On dispersive representation of kaon and eta decays to 3 pions

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Details can be found in [KKNZ '20] = PRD **101** (2020), 074043 [arXiv:1911.11762 [hep-ph]]

Motivation for the studies

Processes in question

- $K^{\pm} \to \pi^0\pi^0\pi^{\pm}$, $K^{\pm} \to \pi^{\pm}\pi^{\pm}\pi^{\mp}$
- $K_L \to 3\pi^0$, $K_L \to \pi^+\pi^-\pi^0$
- $\eta \to 3\pi^0$, $\eta \to \pi^+\pi^-\pi^0$
- $K_S \to \pi^+\pi^-\pi^0$

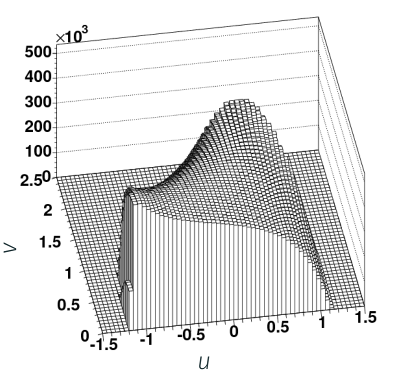
$K^\pm \to \pi^0\pi^0\pi^\pm$, $K^\pm \to \pi^\pm\pi^\pm\pi^\mp$

Experimental situation

- NA48 ['10] analysed $3 \cdot 10^9$ events for $K^\pm \to \pi^\pm\pi^\pm\pi^\mp$
- NA48 ['10] analysed $3 \cdot 10^7$ events for $K^\pm \to \pi^0\pi^0\pi^\pm$

NA48 Data ['07]

Distribution in kinematic variables for $K^\pm \to \pi^\pm\pi^\pm\pi^\mp$



Realm of low-energy QCD (lattice, ChPT, . . .) ⇒ studies of it and $\pi\pi$ scattering from cusp.

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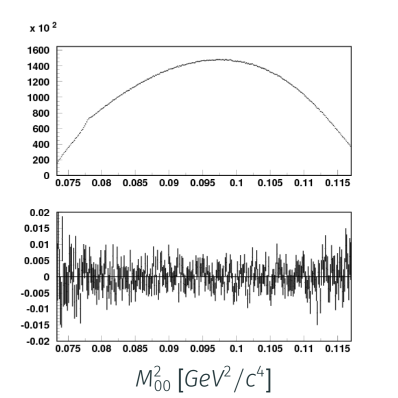
Cusp analysis

- NA48 ['09] analysed $2.8 \cdot 10^7$ events for $K^\pm \to \pi^0\pi^0\pi^\pm$
- used for the determination of $a_0 - a_2$ $\pi\pi$ scattering lengths

Cusp NA48 Data ['10]

Squared mass of $\pi^0\pi^0$ for $K^\pm \to \pi^0\pi^0\pi^\pm$



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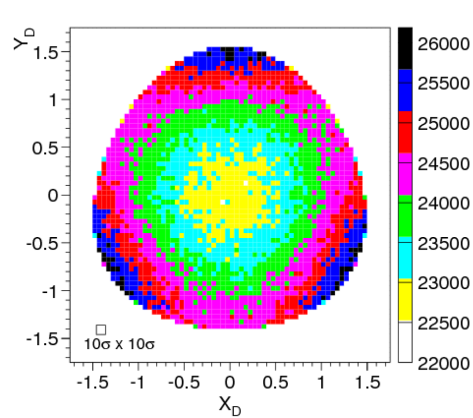
KTeV Data ['08]

Dalitz plot density for $K_L \to 3\pi^0$



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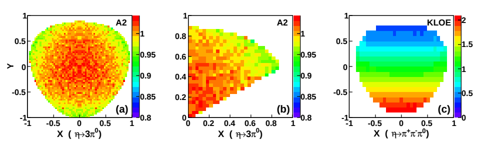
$\eta \to 3\pi$

- KLOE ['16] analysed $4.7 \cdot 10^6$ events for $\eta \to \pi^\pm \pi^\mp \pi^0$
- A2 ['18] analysed $7 \cdot 10^6$ events for $\eta \to 3\pi^0$ including the cusp
- data from other experiments (WASA, MAMI, BES-III, ...)

