# Non-Unitary TQFTs from 3d $\mathcal{N} = 4$ Rank-0 SCFTs

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# Today's Agenda

### Interesting Correspondence between TQFT and SCFTs

In this presentation, we report an interesting correspondence between non-unitary TQFTs and $\mathcal{N} = 4$ Rank-0 SCFTs in 3d.

### Proposals and Dictionaries

From SUSY ptn in "degenerate limits" of $\mathcal{N} = 4$ Rank-0 SCFTs, we obtained partial "modular data" of non-unitary TQFTs.

### The First Application: Bounds for $F_{S^3}$ of all $\mathcal{N} = 4$ SCFTs

As a direct application of our dictionary, we obtained lower bounds for round 3-sphere free energy, $F_{S^3}$, of our rank-0 SCFTs from the modular data.

# $\mathcal{N} = 4$ Supersymmetry: 8 Real Supercharges

## $\mathrm{SO}(4) \cong \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ R-Symmetry

Let $R$ and $R'$ be two Cartans of $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ resp, and let $R_\nu$ and $A$ be two Cartans of $\mathrm{SO}(4)$ R-symmetry. ($\nu$: mixing parameter)

$$R_{\mathrm{IR}} = R_{\nu=0} = R + R', \qquad A = R - R', \qquad R_\nu = R + R' + \nu A,$$

In $\mathcal{N} = 2$ language, $R_\nu$ is $\mathrm{U}(1)$ R-symmetry and $A$ is $\mathrm{U}(1)$ flavor symmetry.

## Multiplets

- V Vector multiplet: $\mathcal{N} = 2$ vector $V$ + $\mathcal{N} = 2$ chiral multiplet $\Phi_{adj}$ in adjoint representation of gauge group
- $h$ Hypermultiplet: $\mathcal{N} = 2$ chiral $\Phi$ + anti-chiral multiplet $\bar{\Phi}$ in $R$ and $\bar{R}$ representations.
- $h_{\frac{1}{2}}$ Half-hyper: If the gauge group is pseudo-real, there is a relation between $\Phi$ and $\bar{\Phi}$. Then, the d.o.f is just a half of $h$.

# Why 8 Real Supercharges?

## Rich Structures with Various Dimensionality

1: 5D, 6D SCFTs predicted by String/M-theory constructions

2: Class-$\mathcal{S}$, Seiberg-Witten, Argyres-Douglas, AGT Correspondences, etc. for 4d $\mathcal{N} = 2$

3: 3d mirror symmetry, Rozansky-Witten, 1d TQM, etc. for 3d $\mathcal{N} = 4$.

4: Rich vacuum moduli structure of SCFTs

## Classification Program of SCFTs by Rank

$$\text{rank} := \dim_{\mathbb{C}}(\mathcal{M}_{\text{Coulomb}}) > 0 \text{ for } D \geq 4$$

In higher dimensions than 3, it is believed that there is no rank-0 SCFTs. NB: 5,6D rank-0 theories are constructed by geometric engineering, but it is uncertain that they are interacting theories [Closset, Schafer-Nameki, Wang 2020]

# Rank of 3d SCFTs

## SCFTs in 3d

1: Non-trivial rank-0 SCFTs are found quite recently.[Gang, Yamazaki 2018]

2: Coulomb and Higgs branches are exchangeable by 3d mirror symmetry [Intriligator and Seiberg 1996]

3: Needs for a notion of rank invariant under 3d mirror symmetry

## Refined Definition

The terminology "Rank" of SCFT is refined to maximum complex dimensions of vacuum moduli space in Coulomb and Higgs branch since

$$\boxed{\text{rank} := \max(\dim_{\mathbb{C}}(\mathcal{M}_{\text{Coulomb}}), \dim_{\mathbb{C}}(\mathcal{M}_{\text{Higgs}}))}$$

Using this modified notion of rank in 3d,

$$\text{Rank-0 Theory} := \text{Theory with no Higgs and Coulomb branches}$$

# Why Rank-0 SCFTs?

## Technical Difficulties for Classification Program

1: No Coulomb and Higgs branch operator

2: Thus, traditional schemes do not work

3: No Flavor Symmetry $\rightarrow$ Only $T_{\mu\nu}$ multiplet $\rightarrow$ Difficult to conformal bootstrap

## Some Properties of Rank-0 Theory

Coulomb/Higgs branch operators are charged under $\mathrm{SU}(2)_{L/R}$ of $\mathrm{SO}(4)$ R-symmetry. Flavor current multiplets includes Coulomb or Higgs branch operators

1: There are no flavor commuting with R-symmetry for Rank-0 theories.

