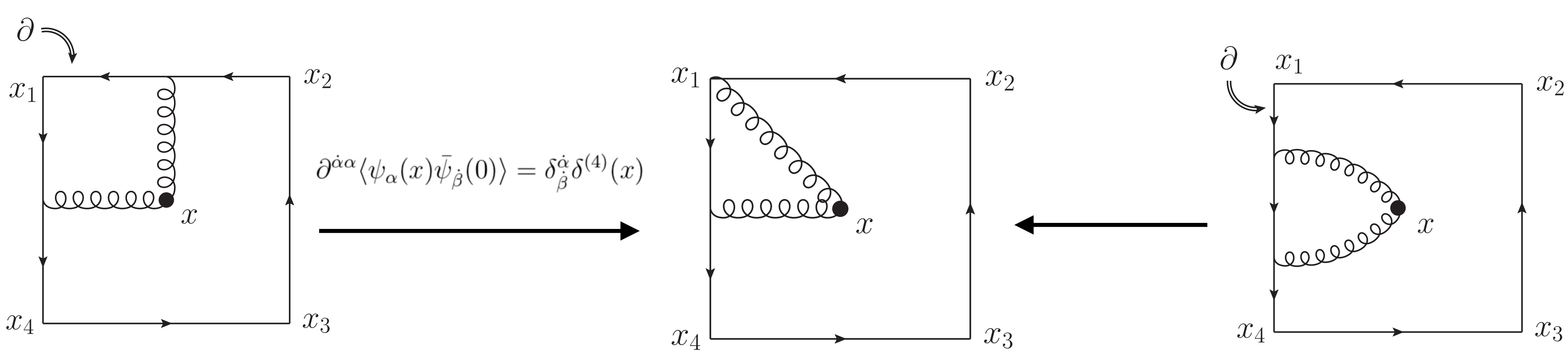
Consistency check

Conformal property integrand (5-pt function) itself is conformal.

Performing a conformal inversion on all points, we numerically checked that integrand transforms covariantly at each point.

Current conservation $\partial_1^{\dot{\alpha}\alpha} \langle L_{\mathrm{YM}}(x)\, J_{\alpha\dot{\alpha}}(x_1) \dots J_{\delta\dot{\delta}}(x_4) \rangle_{\mathrm{Born}} = 0\,.$

The current conservation can be easily seen at the level of the Feynman graphs.



The fermion propagator shrinks to a point upon acting with the derivative, and the shrunk Feynman diagrams cancel in pairs.

Integrating over the insertion point

- Tensor reduction in coordinate space
- Implement IBP reductions of the scalar integrals to a set of master integrals: bubble, three-mass triangle, and four-mass box integrals.
- Removing regularisation $\varepsilon \to 0$. UV divergences of bubble integrals cancel.

$$\boxed{G^{\text{1-loop}}(x_1,\ldots,x_4) = R_c(x)\,\Phi^{(1)}(u,v) + R_u(x)\,\log(u) + R_v(x)\,\log(v) + R_r(x)}$$

R: polynomial tensors carrying Lorentz indices of the four currents

$$u = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}.$$

Bloch-Wigner dilogarithm

$$\Phi^{(1)}(u,v) = \frac{1}{\bar z - z}\left[2\mathrm{Li}_2\left(\frac{z}{z-1}\right) - 2\mathrm{Li}_2\left(\frac{\bar z}{\bar z - 1}\right) - \log\left(\frac{z\bar z}{(1-z)(1-\bar z)}\right)\log\left(\frac{1-z}{1-\bar z}\right)\right]$$