A2 ['09] and KLOE ['16] Data

Comparison of experimental Dalitz plots for $\eta \to 3\pi^0$



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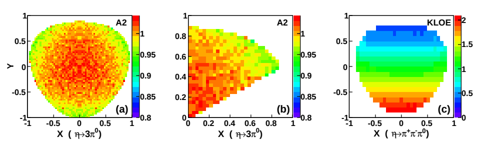

Planned experiments

New analyses and experiments in progress — mainly for η, such as KLOE, BES-III, JLab, ...

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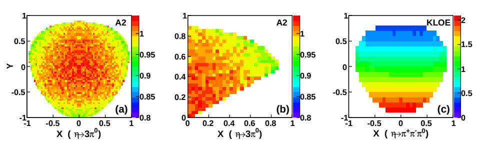


Planned experiments

New analyses and experiments in progress — mainly for η, such as KLOE, BES-III, JLab, ...

Realm of the low-energy QCD $\Rightarrow$ its studies; determination of $m_u - m_d$ and the cusp.

Model-independent description of the data

Kinematic variables used

For $P(k) \to \pi_1(p_1)\pi_2(p_2)\pi_3(p_3)$, we use Mandelstam variables $s_i = (k - p_i)^2$ and then

$$x = \frac{\sqrt{3}}{2m_P Q}(s_2 - s_1), \qquad y = \frac{3}{2m_P Q}((m_P - m_3)^2 - s_3) - 1, \qquad \text{(for } \eta\text{)}$$

where Q is the energy of the reaction, $Q = m_P - m_1 - m_2 - m_3$.

For K, the normalization and the middle point for y are different.

Dalitz plot parametrization

The square of the invariant amplitude of the considered process is usually parametrized using the polynomial (for η)

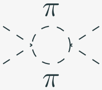
$$|\mathcal{M}|^2 = |A|^2 \left(1 + ay + by^2 + cx + dx^2 + exy + fy^3 + gx^2y + \cdots\right)$$

with real parameters $a, b, \cdots$. (For K, their names are different.)

Model-independent description of the data

Nonanalyticities of the amplitudes

Unitarity tells us that amplitudes can have non-analytic structures.

E.g. the loop contribution 

has two branching points $4m_\pi^2$ and ∞.

For data analyses, only amplitudes within the physical region important.

Various different intermediate particles (branching points) $\rightarrow$ they can manifest themselves as non-analyticities in the physical region.

Cusps in the processes with pions

As $m_{\pi^0} < m_{\pi^\pm}$, in the processes where the physical region starts at the neutral masses, the branch points connected with the charged pions appear. The interference of such rescattering with the analytic one, leads to the cusp effect.

Its strength sensitive to $\pi^+\pi^- \rightarrow \pi^0\pi^0$ scattering $\Rightarrow$ can be used for its analysis.

Model-independent description of the data

Description of the non-analytic effects

Polynomial parametrizations cannot describe them.

$\Rightarrow$ One uses more complicated theory-inspired parametrizations (sometimes resigning on their model independency).

Nowadays, either those that follow from the Non-relativistic effective field theory (see e.g. [Gasser et al. '11, NPB 850, 96]) or various dispersive approaches (see e.g. [Collangelo et al. '18, EPJC 78, 947]).

Our aim

Construction of a fully relativistic parametrization valid at the two-loop level in the low-energy expansion possessing correctly all the analytic properties and taking into account $m_{\pi^0} \neq m_{\pi^\pm}$ isospin breaking.

The exchange of virtual photons, photon emissions, etc. neglected.

Construction of the parametrization

Basic principles

Analyticity ("Cauchy's theorem"$\rightarrow$ "Dispersion relations")

Unitarity ("Optical theorem")

Low-energy expansion ("Low energy behavior of the partial waves")

Real analyticity

Basic principles

Analyticity (”Cauchy’s theorem”$\rightarrow$ ”Dispersion relations”)

- follows from the causality
- from the knowledge of the analytic structure and the positions and the values of the discontinuities along the branch cuts one obtains the full function up to a polynomial

Unitarity (”Optical theorem”)

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Real analyticity

Basic principles

Analyticity (”Cauchy’s theorem”$\rightarrow$ ”Dispersion relations”)

Unitarity (”Optical theorem”)

- follows from conservation of probability
- from the knowledge of the (amplitudes of) intermediate processes one obtains the imaginary part of the full amplitude

Low-energy expansion (”Low energy behavior of the partial waves”)

Real analyticity

Basic principles

Analyticity (”Cauchy’s theorem”$\rightarrow$ ”Dispersion relations”)

