Feigin–Semikhatov Duality and Its applications

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based on joint works with T. Creutzig, N. Genra, and R. Sato
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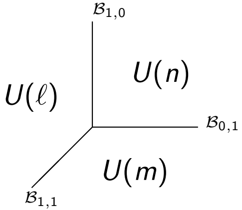
Vertex algebras at the corner

Several two dimensional chiral algebras (vertex algebras) appear as theories appearing at the boundary of other theories in higher dimension.
For example, WZW models at level k i.e., $\mathfrak{g}_k$ appear at the boundary of 3d Chern–Simons theories.
More sophisticated chiral algebras i.e.,$\mathcal{W}$-algebras are also obtained at the junction of the following GL-twisted $\mathcal{N}=4$ four dimensional super Yang–Mills theory. This is known as vertex algebras at the corner obtained by Gaiotto–Rapčák.



$$Y_{\ell,m,n}[\Psi] = \frac{\mathcal{W}_{n-m,1^m|1^\ell}(\mathfrak{u}(n|\ell)_{\Psi+h^\vee})}{\mathfrak{u}(m|\ell)_{\Psi+h^\vee-1}}$$

$(n-m,1^m|1^\ell)$: Boundary condition for $\boxed{\mathfrak{u}(n)\to\mathfrak{u}(m)}$

$\leftrightarrow \boxed{\mathfrak{su}_2 \hookrightarrow \mathfrak{u}(n-m)\subset\mathfrak{u}(n|\ell)}$

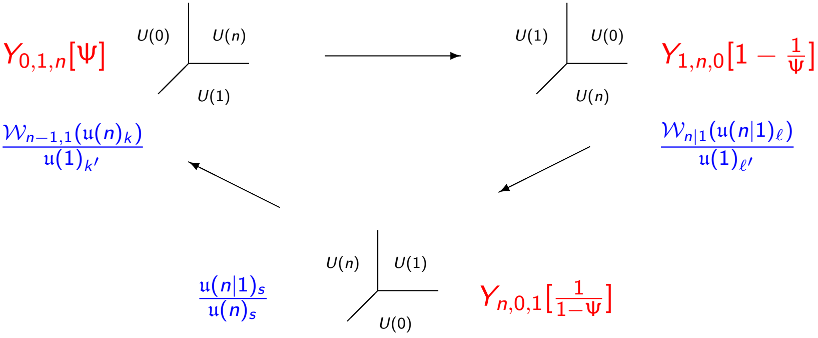
Gaiotto–Rapčák's Triality

Boundary conditions $\{(\mathcal{B}_{p,q},\Psi)\}_{(p,q)\in\mathbb{Z}^2/\{\pm1\}}$ enjoys the action of $PSL_2(\mathbb{Z})$. So we can rotate the Y-diagram and $Y_{\ell,m,n}[\Psi]$ at the corner should be preserved.

The Feigin–Frenkel duality for $\mathcal{W}_n$-algebra is obtained in this way:

$$\mathcal{W}_n(\mathfrak{u}(n)_{\Psi-n}) = Y_{0,0,n}[\Psi] \simeq Y_{0,n,0}[1-\frac{1}{\Psi}] = \mathcal{W}_n(\mathfrak{u}(n)_{\frac{1}{\Psi}-n}).$$

The Feigin–Semikhatov duality is the following case:



Feigin–Semikhatov duality

Theorem 1.1 (Creuztig–Linshaw, Creutzig–Genra–N)

For $(k+n)(\ell+n-1)=1$ [a]

$$\mathbf{FS}: \frac{\mathcal{W}_{n-1,1}(\mathfrak{su}(n)_k)}{\mathfrak{u}(1)_{k'}} \simeq \frac{\mathcal{W}_{n|1}(\mathfrak{su}(n|1)_\ell)}{\mathfrak{u}(1)_{\ell'}}$$

[a] We remove the level $(k,\ell)=(-n+\frac{n}{n-1}, -(n-1)+\frac{n-1}{n})$ when the Heisenberg subalgebra degenerates.

The case $n=2$ is actually the Kazama–Suzuki coset construction of $\mathcal{N}=2$ superconformal algebra:

$$\begin{array}{lccc}\mathbf{KS}: & \mathcal{N}=2\ \mathrm{SCA}_{c.c.=\frac{2k}{k+2}} & \xrightarrow{\simeq} & \frac{\mathfrak{su}(2)_k\times bc}{\mathfrak{u}(1)_{\mathrm{diag}}} \\ & G^+(z) & \mapsto & \sqrt{\frac{2}{k+2}}\,e(z)\otimes b(z) \\ & G^-(z) & \mapsto & \sqrt{\frac{2}{k+2}}\,f(z)\otimes c(z).\end{array}$$