2: SUSY should be no more than $\mathcal{N} = 5$.

# Degenerate Limit of $\mathcal{N} = 4$ Rank-0 SCFTs

### Degenerate Limit or Coulomb/Higgs Branch Limit

The degenerate limits are defined by a certain non-trivial R-charge assignment $\nu \neq 0$ under absence of real mass parameter $m = 0$ or a trivial fugacity variable $\eta = 1$ for $\mathrm{U}(1)_A$

### SCFTs in Degenerate Limits

1: $\mathrm{SCI}(m = 0, \nu = 1) \to$ Hilbert series on Coulomb branch.

2: $\mathrm{SCI}(m = 0, \nu = -1) \to$ Hilbert series on Higgs branch.

3: Rank-0 SCFTs have no Coulomb and Higgs branch $\Rightarrow$ No Local Operators

### TQFTs have NO local operators

Hilbert-series counts gauge invariant local operators. TQFT has no local operators except the identity operator.

# Our Main Proposal [Gang, Kim, Lee, MS, Yamazaki 2021]

**Non-Unitary TQFTs emerge from degenerate limits of $\mathcal{N}=4$ SCFTs**

In a partition function level,

$$\mathcal{Z}^{\mathbb{B}}_{\mathcal{T}_{\text{rank }0}}\left(b^2, m\ (\text{or }\eta), \nu; s\right) \xrightarrow{m\to 0\ (\text{or }\eta\to 1),\ \nu\to\pm 1} \mathcal{Z}^{\mathcal{M}_{g,p}}_{\text{TFT}_{\pm}[\mathcal{T}_{\text{rank }0}]}(s)\ .$$

**Classification of $\mathcal{N}=4$ rank-0 SCFTs**

In terms of mathematically well-defined TQFTs, we initiate classification of $\mathcal{N}=4$ rank-0 theories

**Not Only for Partition Functions**

We also established a dictionary between non-unitary TQFTs and rank-0 SCFTs

# Our Main Result: Dictionaries

| $\mathrm{TFT}_\pm[\mathcal{T}_{\text{rank }0}]$ | $\mathcal{T}_{\text{rank }0}$ |
|---|---|
| $\mathcal{Z}^{\mathcal{M}_{g,p}}_{\mathrm{TFT}_\pm}(s)$ | BPS partition function $\mathcal{Z}^{\mathbb{B}}_{\mathcal{T}_{\text{rank }0}}\big\|_{\nu\to\pm1,m=0}(s)$ with (topology of $\mathbb{B}$) $=\mathcal{M}_{g,p}$ |
| Spin or non-spin | Later Slide |
| Rank $N$ | Witten index |
| Simple objects | Bethe vacua $\{\vec{z}_\alpha\}_{\alpha=0}^{N-1}$ or BPS loop operators $\{\mathcal{O}^\pm_\alpha(\vec{z})\}_{\alpha=0}^{N-1}$ |
| $(S^\pm_{0\alpha})^{-2}$ | $\mathcal{H}_\alpha(m=0,\nu\to\pm1;s=-1)$ |
| $T^\pm_{\alpha\beta}$ (only for non-spin) | $\delta_{\alpha\beta}(\mathcal{F}_\alpha/\mathcal{F}_{\alpha=0})\big\|_{\nu\to\pm1,m=0}$ |
| $(T^2)^\pm_{\alpha\beta}$ | $\delta_{\alpha\beta}(\mathcal{F}_\alpha/\mathcal{F}_{\alpha=0})^2\big\|_{\nu\to\pm1,m=0,s=-1}$ |
| $S^\pm_{00}$ | $\left\|\mathcal{Z}^{S^3_b}_{\mathcal{T}_{\text{rank }0}}(m=0,\nu\to\pm1)\right\|$ |
| $W^\pm_\beta(\alpha)$ | $\mathcal{O}^\pm_\alpha(\vec{z}_\beta)\|_{\nu\to\pm1,m=0}$ |
| $\max_\alpha(-\log\|S^\pm_{0\alpha}\|)$ | $F$ (three-sphere free energy) |

