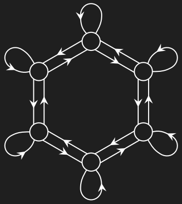
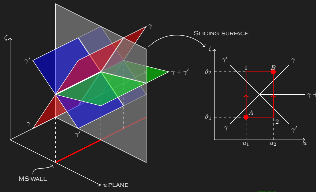
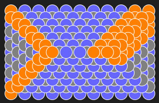
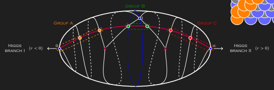








Quiver Yangians and Representations from BPS Crystals

Dmitry Galakhov

Kavli IPMU Institute for the Physics and Mathematics of the Universe

Strings and Fields 2021

Aug 26, 2021

Based on:

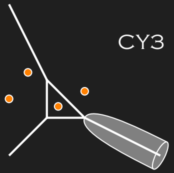
- D.G. 1812.05801
- Wei Li and Masahito Yamazaki 2003.08909
- D.G. and Masahito Yamazaki 2008.07006
- D.G., Wei Li and Masahito Yamazaki 2106.01230
- D.G., Wei Li and Masahito Yamazaki 2108.10286

Toric CY3 and Physics

UV

Type II

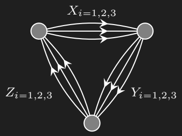
D6/D4/D2/D0 $\longrightarrow$



4d $\mathcal{N} = 1$ SYM

$\downarrow$

1d $\mathcal{N} = 4$ QM



$+ \quad W = \epsilon^{ijk}\,\mathrm{Tr}\, Z_i Y_j X_k$

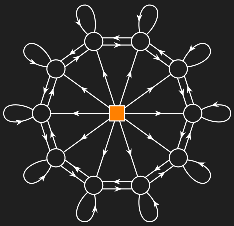
IR

Mathematics
Enumerative Geometry
New Algebras & Reps

Phisics
Non-perturbative BPS Physics
Integrability

[Nakajima; Kontsevich, Soibelman; Alday, Gaiotto, Tachikawa; Douglass, Moore; Schifman, Vasserot, ...]

Quiver Quantum Mechanics



d_a ○ $\rightsquigarrow$ $U(d_a)$ VECTOR MULTIPLET:

ζ_{FI} $\qquad A_0,\ X_3,\ \phi,\ \lambda_\alpha,\ \mathbf{D}$

d_a ○→○ d_b $\rightsquigarrow$ $U(d_b)\times\overline{U(d_a)}$ CHIRAL MULTIPLET:

$\mu_{\mathbb{C}} = \mu_{\mathbb{R}} + \mathrm{i}\mu_{\mathbb{I}}$ $\qquad q,\ \psi_\alpha,\ \mathbf{F}$

n_f ■ $\rightsquigarrow$ $U(n_f)$ FLAVOR SYMMETRY:

$$\phi_f = \mathrm{diag}\left(\mu_1,\ldots,\mu_{n_f}\right)$$

Q_0 – QUIVER VERTICES, Q_1 – QUIVER ARROWS, Q_2 – SUPERPOTENTIAL

$\{a\to b\}$ – A SET OF ARROWS FROM a TO b

BPS STATES = GAUGE INVARIANT GROUND STATES:

$$\mathscr{H}_{\mathrm{BPS}} \subset \mathscr{H}$$

Localization in Quiver Quantum Mechanics

[Denef '02]

[Witten'82, Gaiotto-Moore-Witten'15,...]