Free field realization and Coset construction

$$\mathcal{W}_{n-1,1}(\mathfrak{su}(n)_k) \overset{*1}{\hookrightarrow} \mathfrak{su}(2)_{\substack{k+h^\vee\\-2}} \times \mathfrak{u}(1)^{n-2}_{k+h^\vee} \overset{*2}{\hookrightarrow} \beta\gamma \otimes \mathfrak{u}(1)^{n-1}_{k+h^\vee} \overset{*3}{\hookrightarrow} \widehat{V}_{\mathbb{Z}(1+\sqrt{-1})} \otimes \mathfrak{h}_{k+h^\vee}$$

$$\mathcal{W}_{n|1}(\mathfrak{su}(n|1)_\ell) \overset{*1}{\hookrightarrow} \mathfrak{u}(1|1)_{\ell'} \otimes \mathfrak{u}(1)^{n-2}_{\ell+h^\vee} \overset{*2}{\hookrightarrow} bc \otimes \mathfrak{h}_{\ell+h^\vee} \simeq V_{\mathbb{Z}} \otimes \mathfrak{h}_{\ell+h^\vee}.$$

($*1$: Miura map, $*2$: Wakimoto realization, $*3$: Friedan-Martinec-Shenker bosonization)

At each step, the image is described as the kernel of certain screening charges. In the end, both of them has one Fermionic screening and n bosonic screenings. [1]

Theorem 1.2 (Creutzig–Genra–N)

For $(k+n)(\ell+n-1) = 1$,

$$\textbf{KS}: \mathcal{W}_{n|1}(\mathfrak{su}(n|1)_\ell) \xrightarrow{\simeq} \frac{\mathcal{W}_{n-1,1}(\mathfrak{su}(n)_k) \otimes V_{\mathbb{Z}}}{\mathfrak{u}(1)_{\mathrm{diag}}},$$

$$\textbf{FST}: \mathcal{W}_{n-1,1}(\mathfrak{su}(n)_k) \xrightarrow{\simeq} \frac{\mathcal{W}_{n|1}(\mathfrak{su}(n|1)_\ell) \otimes V_{\sqrt{-1}\mathbb{Z}}}{\mathfrak{u}(1)_{\mathrm{diag}}}.$$

[1] This screening realization coincides with the COHA acting on the cohomology of the moduli space of spiked instantons of Nekrasov, derived by Rapčák, Soibelman, Yang and Zhao.

Cohomological reformulation and Intertewining operators

Let us decompose $\mathcal{W}_{n-1,1}(\mathfrak{su}(n)_k)$ and $\mathcal{W}_{n|1}(\mathfrak{su}(n|1)_\ell)$ as Fock modules:

$$\mathcal{W}_{n-1,1}(\mathfrak{su}(n)_k) \simeq \bigoplus_{a\in\mathbb{Z}} \mathcal{P}_a^{n-1,1} \otimes Fock_{H_1}^{|a\rangle}, \quad \mathcal{W}_{n|1}(\mathfrak{su}(n|1)_\ell) \simeq \bigoplus_{a\in\mathbb{Z}} \mathcal{P}_a^{n|1} \otimes Fock_{H_2}^{|a\rangle}.$$

$$\rightsquigarrow \mathcal{P}_a^{n-1,1} \simeq \mathcal{P}_a^{n|1}!$$

Question 2.1

Can we interchange $Fock_{H_1}^{|a\rangle} \leftrightsquigarrow Fock_{H_2}^{|a\rangle}$ $(a \in \mathbb{Z})$ simultaneously??

- Yes! and given by the **relative semi-infintie cohomology**:

$$H^n_{\mathrm{rel}}\left(\mathfrak{u}(1), Fock_H^{|a\rangle} \otimes Fock_{\sqrt{-1}\,H}^{|b\rangle}\right) \simeq \delta_{n,0}\delta_{a+b,0}\mathbb{C}[|a\rangle \otimes |b\rangle].$$

We set

$$K_{(n-1,1)\to(n|1)} := \bigoplus_{a\in\mathbb{Z}} Fock_{\sqrt{-1}\,H_1}^{|a\rangle} \otimes Fock_{H_2}^{|a\rangle}, \quad K_{(n|1)\to(n-1,1)} := \bigoplus_{a\in\mathbb{Z}} Fock_{\sqrt{-1}\,H_2}^{|a\rangle} \otimes Fock_{H_1}^{|a\rangle}.$$

$$\simeq V_{\mathbb{Z}} \otimes \mathfrak{u}(1)_1 \qquad\qquad \simeq V_{\sqrt{-1}\,\mathbb{Z}} \otimes \mathfrak{u}(1)_1$$

$$\mathcal{W}_{n|1}(\mathfrak{su}(n|1)_\ell) \simeq H^0_{\mathrm{rel}}\left(\mathfrak{u}(1), \mathcal{W}_{n-1,1}(\mathfrak{su}(n)_k) \otimes V_{\mathbb{Z}} \otimes \mathfrak{u}(1)_1\right),$$
$$\mathcal{W}_{n-1,1}(\mathfrak{su}(n)_k) \simeq H^0_{\mathrm{rel}}\left(\mathfrak{u}(1), \mathcal{W}_{n|1}(\mathfrak{su}(n|1)_\ell) \otimes V_{\sqrt{-1}\mathbb{Z}} \otimes \mathfrak{u}(1)_1\right).$$

