A novel computational paradigm for a precise determination of the hadronic contribution to $(g-2)_\mu$ from lattice QCD

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Based on: M. Dalla Brida, LG, T. Harris and M. Pepe, arXiv:2007.02973

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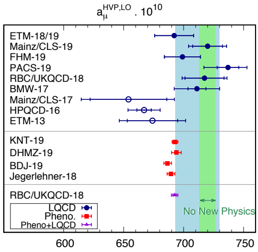
Status of the anomalous magnetic moment of the muon

[Aoyama et al. 20]

Contribution	Section	Equation	Value $\times 10^{11}$	References
Experiment (E821)		Eq. (8.13)	116 592 089(63)	Ref. [1]
HVP LO (e^+e^-)	Sec. 2.3.7	Eq. (2.33)	6931(40)	Refs. [2–7]
HVP NLO (e^+e^-)	Sec. 2.3.8	Eq. (2.34)	−98.3(7)	Ref. [7]
HVP NNLO (e^+e^-)	Sec. 2.3.8	Eq. (2.35)	12.4(1)	Ref. [8]
HVP LO (lattice, *udsc*)	Sec. 3.5.1	Eq. (3.49)	7116(184)	Refs. [9–17]
HLbL (phenomenology)	Sec. 4.9.4	Eq. (4.92)	92(19)	Refs. [18–30]
HLbL NLO (phenomenology)	Sec. 4.8	Eq. (4.91)	2(1)	Ref. [31]
HLbL (lattice, *uds*)	Sec. 5.7	Eq. (5.49)	79(35)	Ref. [32]
HLbL (phenomenology + lattice)	Sec. 8	Eq. (8.10)	90(17)	Refs. [18–30, 32]
QED	Sec. 6.5	Eq. (6.30)	116 584 718.931(104)	Refs. [33, 34]
Electroweak	Sec. 7.4	Eq. (7.16)	153.6(1.0)	Refs. [35, 36]
HVP (e^+e^-, LO + NLO + NNLO)	Sec. 8	Eq. (8.5)	6845(40)	Refs. [2–8]
HLbL (phenomenology + lattice + NLO)	Sec. 8	Eq. (8.11)	92(18)	Refs. [18–32]
Total SM Value	Sec. 8	Eq. (8.12)	116 591 810(43)	Refs. [2–8, 18–24, 31–36]
Difference: $\Delta a_\mu := a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$	Sec. 8	Eq. (8.14)	279(76)	

- SM deviates from E821 result by 3.7σ
- SM error dominated by Hadronic contribution
- E989 expected to reduce exp. uncertainty by 4
- To match E989 final uncertainty, 2‰ precision required for the Hadronic SM contribution
- Lattice has to improve by one order of magnitude!



See also Borsanyi et al. 20

The bottleneck: signal/noise ratio for HVP (HLbL,...)

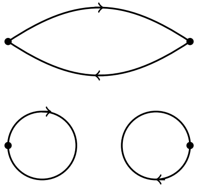
- The HVP contribution to $a_\mu = (g-2)_\mu/2$ reads

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0 K(x_0, m_\mu) G(x_0)$$

where

$$G(x_0) = \int d^3x \langle J_k^{\mathrm{em}}(x) J_k^{\mathrm{em}}(0)\rangle$$

with $K(x_0, m_\mu)$ being a known function

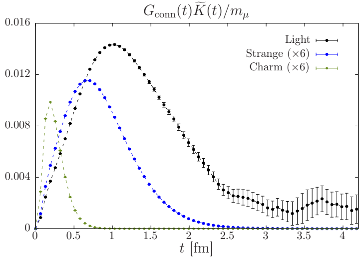


- For the light-connected contribution (by far the largest)

$$\frac{\sigma^2_{G^{\mathrm{conn}}_{u,d}}(x_0)}{[G^{\mathrm{conn}}_{u,d}(x_0)]^2} \propto \frac{1}{n_0}\, e^{2(M_\rho - M_\pi)|x_0|}$$

where M_ρ is the lightest state in that channel. Signal lost after 1.5-2.0 fm due to exponential increase of the statistical error

[Gérardin et al. 19]

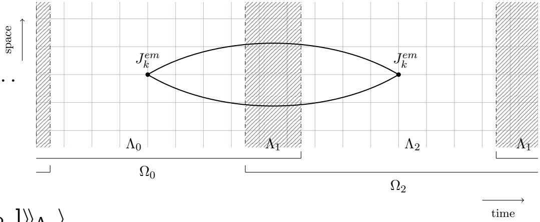


- New computational paradigm needed!