# The First Application: Bounds for $F_{S^3}$ - 1

Topologically Twisted Indices $\Leftrightarrow$ Ground State Degeneracy of TQFT

From the dictionary on partition functions,

$$\mathcal{Z}^{S^1\times\Sigma_g}_{\text{rank-0}}(s)\Big|_{m=0,\nu=\pm1} = |\text{GSD}|_g \in \mathbb{Z}, \qquad |\text{GSD}|_g = \sum_{i=1}^{r} x_i^{1-g}, (x_i > 0)$$

$x_i$ denotes one of $(S_{0\alpha})^2$ with the dictionary

$$\sqrt{x_1} := \mathcal{Z}^{S^3_b}_{\text{rank-0}}(s)\Big|_{m=0,\nu=\pm1}, \qquad \sqrt{x_2} := \sqrt{\min\{x_i\}} = \mathcal{Z}^{S^3}_{\text{rank-0}}(s)\Big|_{IR}$$

Non-unitary condition:

$$x_1 > x_2$$

# The First Application: Bounds for $F_{S^3}$ - 2

**Topologically Twisted Indices $\Leftrightarrow$ Ground State Degeneracy of TQFT**

GSD of TQFTs on $\Sigma^1 \times \Sigma_g$ are

$$|\mathrm{GSD}|_0 = 1 \qquad |\mathrm{GSD}|_1 = \mathrm{Witten\ Index} = r \qquad |\mathrm{GSD}|_2 = k \in \mathbb{Z}$$

For TQFTs, Witten index indicates the number of simple objects including the identity. Therefore, Witten index of TQFTs starts from 2 for non-trivial TQFTs.

**Bounds for $F_{S^3}$ of rank-0 SCFT of Witten index $r$**

$$\sum_{i=1}^{r} x_i = 1 \ \ \& \ \ x_1 > x_2 \Rightarrow x_2 = \min(x_i) < \frac{1}{r} \Rightarrow F_{S^3} = -\frac{1}{2}\log x_2 > \frac{1}{2}\log r$$

since at least one is larger than others from the non-unitary condition.

# The First Application: Universal Bounds for $F_{S^3}$ - 3

## A Minimal Non-Trivial TQFT

A minimal non-trivial TQFT have one simple object in addition to the identity.

$$|\mathrm{GSD}|_0 = x_1 + x_2 = 1 \ \& \ |\mathrm{GSD}|_2 = \frac{1}{x_1} + \frac{1}{x_2} = k \in \mathbb{Z}, \qquad x_1 > x_2$$

We obtained

$$x_1 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{k-4}{k}}, \qquad x_2 = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{k-4}{k}} < \frac{1}{2}, \qquad k \geq 5$$

For $k = 5$, minimum values for $F_{S^3}$ obtained as

$$F_{S^3}^{\text{rank-0}} \geq F_{S^3}^{\min} = -\frac{1}{2}\log x_2^{\min} = -\log\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right) > \frac{1}{2}\log 2$$

It is saturated by $F_{S^3}(\mathcal{T}^{\min}_{\text{rank-0}})$ [Gang, Yamazaki 2018]

# SUSY Partition Function: Versatile Tools

### Main Toolkit: Supersymmetric Partition Functions

1: Partition functions of TQFT from SUSY partition function in degenerate limits of rank-0 theories

2: Modular data from Bethe Vacua Method

### Target Supersymmetric Partition Functions

1 : $S^3$ partition function $\to F_{S^3} = \log \mathcal{Z}_{S^3}$ [Jafferis 2010]

2 : $S_b^3$ Squashed three-sphere partition function [Hama, Hosomichi, Lee 2011]

3 : $S^1 \times S^2$ partition function $\Leftrightarrow$ Superconformal Index (SCI) [Kim 2009, Imamura, Yokoyama 2011, Kapustin, Willett 2011]

4 : $S^1 \times \Sigma_{\mathfrak{g}}$ partition function $\Leftrightarrow$ Topologically Twisted Index ($\mathfrak{g}=0\to$ SCI)

# Evidence for TQFT: Superconformal Index

## SCI in Degenerate Limit: No Local Operators

$$\mathcal{I}^{\text{sci}}(q,\eta=1,\nu=\pm1;s=\pm1)=\mathcal{I}^{\text{sci}}(q,\eta,\nu=0;s=\pm1)\Big|_{\eta\to(\pm q^{\frac{1}{2}})}=1$$

