Counting BPS states with Exponential Networks

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Introduction

The main subject of this talk is the problem of counting BPS states in M-theory compactifications on local Calabi-Yau threefolds. Joint work with **S. Banerjee** and **M. Romo**.

[Banerjee L Romo - 1811.02875, 1910.05296, 2012.09769] also see [L - 2101.01681] [Del Monte L - 2107.14255]

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Goal: given a local threefold $X \times S^1 \times \mathbb{R}^4$, we wish to determine the spectrum of M2 branes on $\mathcal{C}_2 \times \mathbb{R}$, of M5 branes on $\mathcal{C}_4 \times S^1 \times \mathbb{R}$, and of their boundstates.

Math motivations: a new way to computate enumerative (category-theoretic) invariants.

Physics approach: involves supersymmetric QFTs in various dimensions, coupled to each other.

Introduction

This question belongs to a class of problems with universal features:

A moduli space of **stability conditions**: $u \in \mathcal{B}$ defines the notion of stable BPS states.

For a generic choice of $u \in \mathcal{B}$ the **BPS spectrum** is characterized by

- The charge γ of a BPS state is valued in $\Gamma \simeq \mathbb{Z}^k$
- The Dirac pairing of two states is a skew-symmetric bilinear form $\langle \cdot , \cdot \rangle : \Gamma \times \Gamma \to \mathbb{Z}$
- Physical properties (mass, supercharges) of a BPS state are encoded by $Z_\gamma \in \mathbb{C}$
- BPS states are 'counted' by BPS invariants $\Omega(\gamma) \in \mathbb{Z}$.

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Examples:

	$D^b Coh(X)$	$4d\ \mathcal{N} = 2$ QFT	$\cdots$
$\mathcal{B}$	Kähler moduli	Coulomb branch	
Γ	$H^\bullet_{cpt}(X)$	$H_1(\Sigma, \mathbb{Z})$	$\cdots$
$\Omega(\gamma)$	DT invariants	BPS indices	

Introduction – enumerative invariants and spectral networks

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- computation of DT invariants of certain Fukaya categories
- the study of BPS states in class $\mathcal{S}$ theories

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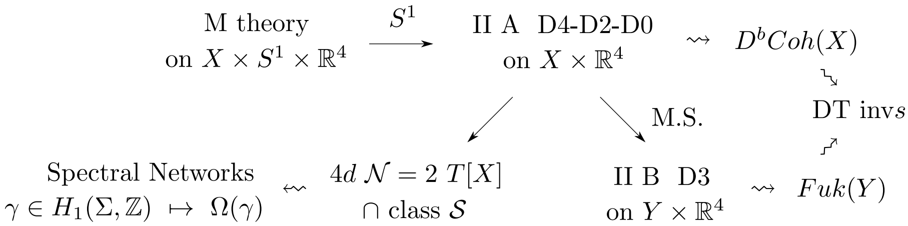
Indeed, M-theory on local CY3 provides a natural home for both



These two frameworks naturally emerge together in our approach to the main problem: computing $\Omega(\gamma, u)$ for M2 & M5 branes in of $X \times S^1 \times \mathbb{R}^4$.

Introduction – defects, mirror curves, BPS cycles

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Approach

Choose L a noncompact Lagrangian in X (for concreteness a toric L [Aganagic Vafa]).
Engineer a defect by introducing a single M5 brane on $L \times S^1 \times \mathbb{R}^2$.

The moduli space of M5 on L, after quantum corrections by holomorphic disks

$$\Sigma : \quad F(x, y) = 0 \quad \subset \quad \mathbb{C}^* \times \mathbb{C}^*$$

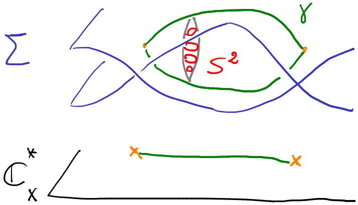
This curve will play a central role, we'll **compute** $\Omega(\gamma, u)$ **from its geometry**, in the spirit of spectral networks.

A key step in this direction is due to [Klemm Lerche Mayr Vafa Warner].