$$u = z\bar z\,, \qquad v = (1-z)(1-\bar z)$$

Finding better representation

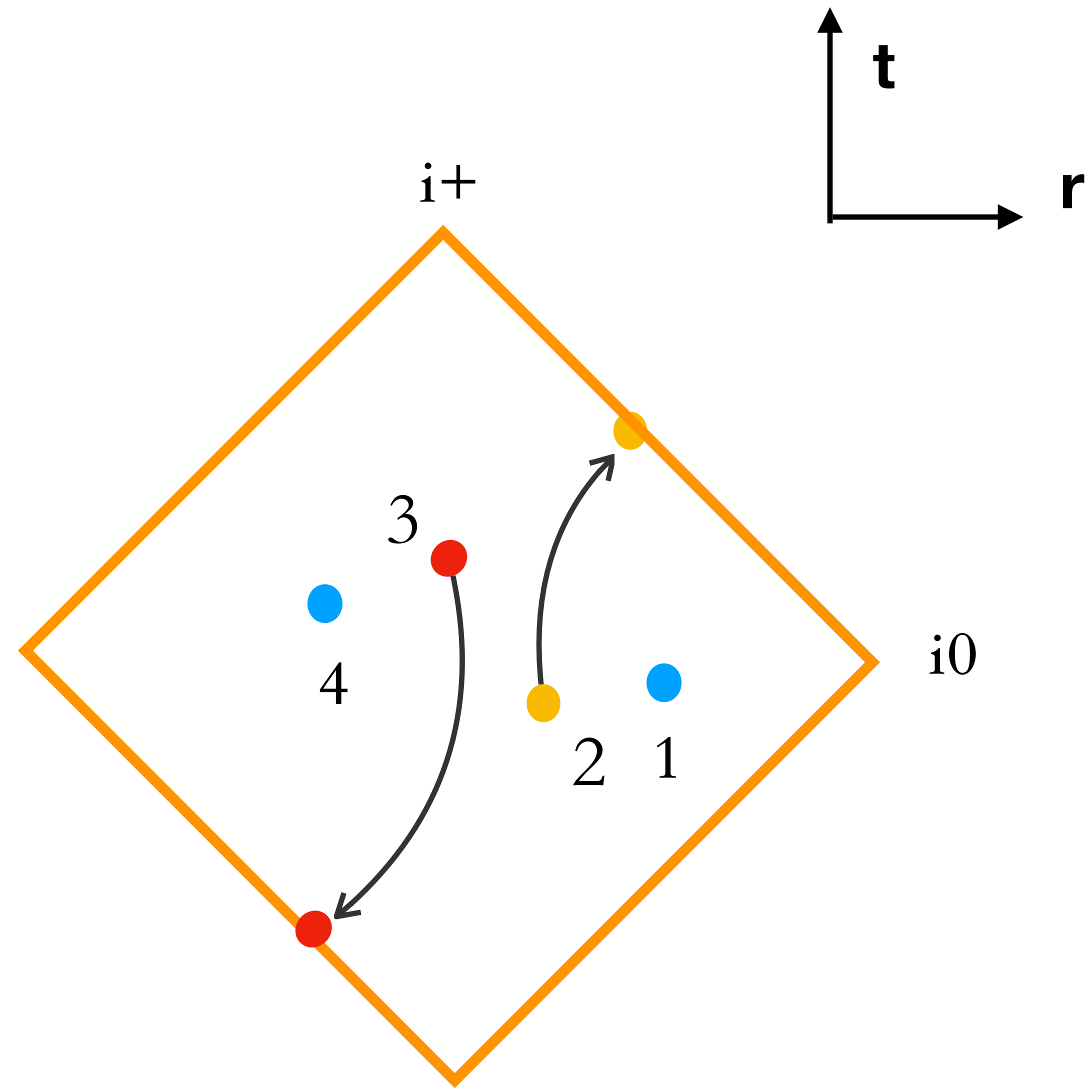
Conformal ansatz for homogeneous polynomials with equal conformal weights at all point: 43 parity even and 23 parity odd independent conformal tensors carrying 8 Lorentz indices: $\alpha\dot{\alpha} \ldots \delta\dot{\delta}$

Matching with the ansatz we find explicitly conformal expressions for the polynomials

$$\boxed{\begin{aligned}(x_{13}^2 x_{24}^2)^4\, G^{\text{1-loop}}(x_1,\ldots,x_4) = \quad & \Big[\partial^2_{uu}\Phi^{(1)}\sum_I c^{(I)}_{uu}(u,v)T_I + \partial^2_{vv}\Phi^{(1)}\sum_I c^{(I)}_{vv}(u,v)T_I \\ & + \partial^2_{uv}\Phi^{(1)}\sum_I c^{(I)}_{uv}(u,v)T_I + \sum_I c^{(I)}_0(u,v)T_I + (x_{13}^2 x_{24}^2)^4\, R'_r\Big]\end{aligned}}$$

{ T_I } contains 54 different conformal tensor structures

Detector limit



Send detectors (points 2 and 3) to null infinity, putting them close to the light-cone

