Recent Developments in N=4 Yang-Mills Amplitudes

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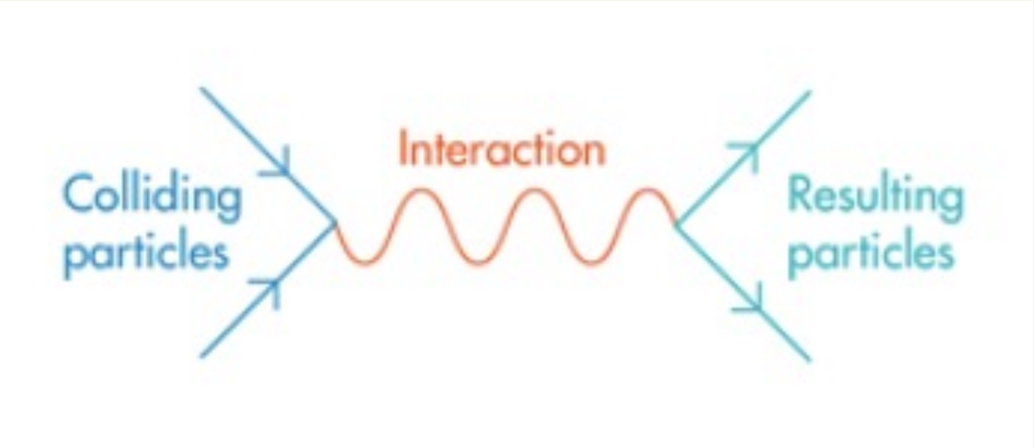
Outline

- Introduction
- Status and tools for amplitudes computations
- 6 and 7-point amplitudes: cluster algebras
- 8 and 9-point amplitudes: new features
- Symbol alphabet from plabic graphs
- Symbol alphabet from tensor diagrams
- Conclusions

Scattering Amplitudes

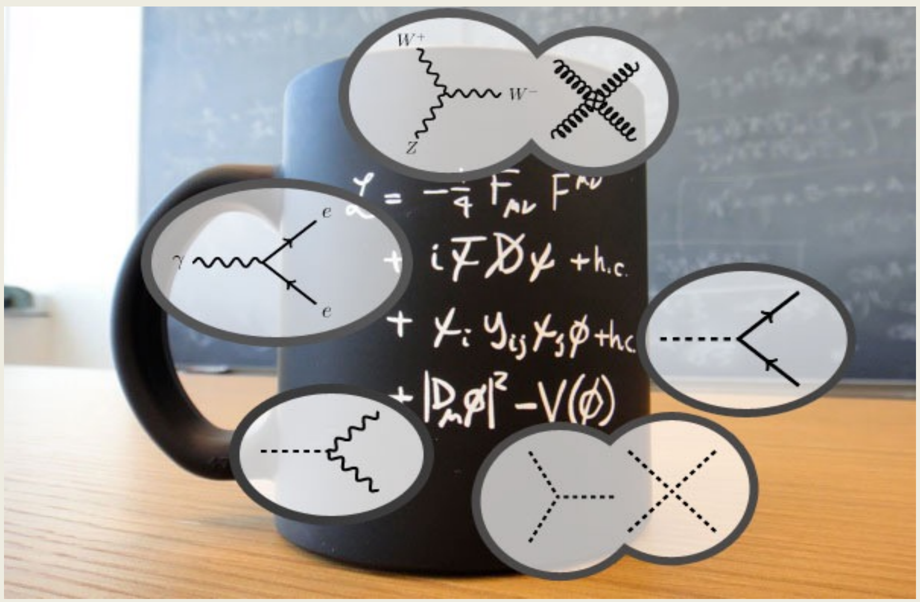
encode the process of elementary particles interaction. They determine probabilities of various outcomes.