First, note that Σ is the **mirror curve** of X [Aganagic Vafa, Aganagic Klemm Vafa, Aganagic Ekholm Ng Vafa]

$$Y \; : \quad uv = F(x,y) \quad \subset \quad \mathbb{C}^2 \times (\mathbb{C}^*)^2$$

Second, BPS states map to D3 on compact sLags $\mathcal{L}_3 \subset Y$, S^2-fibred over arcs in the x-plane



The central charge reduces to periods of a differential on Σ

$$\lambda = \log y \, d \log x \quad \longrightarrow \quad Z_\gamma = \frac{1}{2\pi R} \oint_\gamma \lambda$$

Introduction – the counting problem

The original problem is thus mapped to

$$\begin{aligned}
& M5\text{-}M2 \text{ in } X \times S^1 \times \mathbb{R}^4 \\
\xrightarrow{R\to 0}\ & \text{Type IIA } D4\text{-}D2\text{-}D0 \text{ in } X \times \mathbb{R}^4 \\
\xrightarrow{\text{mirror}}\ & \text{Type IIB } D3 \text{ on calibrated } \mathcal{L}_3 \subset Y \quad \rightsquigarrow \quad \mathcal{F}uk(Y) \\
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This is where lessons from **spectral networks** become useful:

- [Gaiotto Moore Neitzke] solved a similar problem for Hitchin spectral curves
- our setup is different, but underlying physics ideas can be generalized to provide guidance

Introduction – 3d vacua and 3d BPS states

The main input from physics is a different **perspective on** $\Sigma \,:\, F(x,y) = 0$.

IR dynamics of M5 on $L \times S^1 \times \mathbb{R}^2$ is **described by a QFT** $T_{3d}[L]$ (3d $\mathcal{N}=2$ $U(1)$ GLSM)

- $\log x \sim t_{FI}$ is a FI coupling
- $\log y \sim \sigma + \frac{i}{2\pi R}\oint A_3$ is a field (the complexified scalar in the $U(1)$ v.m.)

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But $\lambda=\log y\, d\log x$ is **multi-valued** on Σ. Therefore, so is Z_a!
Physical properties of the BPS states of $T_{3d}[L]$ are really defined on a $\mathbb{Z}$**-covering**

$$\tilde{\Sigma}\xrightarrow{\mathbb{Z}}\Sigma$$

Introduction – 3d vacua and 3d BPS states

On $\tilde{\Sigma}$ vacua of $T_{3d}[L]$ are promoted to towers of points

$$y_j(x) \in \Sigma \quad \rightarrow \quad (j, M) := \log y_j(x) + 2\pi i M \in \tilde{\Sigma}$$

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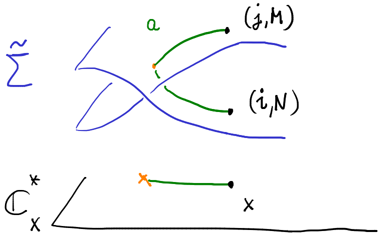

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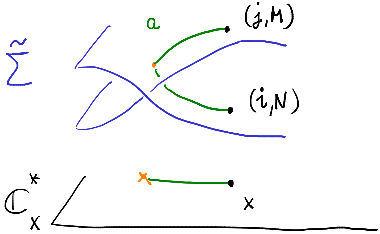

The calibrating BPS equations of an $(i, N) \to (j, M)$ path are [KLMVW] [Eager Selmani Walcher]

$$(\log y_j - \log y_i + 2\pi i(M - N)) \frac{d \log x}{d\tau} = e^{i\vartheta} \qquad (\vartheta = \arg Z_a)$$

These define arcs in $\mathbb{C}^*_x$ which lift to $a \subset \tilde{\Sigma}$.

Physics defines **counts** of these BPS states $\mu(a) \in \mathbb{Q}$ (a 3d lift of [Cecotti Fendley Intriligator Vafa])

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Strikingly, the 3d BPS spectrum $\mu(a)$ encodes BPS invariants $\Omega(\gamma)$ of compact 1-cycles.
This follows from lifting 2d-4d wall-crossing of [Gaiotto Moore Neitzke] to 3d-5d systems on S^1.

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Bringing this to fruition requires

- Some way of computing $\mu(a)$ of 3d BPS states
- Some way of extracting $\Omega(\gamma)$ from $\mu(a)$

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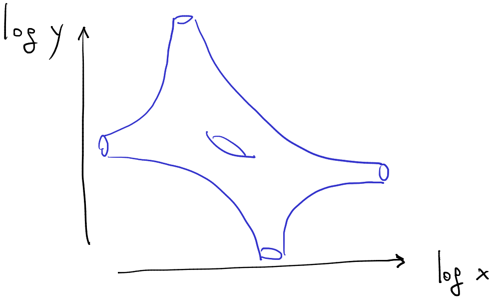
I will describe a construction known as **nonabelianization** for **exponential networks**.
This is $\sim$ a topological redux of 3d tt^* geometry [Cecotti Vafa] [Dubrovin] [CV+Neitzke] [CV+Gaiotto].