Introduce light-cone parametrisation

$$x_2^\mu = x_{2+}\, n^\mu + x_{2-}\, \bar{n}^\mu \ , \ x_3^\mu = x_{3+}\, n'^\mu + x_{3-}\, \bar{n}'^\mu$$

Rescale x2+, x3+ by a factor of λ(→ ∞), the scalar products scale accordingly,

$$x_{12}^2 \to \lambda\, x_{12}^2 \,, x_{13}^2 \to \lambda\, x_{13}^2 \,, x_{24}^2 \to \lambda\, x_{24}^2 \,, x_{34}^2 \to \lambda\, x_{34}^2 \,,$$

$$x_{23}^2 \to \lambda^2\, x_{23}^2 \,, x_{14}^2 \to x_{14}^2 \qquad u \to u \,, \qquad v \to v$$

The tensor structure of the correlation function simplifies significantly .

$$
\begin{aligned}
&(x_{2+}x_{3+})^2 \langle J^\mu(x_1)J^-(x_2)J^-(x_3)J_\mu(x_4)\rangle \to \frac{(n\bar{n})(n'\bar{n}')}{2\,x_{14}^8(nn')^3}\,v^4\times \\
&\Bigg\{\Bigg[4-\frac{2}{u}+4u-10v-\frac{6v}{u}+\frac{4v^2}{u}+\frac{x_{31-}}{x_{34-}}\left(-4-6u-4u^2+2v\right) \\
&+\frac{x_{34-}}{x_{31-}}\left(-6-\frac{4}{u}-4u+\frac{2v}{u}\right)\Bigg]\partial^2_{uu}\Phi^{(1)} \\
&+\Bigg[-4-\frac{4}{u}-\frac{4}{u^2}-4u-4v+\frac{2v}{u^2}+\frac{4v^2}{u^2}+\frac{8v^2}{u}-\frac{6v^3}{u^2}+\frac{x_{31-}}{x_{34-}}\left(\frac{4}{u}+4u+2v+4uv-\frac{2v^2}{u}\right) \\
&+\frac{x_{34-}}{x_{31-}}\left(4+\frac{4}{u^2}+4v+\frac{2v}{u}-\frac{2v^2}{u^2}\right)\Bigg]\partial^2_{vv}\Phi^{(1)} \\
&+\Bigg[-\frac{2}{u^2}-\frac{6}{u}+8u-2v-\frac{4v}{u^2}+\frac{10v^2}{u^2}+\frac{6v^2}{u}-\frac{4v^3}{u^2} \\
&+\frac{x_{31-}}{x_{34-}}\left(2+\frac{4}{u}-2u-4u^2+4v+\frac{6v}{u}+4uv-\frac{2v^2}{u}\right) \\
&+\frac{x_{34-}}{x_{31-}}\left(-2+\frac{4}{u^2}+\frac{2}{u}-4u+4v+\frac{6v}{u^2}+\frac{4v}{u}-\frac{2v^2}{u^2}\right)\Bigg]\partial^2_{uv}\Phi^{(1)} \\
&+\Bigg[\frac{6}{u^3}+\frac{10}{u^2}-\frac{4}{v^2}-\frac{4}{u^2v^2}+\frac{2}{u^3v}+\frac{4}{u^2v}+\frac{4}{uv}-\frac{4v}{u^3} \\
&+\frac{x_{31-}}{x_{34-}}\left(-\frac{2}{u^2}+\frac{2}{v^2}+\frac{2}{uv^2}-\frac{6}{uv}\right)+\frac{x_{34-}}{x_{31-}}\left(-\frac{2}{u^3}+\frac{2}{u^2v^2}+\frac{2}{uv^2}-\frac{6}{u^2v}\right)\Bigg]\Bigg\}
\end{aligned}
$$

$$
\mathcal{G}(u,v,\gamma)=\mathcal{P}(u,v,\gamma)\,\Phi^{(1)}(u,v)+\frac{1}{u^3}r_r(u,v,\gamma) \qquad \gamma=\frac{2(nx_{14})(n'x_{14})}{x_{14}^2\,(nn')}
$$

second-order differential operator

$$
\mathcal{P}(u,v,\gamma)\equiv\frac{1}{u^2}\left[u\,r_{uu}(u,v,\gamma)\,\partial^2_{uu}+r_{uv}(u,v,\gamma)\,\partial^2_{uv}+r_{vv}(u,v,\gamma)\,\partial^2_{vv}\right]
$$