- Can be also applied to modules by

$$K^{\lambda}_{(n-1,1)\to(n|1)} \simeq V_{\mathbb{Z}} \otimes Fock_1^{|\lambda\rangle}, \quad K^{\lambda}_{(n|1)\to(n-1,1)} \simeq V_{\sqrt{-1}\mathbb{Z}} \otimes Fock_1^{|\lambda\rangle}.$$

We consider the modules which decomposes as Fock modules:

$$\mathcal{W}_{n-1,1}\text{-md} = \bigoplus_{\lambda\in\mathbb{C}/\mathbb{Z}} \mathcal{W}_{n-1,1}\text{-md}_\lambda, \quad \mathcal{W}_{n|1}\text{-md} = \bigoplus_{\lambda\in\mathbb{C}/\mathbb{Z}} \mathcal{W}_{n|1}\text{-md}_\lambda.$$

Theorem 2.2 (Creutzig–Genra–N–Sato)

$$\begin{tikzcd} \mathcal{W}_{n-1,1}\text{-md}_\lambda \arrow[rr, bend left, "H^0_{\mathrm{rel},\lambda}"] & \simeq & \mathcal{W}_{n|1}\text{-md}_{\check{\lambda}} \arrow[ll, bend left, "H^0_{\mathrm{rel},\check{\lambda}}"] \end{tikzcd} :$$

$$I_{\mathcal{W}_{n-1,1}}\begin{pmatrix} & M_3 & \\ M_1 & & M_2 \end{pmatrix} \simeq I_{\mathcal{W}_{n|1}}\begin{pmatrix} & H^0_{\mathrm{rel},\lambda_3}(M_3) & \\ H^0_{\mathrm{rel},\lambda_1}(M_1) & & H^0_{\mathrm{rel},\lambda_2}(M_2) \end{pmatrix}$$

Rational case: Comparision of Fusion rings

Creutzig-Linshaw proved a level-rank duality

$$\frac{\mathcal{W}_{n-1,1}(\mathfrak{su}(n)_{k(r)})}{\mathfrak{u}(1)_{H_1}} \simeq \mathcal{W}_r(\mathfrak{su}(r)_{\alpha(n)}), \qquad \begin{array}{l} k(r) = -n + \frac{n+r}{n-1} \\ \alpha(n) = -r + \frac{r+n}{r+1} \end{array}.$$

This implies

$$\mathcal{W}_{n-1,1}(\mathfrak{su}(n)_{k(r)}) \simeq \bigoplus_{i\in\mathbb{Z}_r} \mathbf{L}_{\mathcal{W}}(n\varpi_i) \otimes V_{\frac{ni}{\sqrt{nr}}+\sqrt{nr}\,\mathbb{Z}},$$

$$\mathcal{W}_{n|1}(\mathfrak{su}(n|1)_{\ell(r)}) \simeq \bigoplus_{i\in\mathbb{Z}_r} \mathbf{L}_{\mathcal{W}}(n\varpi_i) \otimes V_{\frac{(n+r)i}{\sqrt{(n+r)r}}+\sqrt{(n+r)r}\,\mathbb{Z}}.$$

$\Rightarrow$ Modules appearing in the branches are simple currents. In general, for a simple current extensions by a lattice theory V_L,

$$\mathcal{E} = \bigoplus_{a\in N/L} S_a \otimes V_{a+L}.$$

Their fusion rings are related by the duality of lattices $N/L \leftrightarrow N'/L$

Theorem 2.3 (cf. Creutzig–Genra–N–Sato)

$$\mathcal{K}(\mathcal{E}) \simeq \left(\mathcal{K}(V) \bigotimes_{\mathbb{Z}[N/L]} \mathbb{Z}[L'/L] \right)^{N/L}, \quad \mathcal{K}(V) \simeq \left(\mathcal{K}(\mathcal{E}) \bigotimes_{\mathbb{Z}[N'/L]} \mathbb{Z}[L'/L] \right)^{N'/L}.$$

Can be applied to the triangle of simple current extensions:

$$\begin{tikzcd}
\mathcal{W}_{n|1}(\mathfrak{su}(n|1)_{\ell(r)}) \arrow[rr, leftrightarrow] & & \mathcal{W}_{n-1,1}(\mathfrak{su}(n)_{k(r)}) \\
& \mathcal{W}_r(\mathfrak{su}(r)_{\alpha(n)}) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\mathcal{K}(\mathcal{W}_{n-1,1}) \simeq \mathcal{K}(\mathfrak{su}(n)_r)$$

$$\mathcal{K}(\mathcal{W}_{n-1,1}) \simeq \left(\mathcal{K}(\mathcal{W}_{n|1}) \underset{\mathbb{Z}[\mathbb{Z}_{n+r}]}{\otimes} \mathbb{Z}[\mathbb{Z}_{n(n+r)}] \right)^{\mathbb{Z}_{n+r}}$$

$$\mathcal{K}(\mathcal{W}_{n|1}) \simeq \left(\mathcal{K}(\mathcal{W}_{n-1,1}) \underset{\mathbb{Z}[\mathbb{Z}_{n}]}{\otimes} \mathbb{Z}[\mathbb{Z}_{n(n+r)}] \right)^{\mathbb{Z}_{n}}$$

Thank you for listening!