Exponential Networks

Consider an algebraic curve Σ defined by $F(x, y) = 0$ in $\mathbb{C}^* \times \mathbb{C}^*$



Exponential Networks – covering layers

We'll view Σ as a ramified covering of the $\mathbb{C}^*$ x-plane with sheets sheets $y_j(x)$, $j = 1, \ldots, d$
Branch points of this ramification structure will be marked by $\times$, branch cuts by $\sim$ on $\mathbb{C}^*$.

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We also consider the differential

$$\lambda = \log y \; d \log x$$

It is multi-valued on Σ, but single-valued on $\tilde{\Sigma} \longrightarrow \Sigma$ with sheets $(j, N) \equiv \log y_j + 2\pi i \, N$.
There will be logarithmic branch cuts on Σ, denoted by $\overset{i}{\text{- - - - -}}$.

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Overall

$$\begin{array}{ccccc} \tilde{\Sigma} & \longrightarrow & \Sigma & \longrightarrow & \mathbb{C}^*_x \\ (j, N) & \mapsto & y_j(x) & \mapsto & x \end{array}$$

Exponential Networks – definitions

$\mathcal{W}(\vartheta)$ is a network of trajectories on $\mathbb{C}^*$ defined by solutions of

$$(\log y_j(x) - \log y_i(x) + 2\pi i\, n)\frac{d\log x}{d\tau} = e^{i\vartheta}$$

parameterized by $\tau \in \mathbb{R}$, and labeled by (ij, n) for some $n \in \mathbb{Z}$ [Eager Selmani Walcher].

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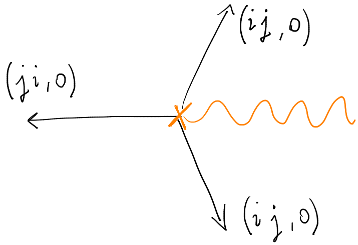
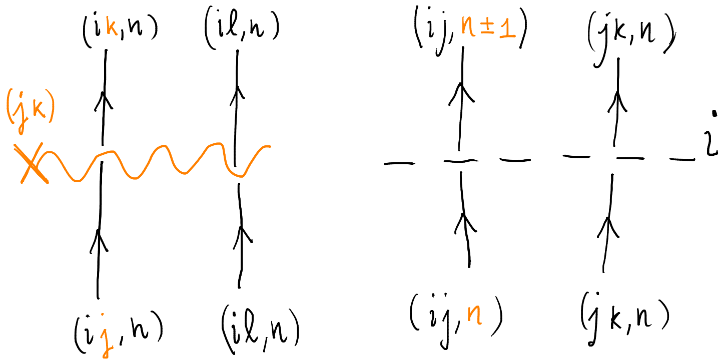
Boundary conditions: trajectories start from branch points × with $y_i = y_j$, and $n = 0$:

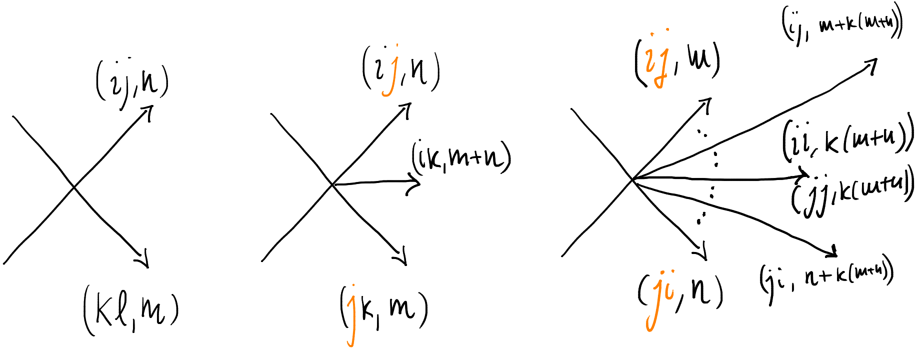


As a trajectory evolves in τ, it may cross branch cuts. This may change labeling



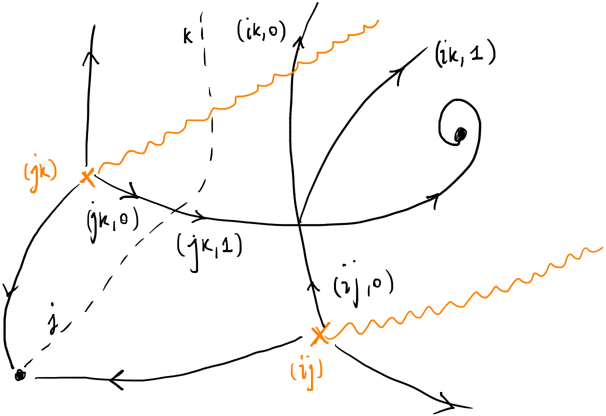
Exponential Networks – definitions

Trajectories may also intersect transversely. New ones may be generated by these rules



Exponential Networks – definitions

Globally, a network $\mathcal{W}(\vartheta)$ is a collection of trajectories and their interactions. As $\tau \to +\infty$ all trajectories end up into punctures, for generic ϑ.



Exponential Networks – definitions

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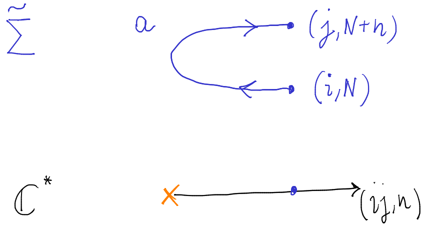
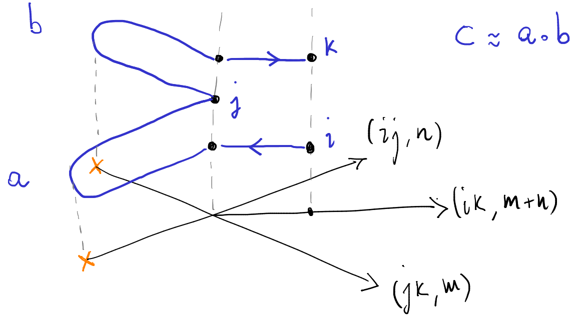
1. For trajectories sourced by branch points, $\mu(a) = 1$ for the class of "simplest lifts" to $\tilde{\Sigma}$



2. For trajectories sourced at intersections, $\mu(a)$ on the newborn trajectories is fixed by combinatorics of concatenations of incoming ones.

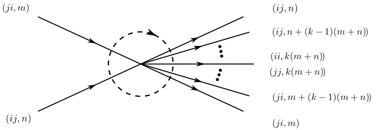
Example: (ij, n)-(jk, m) intersections.

- Incoming data: $\mu(a)$ for $a \in \Gamma_{(ij,n)}$ and $\mu(b)$ for $b \in \Gamma_{(jk,m)}$.
- Outgoing data: $\mu(c) = \sum_{ab \simeq c} \pm\mu(a)\mu(b)$ for all concatenating a, b in class $c \in \Gamma_{(ik,m+n)}$



Exponential Networks – nonabelianization

(ij, n)-(ji, m) intersections are more involved, but soliton data on all descendant trajectories is again fixed.



$$\Theta := \sum_N \sum_{\substack{a:|i,N\rangle \to |j,N+n\rangle \\ b:|i,N+n\rangle \to |j,N+n+m\rangle}} \mu(a)\mu(b) X_{a\circ b} \ , \qquad \bar{\Theta} := \sum_N \sum_{\substack{a:|i,N+m\rangle \to |j,N+n+m\rangle \\ b:|i,N\rangle \to |j,N+m\rangle}} \mu(a)\mu(b) X_{b\circ a}$$

$$\mathcal{E}'_{ii} = e^{\sum_{k\geq 1} \frac{(-1)^{1+k}}{k} \Theta^k} \qquad \mathcal{E}'_{ij,n+k(n+m)} = \exp\left(\sum_N \sum_{a:|i,N\rangle \to |i,N+n\rangle} \mu(a) X_a \cdot (-\bar{\Theta})^k \right)$$

$$\mathcal{E}'_{jj} = e^{\sum_{k\geq 1} \frac{(-1)^{k}}{k} \bar{\Theta}^k} \qquad \mathcal{E}'_{ji,n+k(n+m)} = \exp\left(\sum_N \sum_{b:|j,N\rangle \to |i,N+m\rangle} \mu(b) X_b \cdot (-\Theta)^k \right)$$

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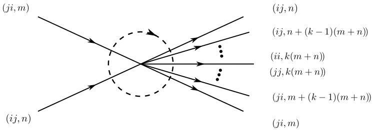

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The definition of exponential networks $\mathcal{W}(\vartheta)$ with soliton data $\mu(a)$ on trajectories is complete.

