VII OFICINA NACIONAL DE TEORIA QUÂNTICA DE CAMPOS
/////////////////// VII NATIONAL WORKSHOP ON QUANTUM FIELD THEORY

Contribution ID: **57** Type: **Palestra/Seminário**

Towards hidden symmetries in gauge theories

Friday, 10 December 2021 09:30 (35 minutes)

The non-abelian generalization of the gauge symmetry proposed by C. N. Yang and R. Mills in 1954 was done à la Maxwell, i.e., in terms of a set of partial differential equations. However, the integral formulation counterpart of this generalization was not known until quite recently.

The critical problem in constructing the integral Yang-Mills equations is the need for a consistent definition of the flux of the non-abelian electric and magnetic fields with which we can build a relationship with the dynamically conserved charges in such a way that these charges are invariant under gauge transformations. Indeed, the naive definition of the flux of the non-abelian fields $\Phi(F) = \int_\Sigma F_{\mu\nu} \frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau} d\sigma d\tau$ is strongly dependent of the gauge choice since under a local gauge transformation $g(x)$, $F_{\mu\nu}(x) \to g(x)F_{\mu\nu}(x)g^{-1}(x)$ and therefore, the flux through a closed surface cannot be directly associated to gauge-invariant charges inside.

The problem of finding the gauge-invariant charges in non-abelian gauge theories is therefore linked to the problem of formulating the integral version of the Yang-Mills equations.

By scanning the $3+1$ dimensional Minkowski space-time with closed 2-dimensional surfaces based at a reference point x_R, which are in turn scanned by a family of homotopically equivalent loops based at x_R, it can be shown that the flux of the "conjugate field-strength" $F^W_{\mu\nu}(x) = W^{-1}F_{\mu\nu}(x)W$ through that closed surface, with W being the holonomy defined along a loop from x_R to x, will transform, under a local gauge transformation $g(x)$, as $\Phi \to g(x_R)\Phi g(x_R)^{-1}$, i.e., bringing the gauge group element to that defined at the reference point.

A relation between the flux of the conjugate field through the closed surface $\partial\Omega$ and quantities evaluated inside the volume Ω can be established and expanding this construction for the dual field strength $\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}F^{\sigma\rho}$, with the use of the (differential) Yang-Mills equations
\begin{eqnarray}
D_\mu F^{\mu\nu} &=& J^\nu_\textrm{e}\\
D_\mu \widetilde{F}^{\mu\nu} &=& J_\textrm{m}^\nu,
\end{eqnarray}
with $D_\mu\star = \partial_\mu \star + ie[A_\mu, \star]$ the covariant derivative, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu]$ the field strength and $J^\mu_{\rm e,m}$ the electric and magnetic currents, we obtain their integral formulation:
\begin{eqnarray}
\oint_{\partial\Omega}W^{-1}F_{\mu\nu}W\frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^\nu}{\partial \tau}d\sigma d\tau = \int_{\Omega}\epsilon_{\lambda\mu\nu\gamma}W^{-1}J_\textrm{m}^{\gamma}W\frac{\partial x^{\lambda}}{\partial \zeta}\frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^\nu}{\partial \tau} d\sigma d\tau d\zeta \nonumber\\
+\int_{\Omega}\int_0^\sigma [F^W_{\mu\nu}(\sigma),F^W_{\alpha\beta}(\sigma^\prime)]\bigg(\frac{\partial x^\beta}{\partial \zeta}(\sigma^\prime)\frac{\partial x^\nu}{\partial \tau}(\sigma)
- \frac{\partial x^\beta}{\partial \tau}(\sigma^\prime)\frac{\partial x^\nu}{\partial \zeta}(\sigma) \bigg)\frac{\partial x^\alpha}{\partial \sigma^\prime}\frac{\partial x^\mu}{\partial \sigma}d\sigma^\prime d\sigma d\tau d\zeta
\end{eqnarray}
\begin{eqnarray}
\oint_{\partial\Omega}W^{-1}\tilde{F}*{\mu\nu}W\frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^\nu}{\partial \tau}d\sigma d\tau = \int*{\Omega}\epsilon_{\lambda\mu\nu\gamma}W^{-1}J_\textrm{e}^{\gamma}W\frac{\partial x^{\lambda}}{\partial \zeta}\frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^\nu}{\partial \tau} d\sigma d\tau d\zeta \nonumber\\
+\int_{\Omega}\int_0^\sigma [\tilde{F}^W_{\mu\nu}(\sigma),F^W_{\alpha\beta}(\sigma^\prime)]\bigg(\frac{\partial x^\beta}{\partial \zeta}(\sigma^\prime)\frac{\partial x^\nu}{\partial \tau}(\sigma)
- \frac{\partial x^\beta}{\partial \tau}(\sigma^\prime)\frac{\partial x^\nu}{\partial \zeta}(\sigma) \bigg)\frac{\partial x^\alpha}{\partial \sigma^\prime}\frac{\partial x^\mu}{\partial \sigma}d\sigma^\prime d\sigma d\tau d\zeta.
\end{eqnarray}