## Superconformal Index: SUSY Partition Function on $S^1\times S^2$

Generic structure of SCI with a gauge group $G$ is the below.

$$\mathcal{I}^{\text{sci}}(q,\eta,\nu=0;s=1)$$
$$=\sum_{\mathrm{m}}\oint_{|a_i|=1}\left(\prod_{i=1}^{\text{rank}G}\frac{da_i}{2\pi i a_i}\right)\Delta_G(\mathrm{m},\mathrm{a};q)q^{\epsilon_0(\mathfrak{n})}\mathcal{I}_0^{cs}(\mathrm{m},\mathrm{a})\text{P.E.}[f_{\text{single}}(q,\mathrm{a},\eta;\mathrm{m})]\ .$$

## Localization Recipes

Each building blocks in the above formula are obtained by localization recipes with matter contents [Kapustin, Willett 2011, Imamura, Yokoyama 2011]

# Modular Data from Bethe Vacua Methods

### SUSY Localization on Squashed Three Sphere

Field contents + charge table $\rightarrow S_b^3$ partition function by localization [Hama, Hosomichi, Lee 2011]

### Method of Bethe Vacua

From the squashed three-sphere partition function, one can consider twisted superpotential and its saddle points, Bethe Vacua, from $S_b^3$ partition function. Then, one can construct handle gluing $\mathcal{H}$, fibering $\mathcal{F}$ operators, and general topologically twisted indices [Closset, Kim, Willett 2017, 2018]

# Bethe Vacua Recipes - 1 : Asymptotic Expansion

## $\mathcal{Z}_{S_b^3}$, Twisted Superpotential $\mathcal{W}$, and Perturbative Expansions around Saddles

$$\mathcal{Z}^{S_b^3}(b,m,\nu) = \int \left( \prod_{i=1}^{\mathrm{rank}(G)} \frac{dZ_i}{\sqrt{2\pi\hbar}} \right) \exp[\log \mathcal{I}_\hbar(Z,m,\nu)] \,, \quad \hbar := 2\pi i b^2 \,.$$

$$\mathcal{W}_n \,:\, \log \mathcal{I}_\hbar(\vec{Z},m,\nu) \xrightarrow{\hbar\to 0} \sum_{n=0}^{\infty} \hbar^{n-1} \mathcal{W}_n(\vec{Z},n,\nu) \,.$$

Around the saddles,

$$\mathcal{Z}^{S_b^3}(b,m,\nu) = |\mathrm{Weyl}(G)| \times \int \prod_{i=1}^{\mathrm{rank}(G)} \frac{d(\delta Z_i)}{\sqrt{2\pi\hbar}} \exp\left( \frac{1}{\hbar} \mathcal{W}_0^{\vec{n}_\alpha}(\vec{Z}^\alpha + \delta\vec{Z},m,\nu) + \sum_{n=1}^{\infty} \hbar^{n-1} \mathcal{W}_n(\vec{Z}+\delta\vec{Z},m,\nu) \right) \xrightarrow{\hbar\to 0} \exp\left( \sum_{n=0}^{\infty} \hbar^{n-1} \mathcal{S}_n^\alpha(m,\nu) \right)$$

# Bethe Vacua Recipes - 2: Bethe Vacua and Entries of Modular Matrices

## Bethe Vacua: Saddle Points

$$\frac{\left\{\vec{z}\ :\ \left(\exp(\partial_{Z_i}\mathcal{W}_0)\big|_{\vec{Z}\to\log\vec{z}}\right) = 1\ ,\ w\cdot\vec{z}\neq\vec{z}\ \ \forall\ \text{non-trivial}\ w\in\text{Weyl}(G)\right\}_{i=1}^{\text{rank}(G)}}{\text{Weyl}(G)}$$

## Handle Gluing $\mathcal{H}$ and Fibering $\mathcal{F}$ Operators: Recipes and Dictionaries

$$\mathcal{H}_\alpha(\eta,\nu;s=-1) = e^{i\varphi}\exp\left(-2\mathcal{S}_1^\alpha(m,\nu)\right)\big|_{m=\log\eta} \Rightarrow (S_{0\alpha}^{\pm})^{-2}$$

$$\mathcal{F}_\alpha(\eta,\nu;s=-1) = \exp\left(\frac{\mathcal{S}_0^\alpha(m,\nu)}{2\pi i}\right) \Rightarrow (T^2)_{\alpha\beta}^{\pm} = \delta_{\alpha\beta}(\mathcal{F}_\alpha/\mathcal{F}_{\alpha=0})^2$$