[Eager Selmani Walcher] [Banerjee L Romo]

Exponential Networks – dependence on ϑ

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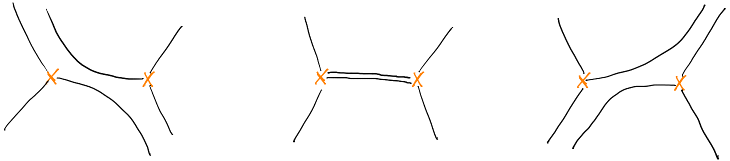

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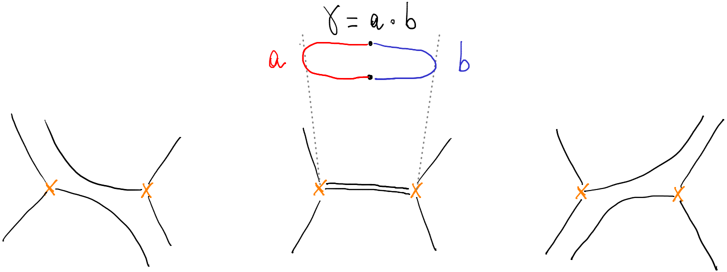

Saddles on $\mathbb{C}^*$ lift to closed cycles on $\tilde{\Sigma}$.

Exponential Networks – dependence on ϑ

Claim: combinatorics of $\mu(a)$ counting **open paths** encode $\Omega(\gamma)$ of **closed** BPS cycles.
Invariance of certain generating series $F = \sum_a \mu(a) X_a$, where $\mu(a)$ jump in a computable way, implies that X_a must jump by Kontsevich-Soibelman morphisms encoding $\Omega(\gamma)$

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The full BPS spectrum can be obtained this way

- Given Σ fixes $u \in \mathcal{B}$ via BPS central charges via $Z_\gamma \sim \oint_\gamma \lambda$.
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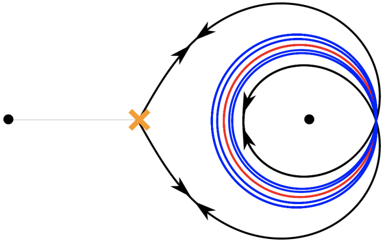
Results

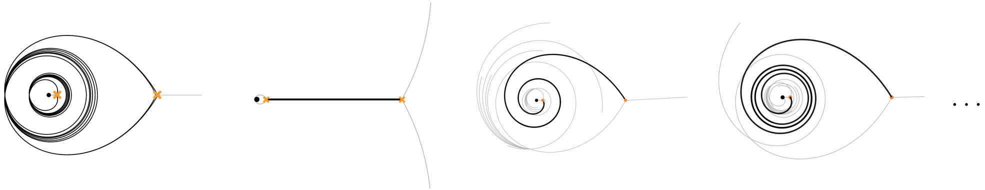
For $\mathbb{C}^3$ with $F = 1 + y + xy^2$ we find BPS saddles



with $Z_\gamma = k\frac{2\pi}{R}$ and $\Omega(\gamma) = -1$, corresponding to k D0 branes with $k \in \mathbb{Z}$.
[Banerjee L Romo – 1811.02875]

Results

For $\mathcal{O}(-1)^2 \to \mathbb{P}^1$ with $F = 1 + y + xy + Qxy^2$ we find BPS saddles



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- $Z_\gamma = k\frac{2\pi}{R} - \frac{i}{R}\log Q$ and $\Omega(\gamma) = 1$, corresponding to D2 bound to k D0's, for $k \in \mathbb{Z}$

[Banerjee L Romo – 1910.05296]

Conclusions

- Further results for $\mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbb{P}^1$ and $K_{\mathbb{F}_0}$ match, and extend, known results for (rank-0) DT invariants of $D^b Coh(X) \simeq \mathcal{F}uk(Y)$. [Banerjee L Romo – 1910.05296, 2012.09769]

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