Multi-level integration

[Parisi, Petronzio, Rapuano 83; Lüscher, Weisz 01; ...; Meyer 02; LG, Della Morte 08 10, ...]

- If the action and the observable can be factorized



$$
\begin{aligned}
S[U] &= S_0[U_{\Omega_0}] + S_2[U_{\Omega_2}] + \ldots \\
O[U] &= O_0[U_{\Omega_0}] \times O_2[U_{\Omega_2}]
\end{aligned}
$$

then

$$\langle O[U] \rangle = \langle \langle\!\langle O_0[U_{\Omega_0}] \rangle\!\rangle_{\Lambda_0} \times \langle\!\langle O_2[U_{\Omega_2}] \rangle\!\rangle_{\Lambda_2} \rangle$$

where

$$\langle\!\langle O_0[U_{\Omega_0}] \rangle\!\rangle_{\Lambda_0} = \frac{1}{Z_{\Lambda_0}} \int \mathcal{D}U_{\Lambda_0}\, e^{-S_0[U_{\Omega_0}]}\, O_0[U_{\Omega_0}]$$

- Two-level integration:

  - n_0 configurations U_{Λ_1}
  - n_1 configurations U_{Λ_0} and U_{Λ_2} for each U_{Λ_1}

- If $\langle\!\langle \cdot \rangle\!\rangle_{\Lambda_i}$ can be computed efficiently with a statistical error comparable to its central value, then the prefactor in the signal/noise ratio changes as

$$n_0 \to n_0 n_1^2$$

at the cost of generating approximatively $n_0 n_1$ level-0 configurations

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- If the action and the observable can be factorized

$$S[U] = S_0[U_{\Omega_0}] + S_2[U_{\Omega_2}] + \ldots$$
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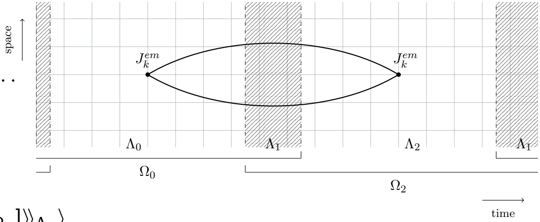

then

$$\langle O[U] \rangle = \langle \langle\!\langle O_0[U_{\Omega_0}] \rangle\!\rangle_{\Lambda_0} \times \langle\!\langle O_2[U_{\Omega_2}] \rangle\!\rangle_{\Lambda_2} \rangle$$

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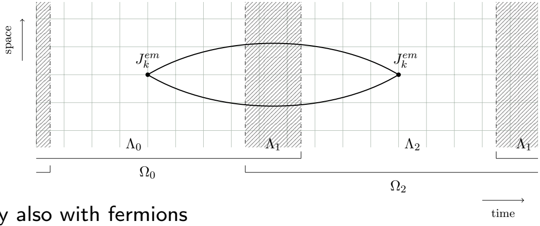
- With more active blocks, at the cost of approximatively $n_0 n_1$ level-0 configurations,

$$n_0 \to n_0 n_1^{n_{\rm block}}$$

and the gain increases exponentially with the distance since $n_{\rm block} \propto |y_0 - x_0|$. For the same relative accuracy of the correlator, the computational effort would then increase approximatively linearly with the distance

Multi-level integration with fermions

[Cè, LG, Schaefer 16; Dalla Brida, LG, Harris, Pepe 20]



- Thanks to
  - ∗ Overlapping Domain Decomp.
  - ∗ Multi-Boson representation

  multi-level integration is now a reality also with fermions

- The effective action (determinant of the Dirac operator) can be decomposed as

$$\det\ Q = \frac{\det(1-\omega)}{\det\ Q_{\Lambda_1}\ \det Q_{\Omega_0}^{-1}\ \det Q_{\Omega_2}^{-1}}$$

  and for 2 flavours, for instance, can be represented as

$$\{\det Q[U]\}^2 = \int \mathcal{D}\phi \ldots \exp\{-S_0[U_{\Omega_0},\ldots] - S_1[U_{\Lambda_1},\ldots] - S_2[U_{\Omega_2},\ldots]\}$$