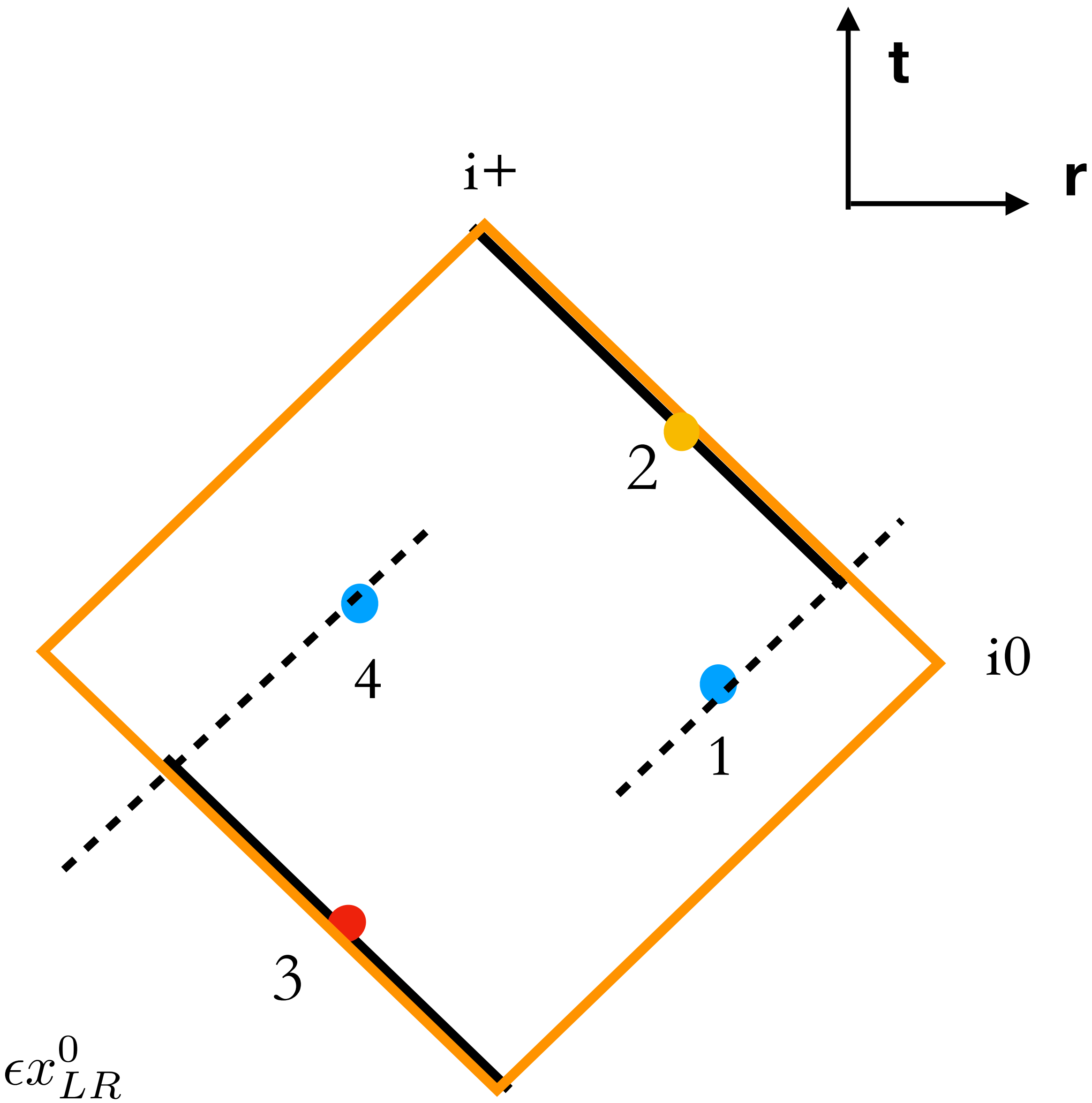
Analytic continuation

Wightman correlation function:

$$\lim_{n\to\infty} x_{2+}^2\, x_{3+}^2 \left\langle J^{\mu}(x_1) J^{-}(x_2) J^{-}(x_3) J_{\mu}(x_4) \right\rangle$$
$$:= \frac{(n\cdot\bar{n})(n'\cdot\bar{n}')}{x^2\,(n\cdot n')} G_W(u,v,\gamma)$$

In Minkowski space, the distance invariants have small imaginary parts which admit the Wightman prescriptions

$$\langle \cdots O_L \cdots O_R \rangle \qquad x_{LR}^2 \to \hat{x}_{LR}^2 = -x_{LR}^2 + i\epsilon x_{LR}^0$$



Causal relationships between scattering points

$\mathcal{G}_W(u,v)$ **develops branch cut when detectors are time-like separated from the source/sink**

Time ordering in scattering regime 1< 2, 3' <4' specifies the analytic continuation path.