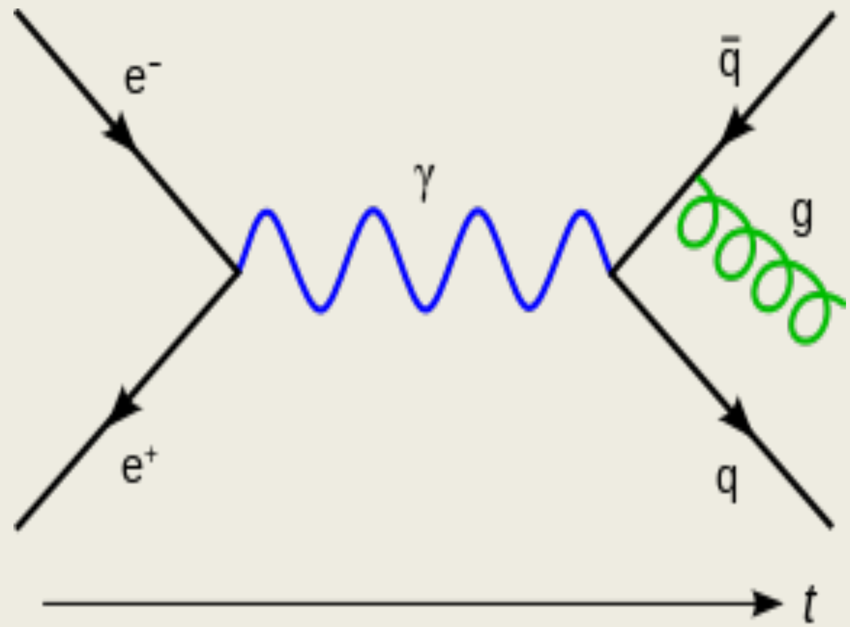


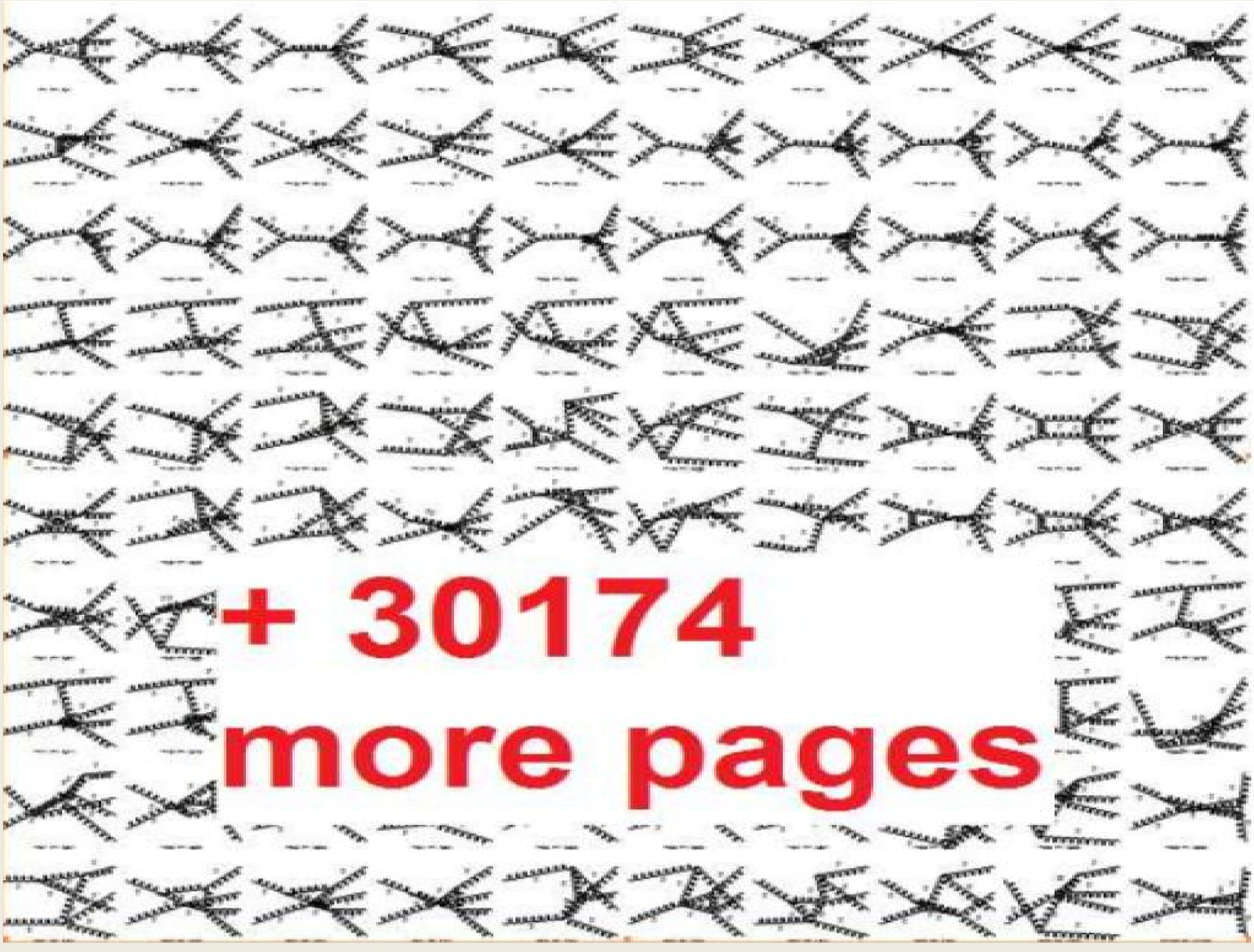
Scattering Amplitudes

are computed by

Feynman diagrams



