Metrics from Machine Learning

Moduli-dependent Calabi-Yau and SU(3)-structure metrics from machine learning

Anderson, Gerdes, Gray, Krippendorf, Raghuram, Ruehle; `arXiv:2012.04656`

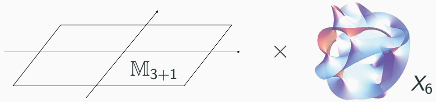
Mathis Gerdes

27 August 2021
SUSY 2021

Calabi-Yau Manifolds - Compactification

Why Calabi-Yau manifolds?
(Compact, complex space with $c_1(X) = 0 \leftrightarrow$ Ricci-flat Kähler metric)

- String theory vacuum configuration $M_{10} = \mathbb{M}_{3+1} \times X_6$
- Compactification with CY manifold gives 4D $\mathcal{N} = 1$ SUSY, hope to reproduce low energy physics (Candelas, Horowitz, Strominger, Witten 1985)



Why do we want the metric?

- Know CY metric exists but no analytic expression (Yau 1978)
- Many quantities without explicit metric using algebraic geometry
- E.g. massive spectrum, D3-brane inflation **do** need metric
  $\longrightarrow$ existing numerical & new ML approaches

Outline

Calabi-Yau Metric

Numerical Approaches

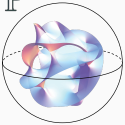
Machine Learning

Machine Learning of the Metric

Calabi-Yau Metric

Quintic Example

We consider a one-parameter family of hypersurfaces in $\mathbb{P}^4$:



$$p_\psi(z) = \sum_{i=0}^{4} \left(z^i\right)^5 + \psi \prod_{i=0}^{4} z^i = 0$$

Know CY metric exists, now seek numerical approximations.

- Kähler: Kähler form $J = i\, g_{i\bar{\jmath}}\, dz^i \wedge d\bar{z}^{\bar{\jmath}}$ is closed:
  $dJ = 0 \leftrightarrow$ locally have Kähler function K s.t. $g_{i\bar{\jmath}} = \partial_i \overline{\partial}_{\bar{\jmath}} K$
- Overlaps: If defined on patches, must match on overlaps
- Ricci-flat: The Ricci curvature vanishes,

$$R_{i\bar{\jmath}} = -\partial_i \overline{\partial}_{\bar{\jmath}} \log \det g \overset{!}{=} 0$$

Fourth order PDE in terms of the Kähler potential!

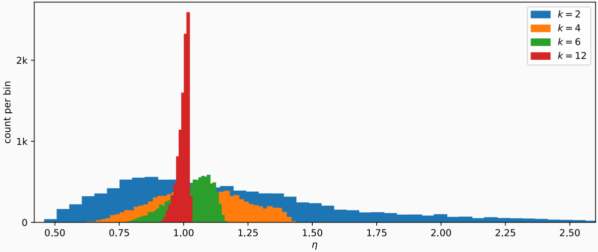
CY Accuracy Measure

Two constructions for top (volume) form:

- Holomorphic $(3,0)$ form Ω: $\mathrm{vol}_\Omega = \Omega \wedge \bar{\Omega}$
- Kähler form J for metric g: $\mathrm{vol}_g = J \wedge J \wedge J \propto \det g$
- For Ricci-flat CY metric, must be proportional: $J \wedge J \wedge J = \kappa\, \Omega \wedge \bar{\Omega}$

Accuracy measure $\sigma = \dfrac{1}{\mathrm{vol}(X)} \displaystyle\int_X \left|1 - \frac{\eta}{\kappa}\right|$ with $\eta = \dfrac{\mathrm{vol}_g}{\mathrm{vol}_\Omega} \overset{\text{CY}}{=} \kappa = \text{const}$

Integral is convex in J! (Headrick, Nassar [0908.2635])



Numerical Approaches

Local vs Spectral Methods

Finite Differences on K3 twofold (Headrick, Wiseman [hep-th/0506129])

- Using $J = \partial\bar{\partial}K$ in $J^3 = \kappa\,\Omega \wedge \bar{\Omega}$ gives a Monge-Ampère equation (PDE linear in determinant of Hessian of K)
- Applying relaxation method to grid approximation of K
- Curse of dimensionality for higher dimensions & need explicit patches!