Localization:

$$\psi_i \rightsquigarrow dx^i, \quad \psi_i^\dagger \rightsquigarrow \iota_{\partial/\partial x^i}, \quad \mathcal{Q}_\alpha,\ \bar{\mathcal{Q}}_{\dot\alpha} \rightsquigarrow \text{differentials}, \quad \mathcal{H} \rightsquigarrow \text{Laplacian}$$

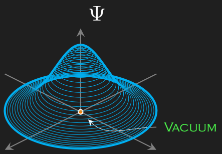
de Rham Dolbeault equivariant sup. twist

$$\mathcal{Q} = e^{-h}\left(d + \bar\partial + \iota_V + dW\wedge\right) e^{h}$$

Morse height function

$$\mathscr{H}_{\text{BPS}} = H^*_G\left(\text{Target Space}, \mathcal{Q}\right) \approx \bigoplus_{p\in\mathcal{I}} \mathbb{C}\Psi_p$$

$$\mathcal{I} = \{\text{crit. fixed points}\} = \{\text{classical vacua}\}$$



Fixed Points

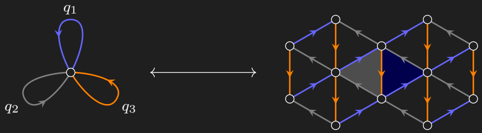
D-term + F-term:

$$\sum_{x \in Q_0} \sum_{I \in \{a \to x\}} q_I {q_I}^\dagger - \sum_{y \in Q_0} \sum_{J \in \{y \to a\}} q_J^\dagger q_J = \zeta_a \mathrm{Id}_{d_a \times d_a}, \quad \forall\ a \in Q_0;$$

$$\Phi_b q_I - q_I \Phi_a - \mu_I q_I = 0, \quad \forall\ a, b \in Q_0,\ I \in \{a \to b\};$$

$$\partial_{q_I} W = 0, \quad \forall\ I \in Q_1.$$

Periodic quiver:



$$W = \sum_{\text{faces}} (-1)^{\text{ori}}\, \mathrm{Tr} \prod_{\text{loop}} q = \triangle - \triangle = \mathrm{Tr}\ q_1 [q_2, q_3]$$

Constraints on flavor charges (masses):

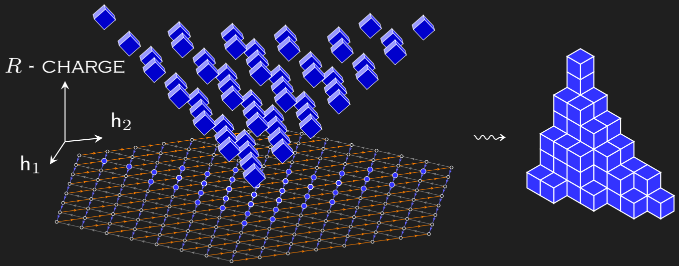
$$\left.\begin{array}{lll} \text{Loop:} & \sum_{\text{loop}} \mu_I = 0, & \forall \text{faces}; \\ \text{Vertex:} & \mu_I \sim \mu_I - \epsilon_a + \epsilon_b, & \forall I \in \{a \to b\}. \end{array}\right\} \mu_I = x_I \mathsf{h}_1 + y_I \mathsf{h}_2.$$

Equivariant toric action on CY3:

$$(z_1, z_2, z_3) \longmapsto \left(e^{\mathsf{h}_1} z_1, e^{\mathsf{h}_2} z_2, e^{-\mathsf{h}_1 - \mathsf{h}_2} z_3\right)$$

Crystals

Quiver path algebra: $\mathbb{C}Q/\langle dW\rangle \rightsquigarrow \prod q$ – "Baryons"
Crystal = Possible baryons:



$\square$ – atom of a crystal

Color of $\square$ denoted $\hat{\square} \in Q_0$ is a color of atom projection to $(\mathsf{h}_1, \mathsf{h}_2)$

Melting rule: K – molten crystal

For any atom $\square$ such that $I \cdot \square \in \mathrm{K}$ for some arrow I,
then $\square$ is also contained in K

[Szendroi; Mozgovoy, Reyneke; Nagao, Nakajima; Ooguri-Yamazaki; Jafferis, Chuang, Moore; Sulkowski; Aganagic, Schaeffer; Aganagic, Vafa; ...]