Time integration: integrated discontinuities

$$x_1 = x, x_4 = 0, x_{2+} = x'_{3+} \to \infty$$

$$x_{i-} = n \cdot x_i, \quad x'_{i-} = n' \cdot x_i$$

Detector-time integration

time-integration contour specified by Wightman ordering encoded in ie prescriptions

$$\int_{-\infty}^{\infty} dx_{2-} dx'_{3-} \, \mathcal{G}(u, v)$$

no end-point singularity; branch cuts starting at

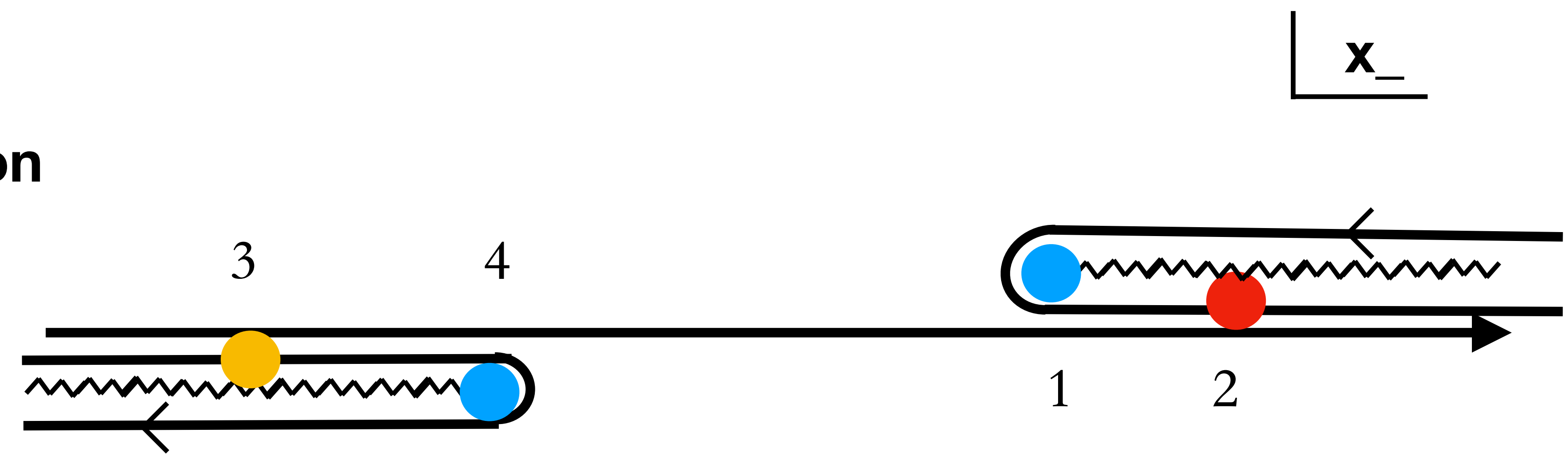
$$x_{2-} = x_-, x'_{3-} = 0$$

cross ratio after sending detectors to null infinity :

$$u = \frac{\hat{x}^2 \, (n \cdot n')}{2(x_- - x_{3-} - i\epsilon)(x'_{2-} - i\epsilon)}$$

$$v = \frac{(x_- - x_{2-} - i\epsilon)(x'_{3-} - i\epsilon)}{(x_- - x_{3-} - i\epsilon)(x'_{2-} - i\epsilon)}$$

Contour deformation



Detector-time integration

integrated double discontinuity:

$$\int_{C_3} dx_{2-} \int_{C_2} dx'_{3-} \boxed{\mathrm{disc}_{x_{2-}=x_-}\, \mathrm{disc}_{x'_{3-}=0} [\mathcal{G}(u,v)]} \qquad v = \frac{(x_- - x_{2-} - i\epsilon)(x'_{3-} - i\epsilon)}{(x_- - x_{3-} - i\epsilon)(x'_{2-} - i\epsilon)}$$