Spectral method based on Donaldson's work [math/0512625]

- Algebraic expression for K parametrized by a Hermitian matrix H_k
- Parameter k is the cut-off of a spectral expansion
- Donaldson's algorithm obtains the *balanced metric* which satisfies

$$H_k = T(H_k)^{-1}$$

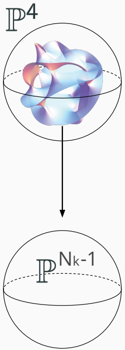
- Obtain balanced metrics through iteration of T for fixed degree k
- Balanced metrics $\xrightarrow{k\to\infty}$ CY metric

Algebraic Kähler Potential

- Have Fubini-Study metric from $\mathbb{P}^4$: $K_{FS} = \log \sum_i \left|z^i\right|^2$
- Generalize by replacing z^i with basis of homogeneous polynomials $s^\alpha(z)$ of degree k and inserting a Hermitian matrix H:

$$K_h = \log \sum_{\alpha\bar{\beta}} s^\alpha(z)\, H_{\alpha\bar{\beta}}\, \bar{s}^{\bar{\beta}}(\bar{z})$$

- Geometric interpretation: H parametrizes embedding of X into $\mathbb{P}^{N_k - 1}$
- Ansatz is the pullback of the FS metric from $\mathbb{P}^{N_k - 1}$
- The s^α must form basis of sections of $\mathcal{O}_X(k)$; reduction from polynomial basis on $\mathbb{P}^4$ given $p_\psi(z) = 0$ not unique
- Algebraic construction always satisfies overlap and Kählerity constraints!

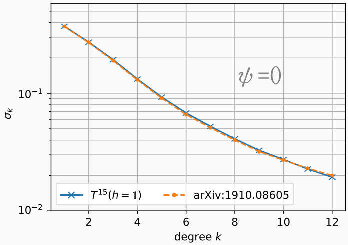


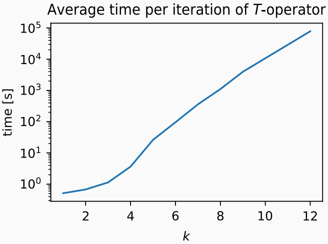
Donaldson's Algorithm

- Donaldson's algorithm iteratively applies T-operator

$$H_{\alpha\bar{\beta}} \quad \mapsto \quad T(H)^{\alpha\bar{\beta}} = \frac{N_k}{\mathrm{vol}(X)} \int_X \frac{s^\alpha \, \bar{s}^{\bar{\beta}}}{s^\gamma \, H_{\gamma\bar{\delta}} \, \bar{s}^{\bar{\delta}}} \, d\mathrm{vol}$$

- Balanced metrics converge to CY metric like $O(k^{-2})$
- If we optimize σ accuracy, expect in general exponential convergence (Headrick, Nassar [0908.2635])
- Computational cost of algorithm grows like $O(k^{16})$
- Must evaluate for each new set of moduli

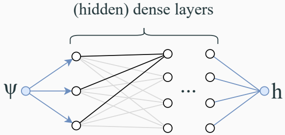


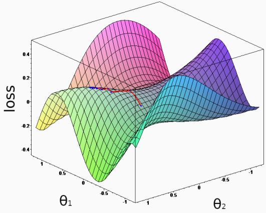


Machine Learning

Deep Learning

- Parametrized *network*: $(\theta, x) \mapsto \hat{y}$
- Often *deep* chain of linear combinations and non-linear *activation functions*
- *Learn* parameters θ by following gradient of some loss (energy) function, e.g. $\mathcal{L}(\theta, x) = |y(x) - \hat{y}(\theta, x)|^2$
- Automatic differentiation: For any $f(\alpha, \beta)$, can programatically get $\partial_\alpha f$
- We need to work with complex variables, holomorphic derivatives
- Our loss contains $\eta \propto \det \partial\bar{\partial} K$
- We used PyTorch, Tensorflow and JAX

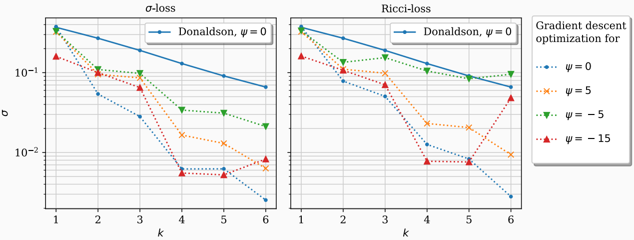




Machine Learning of the Metric

Existing ML Approaches

- Balanced metrics converge slowly: minimize σ for algebraic ansatz (H) at fixed moduli (Headrick, Nassar [0908.2635])