Euler classes

Quiver representation in Crystal basis:

$$V^a = \bigoplus_{\square \in \mathrm{K},\, \hat{\square}=a} \mathbb{C}|\square\rangle,\ a \in Q_0,\quad \langle q_I\rangle|\square\rangle = |I\cdot\square\rangle$$

Fixed point structure:

$$\underbrace{\text{Fixed Point K}}_{\langle\text{Baryons}\rangle} + \underbrace{(\text{Tangent Space}/G)}_{\text{Mesons}}$$

Meson space:

$$\mathcal{M}_{\text{meson}} = \text{Span}\,\{q_\alpha, \mu_\alpha\}_{\alpha=1}^{N_{\text{meson}}}$$

IR meson flavor charges:

$$\mu\left(\delta q_{I\in\{x\to y\}}\right) = h_y - h_x - \mu_I$$

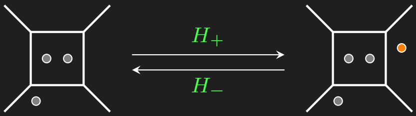
Wave function and BPS Hilbert space:

$$\Psi(\mathrm{K}) := \mathrm{Eul}\left(\mathcal{M}_{\text{meson}}\right) = (-1)^{\left\lfloor \sum_{\alpha:\ \mu_\alpha=0} \frac{1}{2}\right\rfloor} \prod_{\alpha:\ \mu_\alpha\neq 0} \mu_\alpha,$$

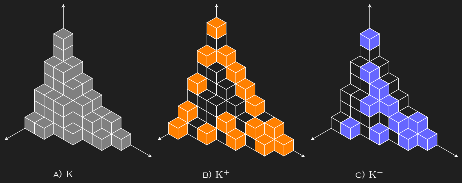
$$\mathscr{H}_{\text{BPS}} = \bigoplus_{\mathrm{K}} \mathbb{C}\Psi(\mathrm{K}).$$

Hecke modification

Adding/deleting branes → Hecke modifications:



Vacant positions:



Raising/lowering generators:

$$e^{(a)}(z) = \left[\mathrm{Tr}\,(z - \Phi_a)^{-1}, \sum_{\square \in K^+} |K + \square\rangle\langle K + \square| H_+ |K\rangle\langle K| \right]$$

$$f^{(a)}(z) = -\left[\mathrm{Tr}\,(z - \Phi_a)^{-1}, \sum_{\square \in K^-} |K - \square\rangle\langle K - \square| H_- |K\rangle\langle K| \right]$$

BPS algebra

$$
\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\;,\\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(z-w)\,e^{(b)}(w)\,\psi^{(a)}(z)\;,\\
e^{(a)}(z)\,e^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(z-w)\,e^{(b)}(w)\,e^{(a)}(z)\;,\\
\psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(z-w)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\;,\\
f^{(a)}(z)\,f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(z-w)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\;,\\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b}\,\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\;,
\end{aligned}
$$

Bond factor:

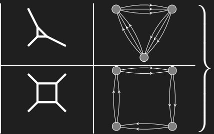
$$
\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+\mu_I)}{\prod_{I\in\{a\to b\}}(u-\mu_I)}
$$

What algebras do we get?

$$
xy = z^n w^m \quad \rightsquigarrow \quad \mathsf{Y}\left(\widehat{\mathfrak{gl}}_{n|m}\right)
$$

$K_{\mathbb{P}^2}$

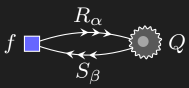
$K_{\mathbb{P}^1\times\mathbb{P}^1}$



NEW!

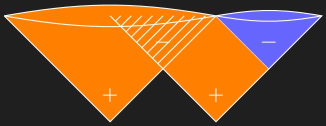
Framing and New Reps

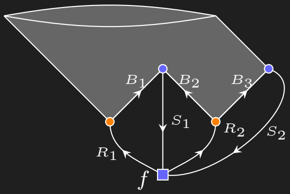
Framing Node ≈ "Frozen" Gauge Node:



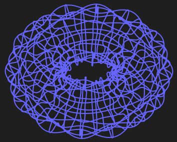
R – positive crystal 

, S – negative crystal 





$$\Delta W = \mathrm{Tr}\left[S_1(B_1R_1 - B_2R_2) + S_2B_3R_2\right]$$

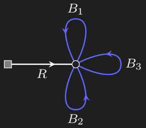
Even "bizarre" crystals: 

However the resulting Rep may be **reducible**...