# Bethe Vacua Recipes - 3 : Topologically Twisted Index

### Twisted Index on $S^1 \times_p \Sigma_\mathfrak{g}$

$$\mathcal{Z}^{\mathcal{M}_{g,p}}(m,\nu,s) = \sum_{\vec{z}_\alpha \,:\, \text{Bethe-vacua}} (\mathcal{H}_\alpha(\eta = e^m, \nu; s))^{g-1} (\mathcal{F}_\alpha(m,\nu;s))^p$$

### Twisted Index on $S^1 \times \Sigma_\mathfrak{g}$ → Ground State Degeneracy of TQFTs

$$\mathcal{Z}^{\mathcal{M}_{g,0}}(m,\nu,s) = \mathcal{Z}^{S^1\times\Sigma_\mathfrak{g}}(m,\nu,s) = \sum_{\vec{z}_\alpha \,:\, \text{Bethe-vacua}} (\mathcal{H}_\alpha(\eta = e^m, \nu; s))^{g-1}$$

$$\mathcal{Z}^{S^1\times\Sigma_\mathfrak{g}}(m=0,\nu=\pm1,s) = |\text{GSD}|_g \in \mathbb{Z}$$

### For TQFT: Trivial SCI

$$\mathcal{I}^{\text{sci}} = \mathcal{Z}^{S^1\times\Sigma_{\mathfrak{g}=0}}(m,\nu,s) = \sum_{\vec{z}_\alpha \,:\, \text{Bethe-vacua}} (\mathcal{H}_\alpha(\eta = e^m, \nu; s))^{-1} = 1$$

# Rests of Modular Data

### Basic Operators and Fusions in TQFTs

Basic operators *w.r.t.* $A$ and $B$ cycles act on the Hilbert space of TQFTs

$$\mathcal{O}^A_\beta|\alpha\rangle = W_\beta(\alpha)|\alpha\rangle, \qquad \mathcal{O}^B_\beta|\alpha\rangle = \mathcal{O}^B_\beta\mathcal{O}^B_\alpha|0\rangle = \sum_{\gamma=0}^{N-1} N^\gamma_{\alpha\beta}|\gamma\rangle,$$

where $W_\beta(\alpha) = S_{\alpha\beta}/S_{\alpha 0}$.

### Fusion Rings and Verlinde Formula

Fusions of operators form a commutative ring, and the fusion coefficient can be obtained from $S$ matrix by Verlinde formula

$$\mathcal{O}^B_\beta\mathcal{O}^B_\alpha = \sum_{\gamma=0}^{N-1} N^\gamma_{\alpha\beta}\mathcal{O}^B_\gamma, \qquad N^\gamma_{\alpha\beta} = \sum_{\delta=0}^{N-1}\frac{S_{\delta\alpha}S_{\delta\beta}\bar{S}_{\delta\gamma}}{S_{0\delta}}$$

# Unitarity Condition of Modular Data

### Unitarity $S^{-1} = S^{\dagger}$

Identity operator has minimum categorical dimensions $\{S_{0\alpha}\}$

$$|S_{00}| \leq |S_{0\alpha}|$$

### Non-Unitary Condition

Identity operator has not minimum categorical dimensions $\{S_{0\alpha}\}$

$$|S_{00}| > |\min(S_{0\alpha})|$$

Combining with dictionary for $F_{S^3}$, this condition is related $F$-maximization.

# Classification of TQFTs and $\mathcal{N} = 4$ Rank-0 SCFTs

## Classification by Rank(Witten Index) of TQFTs

1: All 35 Unitary Modular Tensor Categories ($r \leq 4$) up to ribbon tensor equivalences are classified [Rowell, Stong, Wang 2007]

2: Rank 5 Modular Tensor Categories [Bruillard, Ng, Rowell, Wang 2015]

3: Partial list of primitive UMTC of $r = 7, 8, 9$ [Wen 2015]

4: Classification of Fermionic MTC [Bruillard, Galindo, Hagge, Ng, Plavnik, Rowell, Wang 2016]

## Classification of $\mathcal{N} = 4$ Rank-0 SCFTs

Using the correspondence, $\mathcal{N} = 4$ Rank-0 SCFTs can be classified by classification of TQFT by Witten index.