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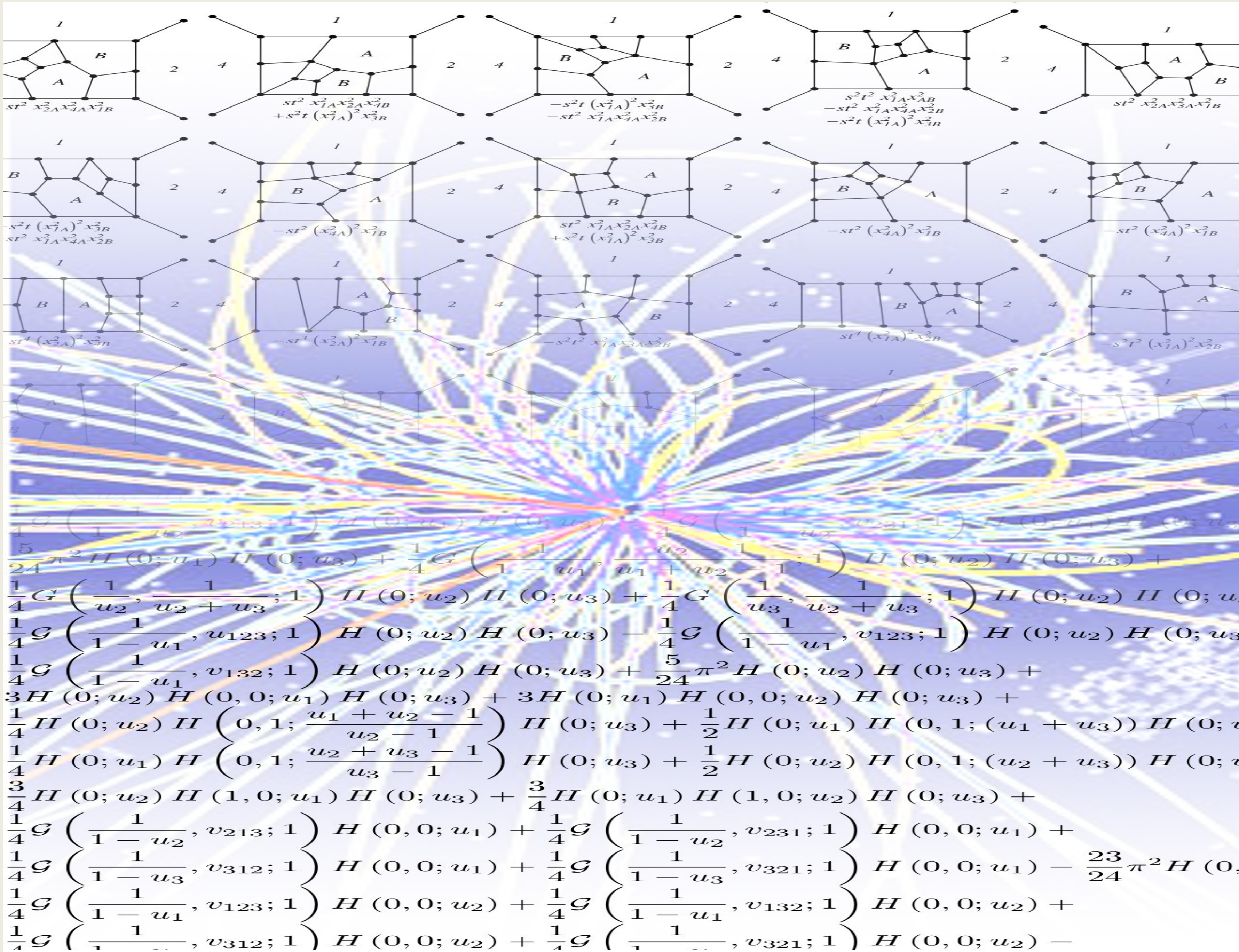
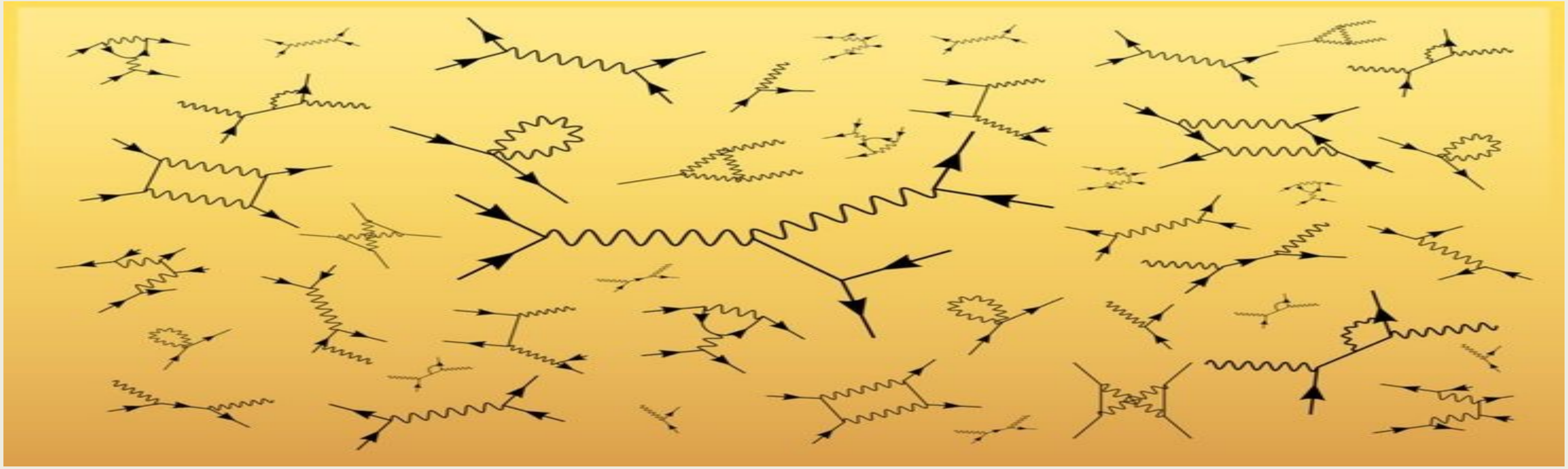