- Factorization thanks to different representations of various quark-path contributions:
  - ∗ Pseudo-fermions for paths with no loops around Λ_1
  - ∗ Multi-Bosons for paths with 1–N loops (N is the number of Multi-Bosons)
  - ∗ Reweighting factor for paths with more than N loops

First multi-level computation of HVP

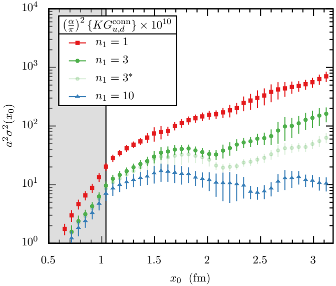
[Dalla Brida, LG, Harris, Pepe 20]

- Wilson glue with $O(a)$-improved Wilson quarks

$$\beta = 5.3\,, \quad (T/a)\times(L/a)^3 = 96\times 48^3$$

$$a = 0.065\ \mathrm{fm}\,, \quad M_\pi = 270\ \mathrm{MeV}$$

$$n_0 = 25\,, \qquad n_1 = 10$$



- Domain Decomposition adopted:

$$\Lambda_0 : x_0/a \in [0,39]\,, \quad \Lambda_1 : x_0/a \in [40,47]\cup[88,95]\,, \quad \Lambda_2 : x_0/a \in [48,87]$$

- Variance reduction due to 2-level integration expected to grow exponentially with x_0
- Indeed the sharp rise of the variance of $K(x_0, m_\mu)G^{\rm conn}_{u,d}(x_0)$ with the distance x_0 when computed by a standard 1-level integration (red points) is automatically flattened out by the two-level integration (blue-points) without the need for modeling the long-distance behaviour of $G^{\rm conn}_{u,d}$

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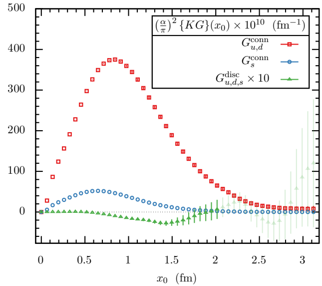
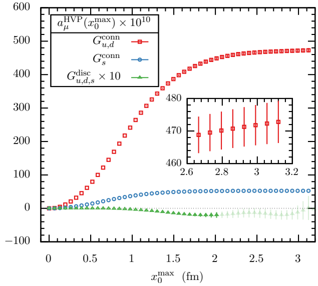
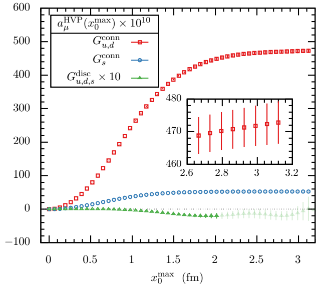


- With 2-level integration achieved 1% precision with just $n_0\cdot n_1 = 250$ configurations!
- The combined usage of split-even estimators and two-level integration solves the problem of the computation of the disconnected contribution
- With lighter quarks, the gain due to the 2-level integration is even more dramatic since $(M_\rho - M_\pi)$ increases significantly



Conclusions & Outlook

- Multi-level Monte Carlo integration accelerates the inverse scaling of the statistical error with the cost of the simulation
- In this study 1% precision was reached for $a_\mu^{\rm HVP}$ with just $n_0 \cdot n_1 = 250$ configurations. At this light-quark mass a 2‰ statistical precision is reachable by increasing n_0 and n_1 by a factor of about 4–6 and 2–4 respectively



- When the up and the down quarks becomes lighter, the gain due to the multi-level integration is expected to increase exponentially in the quark mass, hence improving even more dramatically the scaling of the cost with respect to a standard Monte Carlo
- The change of computational paradigm presented here thus removes the main barrier for making affordable, on computers available today, the goal of a 2‰ precision on $a_\mu^{\rm HVP}$
- Here we focused on the main bottleneck in the computation of the HVP. It goes without saying that the very same variance-reduction pattern is expected to work out also for the calibration of the lattice spacing, the calculation of electromagnetic corrections and HLbL