# Summary: Dictionaries

| $\mathrm{TFT}_\pm[\mathcal{T}_{\mathrm{rank}\ 0}]$ | $\mathcal{T}_{\mathrm{rank}\ 0}$ |
|---|---|
| $\mathcal{Z}^{\mathcal{M}_{g,p}}_{\mathrm{TFT}_\pm}(s)$ | BPS partition function $\mathcal{Z}^{\mathbb{B}}_{\mathcal{T}_{\mathrm{rank}\ 0}}\big\vert_{\nu\to\pm1,m=0}(s)$ with (topology of $\mathbb{B}$) $=\mathcal{M}_{g,p}$ |
| Spin or non-spin | Next Slide |
| Rank $N$ | Witten index |
| Simple objects | Bethe vacua $\{\vec{z}_\alpha\}_{\alpha=0}^{N-1}$ or BPS loop operators $\{\mathcal{O}^\pm_\alpha(\vec{z})\}_{\alpha=0}^{N-1}$ |
| $(S^\pm_{0\alpha})^{-2}$ | $\mathcal{H}_\alpha(m=0,\nu\to\pm1;s=-1)$ |
| $T^\pm_{\alpha\beta}$ (only for non-spin) | $\delta_{\alpha\beta}(\mathcal{F}_\alpha/\mathcal{F}_{\alpha=0})\big\vert_{\nu\to\pm1,m=0}$ |
| $(T^2)^\pm_{\alpha\beta}$ | $\delta_{\alpha\beta}(\mathcal{F}_\alpha/\mathcal{F}_{\alpha=0})^2\big\vert_{\nu\to\pm1,m=0,s=-1}$ |
| $S^\pm_{00}$ | $\left\vert\mathcal{Z}^{S^3_b}_{\mathcal{T}_{\mathrm{rank}\ 0}}(m=0,\nu\to\pm1)\right\vert$ |
| $W^\pm_\beta(\alpha)$ | $\mathcal{O}^\pm_\alpha(\vec{z}_\beta)\vert_{\nu\to\pm1,m=0}$ |
| $\max_\alpha(-\log\vert S^\pm_{0\alpha}\vert)$ | $F$ (three-sphere free energy) |

# Minimal Rank-0 Theory $\mathcal{T}_{\text{min}}$

**UV $\mathcal{N}=2$ $\mathrm{U}(1)_{k=-3/2}$ with a chiral multiplet of charge 1**

In IR, supersymmetries are enhanced to $\mathcal{N}=4$.

**What is minimal? Degrees of Freedom**

At the first shot, it seems that a theory with a free chiral multiplet has minimal degrees of freedom.

**Comparison of central charge [Gang, Yamazaki 2018]**

Central charge is one of measures of degrees of freedom of conformal theories.

$$\frac{C_T(\mathcal{T}_{\text{min}})}{C_T(\text{free theory with single }\Phi)} \simeq 0.992549$$

# Lee-Yang TQFT from Degenerate Limit of $\mathcal{T}_{\text{min}}$

## Check of SCI

SCI with conformal R-charges is

$$\mathcal{I}^{\text{sci}}_{\mathcal{T}_{\text{min}}}(q, u, \nu = 0; s = 1) = 1 - q + \left(\eta + \frac{1}{\eta}\right) q^{3/2} - 2q^2 + \cdots \ .$$

Consider non-trivial R-charge mixing, $\nu = \pm 1$, SCI is

$$\begin{aligned} &\mathcal{I}^{\text{sci}}_{\mathcal{T}_{\text{min}}}(q, \eta, \nu, s = 1)\big|_{\nu \to \pm 1} \\ &= 1 + \left(-1 + \eta^{\mp 1}\right) q + \left(-2 + \eta + \frac{1}{\eta}\right) q^2 + \left(-2 + \eta + \frac{1}{\eta}\right) q^3 + \cdots \ . \end{aligned}$$

Taking $\eta \to 1$, SCI yields 1. This would be reproduced from twisted index with $\mathfrak{g} = 0$.