- Donaldson's algorithm is expensive: Extrapolate results to larger k (Ashmore, He, Ovrut [1910.08605])
- Reduce # parameters & faster numerics: Replace polynomial in algebraic ansatz $K_h = \log \sum_{\alpha\bar{\beta}} s^\alpha H_{\alpha\bar{\beta}} \bar{s}^{\bar{\beta}}$ with a network (Douglas, Lakshminarasimhan, Qi [2012.04797])

$$(z, \bar{z}) \xmapsto{\text{net}} s^\alpha(z) \, H_{\alpha\bar{\beta}} \, \bar{s}^{\bar{\beta}}(\bar{z})$$

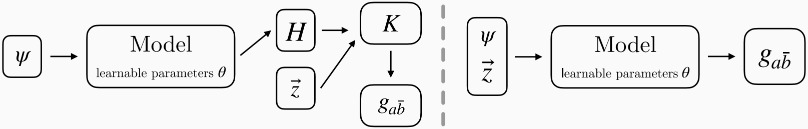
New Approaches

Moduli-dependent approximation of algebraic metrics

- Interpolate parameters H from Donaldson's algorithm between values of ψ (at a fixed degree k)
- Optimize networks $\psi \mapsto H$ **without** training data from Donaldson's algorithm

Networks to predict the metric directly

- Optimize networks for CY metric: $(\psi, z) \mapsto g_\psi(z)$
- No longer automatically Kähler & satisfy overlap condition
- Advantage: also works for metrics with SU(3)-structure ($dJ \neq 0$)

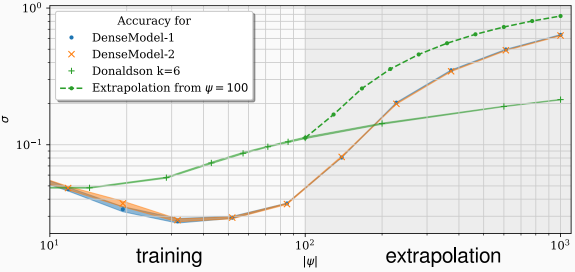


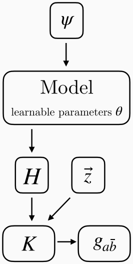
Moduli Dependent Algebraic Metric

- In all approaches used the Monge Ampère loss which is the Monte Carlo approximation of the σ accuracy:

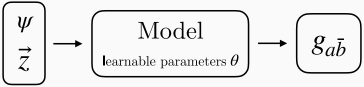
$$\mathcal{L}_{MA} = \frac{1}{N} \sum_{n=1}^{N} \left| \frac{\eta(z_n)}{\kappa} - 1 \right|^2 w(z_n)$$

- Weights $w(z_n)$ known if we sample points z as intersections of X with a random line in $\mathbb{P}^4$ [hep-th/0612075]
- Instead of $|\dots|^2$ could use any convex function

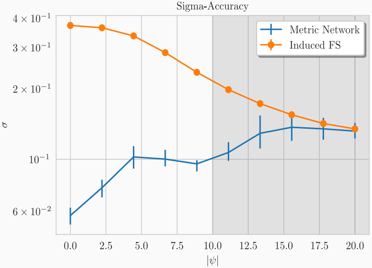


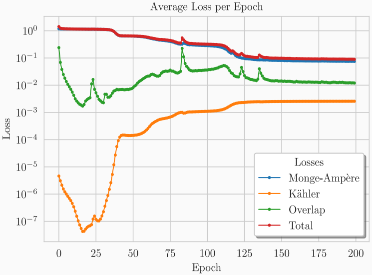


Network for Metric Directly



- Similar accuracy as Donaldson's balanced metric at same degree but lower numerical cost
- Need to include all losses: $\mathcal{L} = \lambda_1 \mathcal{L}_{MA} + \lambda_2 \mathcal{L}_{dJ} + \lambda_3 \mathcal{L}_{\text{overlap}}$
- Use perturbation around FS metric as ansatz: $g = g_{FS}\,(1 + g_{NN})$
- Weighing of different losses not obvious, application dependent
- Very general approach: non-Kähler, better parametrizations possible





Summary and Future Work

- Moduli-dependent networks which approximate the metric simultaneously for a range of values
  - Order of minutes where Donaldson's algorithm would take tens of hours (for same σ accuracy)
  - Standalone, does not need existing results for training
- Networks which predict the metric directly
  - General approach, can apply to SU(3)-structure
- NN and ML frameworks well suited for problem
- Never explicitly exploited symmetry, but application to more moduli work for the future
- Find better basis of sections (basis for H_k), or replace with networks for polynomial
- Apply to physics questions

Thank You