Unitarity (”Optical theorem”)

Low-energy expansion (”Low energy behavior of the partial waves”)

- easily understood as chiral counting of partial waves $O(E^n)$
- partial waves are dominantly real (imaginary part follows from the unitarity and so is of a higher chiral order)
- dominantly, (real parts of) S and P waves are of the order $O(E^2)$, whereas the other partial waves start at the order $O(E^4)$

Real analyticity

Basic principles

Analyticity ("Cauchy's theorem" $\rightarrow$ "Dispersion relations")

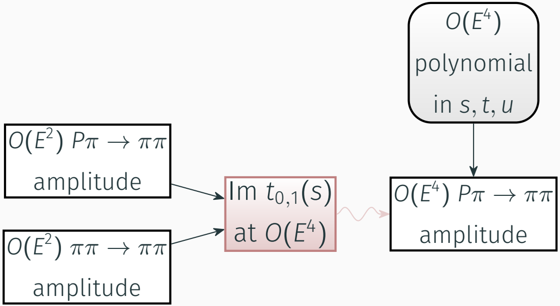
Unitarity ("Optical theorem")

Low-energy expansion ("Low energy behavior of the partial waves")

Real analyticity

- the amplitudes are real on the real axis with the exception of the branch cuts
- the discontinuity along the branch cut is equal to the imaginary part

The reconstruction procedure (for a scattering process)



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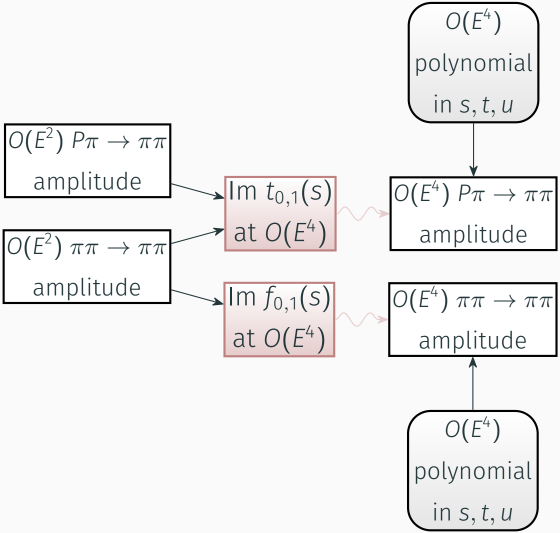

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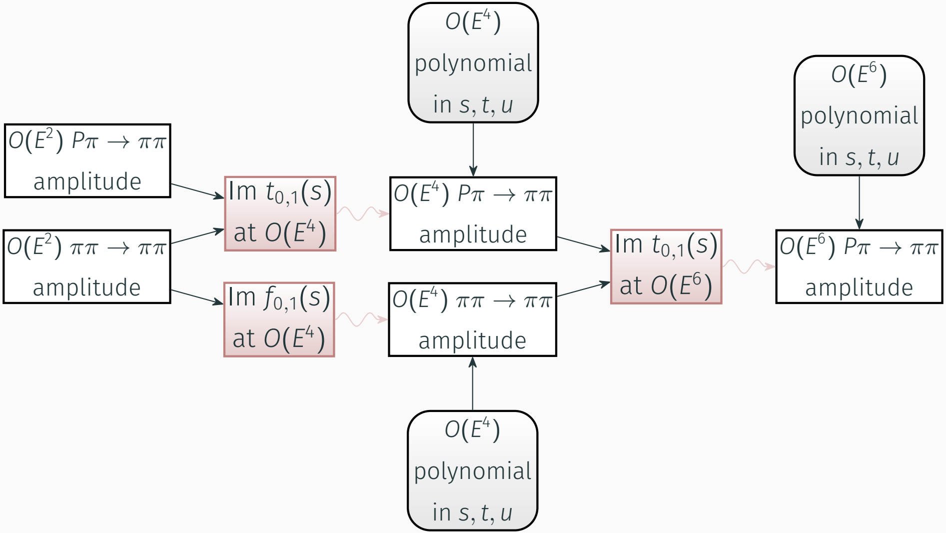

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Complications

Decay process (not the scattering one)

$\Rightarrow$ the analytic continuation in the Mandelstam variables into the decay region

$\Rightarrow$ P unstable $\Rightarrow$ no real analyticity $\rightarrow$ analytic continuation in the mass M_P to the stable values $M_P < 3m_{\pi^0}$