# Modular Data from Bethe-Vacua

## $S_b^3$ Partition Function from Localization Formula

From the Localization Recipes in previous slides with $k = -3/2$, $R_\Phi = \Delta_\Phi = 1/2$, charge $+1$, one can construct $\mathcal{Z}^{S_b^3}$ as

$$\mathcal{Z}^{S_b^3}_{\mathcal{T}_{\min}}(b, m, \nu) = \int \frac{dZ}{\sqrt{2\pi\hbar}} \exp\left[-\frac{Z^2 + 2Z\left(m + (i\pi + \frac{\hbar}{2})\nu\right)}{2\hbar}\right] \psi_\hbar(Z)$$

## Bethe-Vacua

$$\text{Bethe-vacua of } \mathcal{T}_{\min} \ : \ \left\{ z \ : \ \frac{(z-1)}{z^2} e^{-m-i\pi\nu} = 1 \right\}$$

# Modular Data from Bethe-Vacua

### Handle-Gluing and Fibering Operators

$$\left\{\mathcal{F}_\alpha(m=0,\nu\to\pm 1,s=-1)\right\}_{\alpha=0,1} \longrightarrow \left\{\exp\left(-\frac{7i\pi}{60}\right),\exp\left(\frac{17i\pi}{60}\right)\right\}$$

$$\left\{\mathcal{H}_\alpha(m=0,\nu\to\pm 1,s=-1)\right\}_{\alpha=0,1} \longrightarrow \left\{\frac{5-\sqrt{5}}{2},\frac{5+\sqrt{5}}{2}\right\}$$

### Modular Matrices $(S,T)$ obtained but for $S_{11}$

$$S=\begin{pmatrix}\sqrt{\frac{1}{10}\left(\sqrt{5}+5\right)} & -\sqrt{\frac{1}{10}\left(5-\sqrt{5}\right)}\\ -\sqrt{\frac{1}{10}\left(5-\sqrt{5}\right)} & -\sqrt{\frac{1}{10}\left(\sqrt{5}+5\right)}\end{pmatrix}, \qquad T=\begin{pmatrix}1 & 0\\ 0 & \exp(-\frac{2\pi i}{5})\end{pmatrix}$$

# Anyons of Lee-Yang TQFT from Loop Operators

## Identification of Simple Objects

For a $U(1)$ gauge theory, the supersymmetric dyonic loop operator $\mathcal{O}_{(p,q)}$ of (electric charge, magnetic charge)=$(p,q)$ is

$$\mathcal{O}_{(p,q)} = z^p (1 - z^{-1})^q \ .$$

Two simple objects in Lee-Yang MTC identified as a supersymmetric loop operators as

$$\mathcal{O}_{\alpha=0} = \mathcal{O}_{(0,0)} = (\text{identity operator}) \ , \quad \mathcal{O}_{\alpha=1} = \mathcal{O}_{(p,q)=(1,0)} \ .$$

## Completing $S$-Matrix

Completing $S$-matrix other than the first row/column, the belows are required to compute.

$$W_{\beta=0,1}(0) = 1 \ , \quad W_{\beta=0}(1) = z_0 = \frac{1}{2}(\sqrt{5} - 1) \ , \quad W_{\beta=1}(1) = z_1 = \frac{1}{2}(-\sqrt{5} - 1)$$