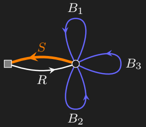
Simple Example

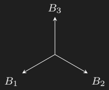
From CY3 to CY2

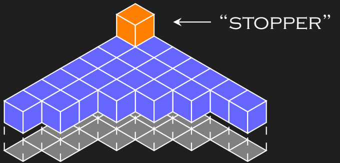
D4 wrapping	$\mathbb{C}^2 \subset \mathbb{C}^3$	wrapped by D6
$\mathrm{Hilb}^n(\mathbb{C}^2)$		$\mathrm{Hilb}^n(\mathbb{C}^3)$
ordinary partitions		plane partitions

D6 on $\mathbb{C}^3$:  ,

$W = \mathrm{Tr}\ B_3\,[B_1, B_2]$

Extra D4:  ,





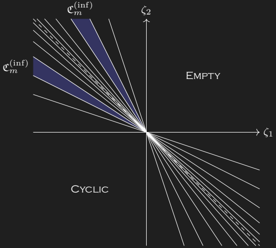
$\Delta W = \mathrm{Tr}\ S B_3 R$

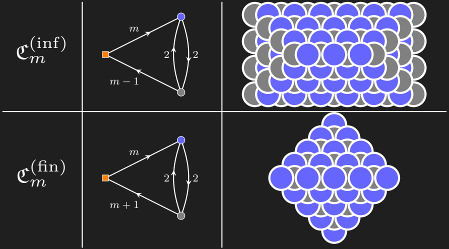
$$W = \mathrm{Tr}\ B_3\,([B_1, B_2] + RS)$$

$$\partial_{B_3} W = [B_1, B_2] + RS = 0 \quad \rightsquigarrow \quad \text{ADHM for } \mathrm{Hilb}^n(\mathbb{C}^2)$$

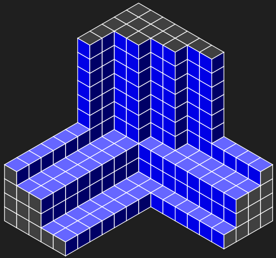
Other Examples

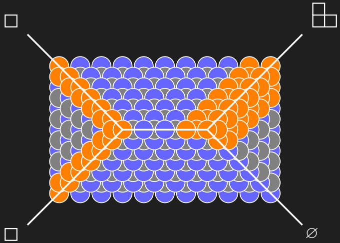
Wall-crossing:





Open BPS counting:



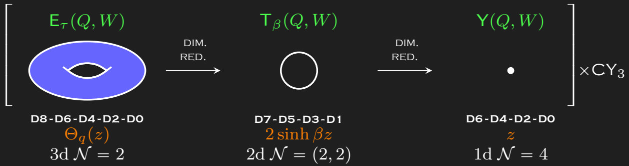


And more exotic things...

Generalized Cohomology

Generalized Euler class:

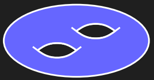
$$\mathrm{Eul}_\zeta(\mathfrak{R}_{\mathrm{IR}}) = (-1)^{\left\lfloor \sum_{a:\ h_a=0} \frac{1}{2} \right\rfloor} \prod_{a:\ h_a\neq 0} \zeta(h_a)\,,$$



Generalized genus:

$$\zeta^{-1}(u) = u + \frac{\varphi(\mathbb{CP}^2)}{3}u^3 + \frac{\varphi(\mathbb{CP}^4)}{5}u^5 + \dots$$

Generalized cohomology theories (GCT):

 $\rightsquigarrow$ (GCT?)

Open Problems

- Wall-crossing, what about non-crystal phases?
- Calabi-Yau 4- and 5-folds (???), 6-folds(???), ...
- Non-toric Calabi-Yau n-folds
- Generalized cohomology

Thank you for your attention