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Analytic continuation in M_P

- non-trivial; has to correspond to the physical sheet of the amplitudes
- we demand the validity of dispersion relations (the right analytic properties) only in the region where the low-energy expansion holds, thus, they should correspond to the analyticity properties of the amplitudes constructed from the Feynman diagram methods
  $\Rightarrow$ from the study of analytic properties of all the possible diagrams that could contribute, we obtain the continuation

Analytic continuation in M_P

Results for the analytic continuation in M_P

- one-loop computation $\to$ simple
- for $m_{\pi^0} = m_{\pi^\pm}$ at two loops $\to$ discussed already in Kacser, Bronzan '63
- $m_{\pi^0} \neq m_{\pi^\pm}$ at two loops in $K_L \to 3\pi^0$ and $\eta \to 3\pi^0$ $\to$ discussed in [KKNZ '20].
  Can be understood as Re $\to$ Disp$= \frac{1}{2}(f(s+i0)+f(s-i0))$, Im $\to$ Abs$= \frac{1}{2}(f(s+i0)-f(s-i0))$ and $M_P^2 \to M_P^2 + i0$.

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Not yet done (more complicated analytic structure)

- two-loop for different pion masses in $K_0 \to \pi^+\pi^-\pi^0$ and $\eta \to \pi^+\pi^-\pi^0$
- two-loop for different pion masses in $K^\pm \to 3\pi$

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Two-loop parametrization of $K \to 3\pi$ and $\eta \to 3\pi$ amplitudes

Parametrization for $\eta \to 3\pi^0$ with $m_{\pi^0} \neq m_{\pi^\pm}$

$$\mathcal{M}^\eta_{00} = \mathcal{P}^\eta_{00}(s,t,u) + \mathcal{U}^\eta_{00}(s,t,u) + O(E^8),$$

$$\mathcal{P}^\eta_{00}(s,t,u) = A_\eta + C_\eta \frac{\left[(s-s_0)^2 + (t-s_0)^2 + (u-s_0)^2\right]}{F_\pi^4} + E_\eta \frac{\left[(s-s_0)^3 + \ldots\right]}{F_\pi^6}$$

$$\mathcal{U}^\eta_{00}(s,t,u) = 16\pi\left[\mathcal{W}_{\eta;00}(s) + \mathcal{W}_{\eta;00}(t) + \mathcal{W}_{\eta;00}(u)\right]$$

$\mathcal{W}_{\eta;00}(s)$ contains various kinematic functions $\tilde{K}_i$, $i = 1, \ldots, 17$ multiplied by polynomials in s containing the parameters A_η, C_η (linearly) and depends on other parameters of the one-loop $\eta \to \pi^\pm\pi^\mp\pi^0$ and $\pi\pi$ scattering processes.

Example of the kinematic functions

$$\tilde{K}_1(s) = \frac{1}{2}\frac{s}{s-4m^2_{\pi^\pm}}\left[16\pi^2\bar{J}^2(s) - 4\bar{J}(s) + \frac{1}{4\pi^2}\right],$$

where $\bar{J}(s)$ is the one-loop pion function (note that also $\tilde{K}_0(s) = \bar{J}(s)$).

However, a majority of them has not so simple form and have to be computed using dispersive integrals $\tilde{K}_i(s) = \frac{s}{\pi}\int_{4M_\pi^2}^\infty \frac{dx}{x}\frac{\tilde{k}_i(x)}{x-s-i0}$ with some $\tilde{k}_i$.

Conclusions

- We have obtained a parametrization of $K \to 3\pi$ and $\eta \to 3\pi$ decay amplitudes possessing the correct analytic properties at two loops (and all the ingredients in closed analytic form).
- For the processes $K_L \to 3\pi^0$ and $\eta \to 3\pi^0$ it also includes the isospin breaking connected with $m_{\pi^0} \neq m_{\pi^\pm}$.
  Such an extension for the other processes under progress.
- The connection with ChPT can be easily restored and one can e.g. study the effects of final state rescattering on its result (as illustrated in [KKNZ '11]).
- Preparations of the fit of data for $\eta \to 3\pi^0$ (with the cusp) and $\eta \to \pi^+\pi^-\pi^0$ under progress.
- An addition of $1/N_C$ expansion enables extension to $\eta' \to \eta\pi\pi$.

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Thank you for your attention.