# Infinitely Many Examples

## Table of Examples [Gang, Kim, Lee, MS, Yamazaki 2021]

| $\mathcal{T}_{\text{rank } 0}$ | $\text{TFT}_\pm[\mathcal{T}_{\text{rank } 0}]$ | Set of $\{\lvert S^\pm_{0\alpha}\rvert\}$ | $\exp(-F)$ |
|---|---|---|---|
| $\mathcal{T}_{\min}$ | (Lee-Yang) | $\{\sqrt{\frac{5+\sqrt{5}}{10}}, \sqrt{\frac{5-\sqrt{5}}{10}}\}$ | $\sqrt{\frac{5-\sqrt{5}}{10}}$ |
| $(U(1)_1 + H)$ | $\text{Gal}_d(SU(2)_6)/\mathbb{Z}_2^f$ (with $d = \zeta_6^3$) | $\{2\zeta_6^1, 2\zeta_6^3\}$ | $2\zeta_6^1$ |
| $SU(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}$ ($\lvert k\rvert > 1$) | $\text{Gal}_d(SU(2)_{4\lvert k\rvert-2})/\mathbb{Z}_2^f$ (with $d = \zeta_{4\lvert k\rvert-2}^{2\lvert k\rvert-1}$) | $\{2\zeta_{4\lvert k\rvert-2}^{2n-1}\}_{n=1}^{\lvert k\rvert}$ | $2\zeta^1_{4\lvert k\rvert-2}$ |
| $T[SU(2)]_{k_1,k_2}$ | See the caption | $\{(\frac{1}{\sqrt{2}}\zeta^n_{\lvert k_1k_2-1\rvert-2})^{\otimes 2}\}_{n=1}^{\lvert k_1k_2-1\rvert-1}$ | $\frac{1}{\sqrt{2}}\zeta^1_{\lvert k_1k_2-1\rvert-2}$ |
| $\frac{T[SU(2)]}{SU(2)^{\text{diag}}_{\lvert k\rvert=3}}$ | $(\text{Lee-Yang})^{\otimes 2}\otimes U(1)_2$ | $\{\frac{1}{\sqrt{10}}^{\otimes 4}, \frac{5+\sqrt{5}}{10\sqrt{2}}^{\otimes 2}, \frac{5-\sqrt{5}}{10\sqrt{2}}^{\otimes 2}\}$ | $\frac{5-\sqrt{5}}{10\sqrt{2}}$ |
| $\frac{T[SU(2)]}{SU(2)^{\text{diag}}_{\lvert k\rvert=4}}$ | $\frac{\text{Gal}_{\zeta_{10}^7}(SU(2)_{10})\times SU(2)_2}{\mathbb{Z}_2^{\text{diag}}}$ | $\{\frac{1}{2}, \frac{1}{2\sqrt{3}}^{\otimes 5}, \frac{3+\sqrt{3}}{12}^{\otimes 2}, \frac{3-\sqrt{3}}{12}^{\otimes 2}\}$ | $\frac{3-\sqrt{3}}{12}$ |
| $\frac{T[SU(2)]}{SU(2)^{\text{diag}}_{\lvert k\rvert=5}}$ | $\text{Gal}_d((G_2)_3)\otimes U(1)_{-2}$ $\left(d = \sqrt{\frac{5}{84}+\frac{1}{4\sqrt{21}}}\right)$ | $\{\frac{1}{\sqrt{6}}^{\otimes 2}, \frac{1}{\sqrt{14}}^{\otimes 6}, \sqrt{\frac{5}{84}\pm\frac{1}{4\sqrt{21}}}^{\otimes 2}\}$ | $\sqrt{\frac{5}{84}-\frac{1}{4\sqrt{21}}}$ |
| $\frac{T[SU(2)]}{SU(2)^{\text{diag}}_{\lvert k\rvert\geq 6}}$ | ? | $\{\frac{1}{\sqrt{2\lvert k\rvert-4}}^{\otimes(\lvert k\rvert-3)}, \frac{1}{\sqrt{2\lvert k\rvert+4}}^{\otimes(\lvert k\rvert+1)}, (\frac{1}{\sqrt{8\lvert k\rvert-16}}+\frac{1}{\sqrt{8\lvert k\rvert+16}})^{\otimes 2}, (\frac{1}{\sqrt{8\lvert k\rvert-16}}-\frac{1}{\sqrt{8\lvert k\rvert+16}})^{\otimes 2}\}$ | $\frac{1}{\sqrt{8\lvert k\rvert-16}}-\frac{1}{\sqrt{8\lvert k\rvert+16}}$ |

# Summary

Non-Unitary TQFT data can be obtained from SCFTs in the degenerate limits

Lower bound for $F_{S^3}$ is obtained. $\mathcal{T}_{\min}$ saturates the bound

The dictionaries are supported by infinitely many examples

For more detail, please take a look at our paper arXiv:2103.09283 [hep-th]

# Natural Questions

### Classification of Rank-0 SCFTs and Dualities

Does the same TQFT imply dualities between rank-0 SCFTs?

### Relation with Rozansky-Witten Theory

In RW theories, one can obtained Unitary TQFTs. We conjectures our Non-Unitary TQFTs are Galois Conjugate of them.

### Relation with 4d $\mathcal{N} = 2$ Argyres-Douglas Theories

# Thank You

THANK YOU for YOUR ATTENTION