In order to obtain the conserved charges, we consider the generalization of the holonomy operator by assigning to each loop parameterized by τ, scanning a closed 2-dimensional surface with base-point at x_R, the quantity $\mathcal{B} = \oint_\gamma W^{-1}B_{\mu\nu}W\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}d\sigma$ and define the 2-holonomy by the differential equation
\begin{equation}
\frac{dV}{d\tau}+ieV\mathcal{B} = 0,
\end{equation}

whose solution is the ordered series
\begin{equation}
V[\partial \Omega] = V_\circ\;P_2\;e^{-ie\oint W^{-1}B_{\mu\nu}W\frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^\nu}{\partial \tau}d\sigma d\tau}.
\end{equation}
This same operator can be obtained if we consider the 2-dimensional surface where it is calculated to be the result of continuous deformations from an infinitesimal surface at x_R. This leads to a definition of the 2-holonomy as the ordered series
\begin{equation}
V[\Omega] = P_3\;e^{ie\int_{0}^{2\pi}\mathcal{A}(\zeta)d\zeta}\;V_\circ
\end{equation}
with
\begin{eqnarray}
\mathcal{A} &=& \int_\Sigma VW^{-1}\left(D_\lambda B_{\mu\nu}+D_\mu B_{\nu\lambda}+D_\nu B_{\lambda \mu}\right)WV^{-1}\frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^\nu}{\partial \tau}\frac{\partial x^\lambda}{\partial \zeta}d\sigma d\tau\\
&+&ie\int_\Sigma V \int_{0}^{\sigma}\left[\mathcal{F}{\mu\nu}^W(\sigma'),B^W{\mu\nu}(\sigma)\right]\left(\frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^{\nu}}{\partial \zeta}\frac{\partial x^\alpha}{\partial \sigma'}\frac{\partial x^\beta}{\partial \tau} - \frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^{\nu}}{\partial \tau}\frac{\partial x^\alpha}{\partial \sigma'}\frac{\partial x^\beta}{\partial \zeta}\right)V^{-1}d\sigma d\tau
\end{eqnarray}
where $\mathcal{F}_{\mu\nu} = F_{\mu\nu} - B_{\mu\nu}$.

The fact that the operator V can be calculated in these two different but equivalent approaches lead us to the identity
\begin{equation}
P_3\;e^{ie\int_{0}^{2\pi}\mathcal{A}(\zeta)d\zeta}= P_2\;e^{-ie\oint W^{-1}B_{\mu\nu}W\frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^\nu}{\partial \tau}d\sigma d\tau}.
\end{equation}

For $B_{\mu\nu} = \alpha F_{\mu\nu} + \beta\widetilde{F}_{\mu\nu}$, the above equation, which is the non-abelian Stokes theorem, leads to the integral Yang-Mills equations.

Two given closed surfaces in space-time can be regarded as points in the loop space $L^2\Omega$ and the volume between them will define a path in this space.

A consequence of the integral Yang-Mills equations is that the operator $V[\Omega]$ is path-independent in $L^2\Omega$, i.e., it does not change under a reparameterization of the volume enclosed by $\partial\Omega$.
By appropriately splitting space-time into space and time one can then show that V evolves from a $t = 0$ volume Ω_0 to a $t > 0$ volume Ω_t as
\begin{equation}
V[\Omega_t] = UV[\Omega_0]U^{-1},
\end{equation}
i.e., it undergoes a unitary tranformation, thus preserving its eigenvalues which can be identified with the conserved charges.

The integral Yang-Mills equations can be regarded as a zero-curvature equation in the loop space $L^2\Omega$ and the conserved charges are a consequence of the hidden gauge symmetry there.

Primary author: Prof. LUCHINI, Gabriel (Universidade Federal do Espírito Santo)

Co-author: Prof. FERREIRA, Luiz Agostinho (Instituto de Física de São Carlos / USP)

Presenter: Prof. LUCHINI, Gabriel (Universidade Federal do Espírito Santo)