$\circlearrowleft$ / $\circlearrowright$ **brings v to adjacent Riemann sheets**

$$\boxed{\mathrm{disc}_{x_{2-}=x_-} = \mathrm{disc}_v^{\circlearrowright} \quad \mathrm{disc}_{x'_{3-}=0} = \mathrm{disc}_v^{\circlearrowleft}}$$

$$\mathrm{dDisc}_v[\mathcal{G}(u,v)] \qquad \mathrm{dDisc}_v[v^j] = \pi^2 \sin^2(\pi j)\, |v|^j$$

$$G_{\rm QQC}(\gamma) \equiv \frac{(n\bar{n})(n'\bar{n}')}{x^2(nn')} \int dx_{2-}dx_{3-}\ \mathcal{G}(u,v,\gamma)$$

double discontinuity at v =0

$$= \int_0^1 \frac{dt}{t^2}\frac{d\bar{t}}{\bar{t}^2} \boxed{\mathrm{dDisc}_v} \left[\mathcal{G}(u = \frac{1}{\gamma}t\bar{t}, v = (1-t)(1-\bar{t}))\right]$$

G contains polylogarithmic function whose arguments are (z, zb): u= z*zb, v=(1-z)(1-zb). Discontinuities should be taken at zb =1.

One of the time-integrals can be carried out explicitly. We are left with an one-fold integral over double discontinuity

$$G_{\rm QQC}(\gamma) = \int_0^1 d\bar{z}\left\{\mathrm{dDisc}_{\bar{z}=1}\left[B_0(\bar{z},\gamma)\ln(1-\bar{z}) + A_0(\bar{z},\gamma)\right]\right.$$
$$\left. + \ln(1-\bar{z})\,\mathrm{dDisc}_{\bar{z}=1}\left[B_1(\bar{z},\gamma)\ln(1-\bar{z}) + A_1(\bar{z},\gamma)\right]\right\}$$

$A_{0,1}$ and $B_{0,1}$ contain only isolated poles but no branch cut at zb= 1. Taking double discontinuity generates distributional terms at zb = 1.

The integral picks up the residue of A1 and B0 at zb = 1.

$$
\begin{aligned}
G_{\mathrm{QQC}}(\gamma) &= 2\pi^2 \left[\mathrm{Res}_{\bar{z}=1} B_0(\bar{z},\gamma) - \mathrm{Res}_{\bar{z}=1} A_1(\bar{z},\gamma)\right] \\
&= -\frac{32\pi^2}{\gamma^3}\left[\mathrm{Li}_2\left(1-\frac{1}{\gamma}\right) + \frac{3}{2}\log(\gamma) - \frac{\pi^2}{6} + \frac{\gamma}{2} + \frac{7}{4}\right]
\end{aligned}
$$

Fourier transform into momentum space

$$
\begin{aligned}
F_{\mathrm{QQC}}(z) &= -\frac{32\pi^2}{(nn')^2}\int \frac{d^4x_1 e^{iqx_{14}}}{x_{14}^6} G_{\mathrm{QQC}}(\gamma) \\
&= -\frac{2\pi^5 q^2}{(nn')^2}\frac{(2+z)z + 2\log(1-z)}{z(1-z)}
\end{aligned}
$$

we find full agreement with momentum space calculation.

At N^k LO,

off-shell position-space calculation

(k+1)-loop integral over k insertion points + 2-fold time integral

IR finite, avoids regularisation. calculation carried in four dimension

systematic procedure

- can be repeated at higher-loop order
- novel methods for computing finite integrals applicable

on-shell phase-space calculation

(k+3)- body real phase-space integral + virtual loop integrals

Separate pieces are IR divergent, regularised in D= 4- 2e

difficulty in atomisation

- multi-body phase-space integration over phase-space constraints
- D-dimensional integral reduction

Future directions

We developed a new and efficient approach, for systematic computation of energy-energy correlations at higher loop orders.

- **further development of our method**

  - at higher loop order/ with higher detector multiplicity

  - analogous study on other observables

- **Opens door to novel ideas on the studies of event shape**

  - Event shape should naturally admits a definition in terms of correlation among energy-flow (ANEC) operators

  -Goal driven study: construct novel event shape observables that fulfil this requirement

Thank you for your attention !