$\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_2)H(0;u_3$
$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)H(0;u_2)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H(0;u_2)H(0;u_3$
$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H(0;u_2)H(0;u_3)+\frac{5}{24}\pi^2H(0;u_2)H(0;u_3)+$
$3H(0;u_2)H(0,0;u_1)H(0;u_3)+3H(0;u_1)H(0,0;u_2)H(0;u_3)+$
$\frac{1}{4}H(0;u_2)H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)H(0;u_3)+\frac{1}{2}H(0;u_1)H(0,1;(u_1+u_3))H(0;u$
$\frac{1}{4}H(0;u_1)H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)H(0;u_3)+\frac{1}{2}H(0;u_2)H(0,1;(u_2+u_3))H(0;u$
$\frac{3}{4}H(0;u_2)H(1,0;u_1)H(0;u_3)+\frac{3}{4}H(0;u_1)H(1,0;u_2)H(0;u_3)+$
$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H(0,0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H(0,0;u_1)+$
$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H(0,0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H(0,0;u_1)-\frac{23}{24}\pi^2H(0,$
$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H(0,0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H(0,0;u_2)+$

Goals of the "Amplitude Program"

- to explore the remarkable, powerful and long-hidden mathematical structure of amplitudes
- to exploit this structure as much as possible to make previously impossible computations possible (including many relevant to LHC)



N=4 Yang-Mills theory

- N=4 Yang-Mills is a quantum field theory which describes “real world” particle interactions. Many techniques we develop for YM have been applied to QCD and Standard Model.
- N=4 Yang-Mills is a string theory (via AdS/CFT), so learning about N=4 Yang-Mills teaches us about quantum gravity.
- N=4 Yang-Mills is solvable (integrable) which is a non-trivial fact for four-dimensional quantum field theory.

Planar N=4 Yang-Mills Amplitudes

- Planar N=4 Yang-Mills scattering amplitudes have been computed to very high loop order.

- They have many remarkable properties, that have sparked interest from mathematicians working on combinatorics, algebraic geometry and number theory.

- At the same time, several methods that have been developed for N=4 Yang-Mills are directly applicable to, and have greatly aided, QCD computations.

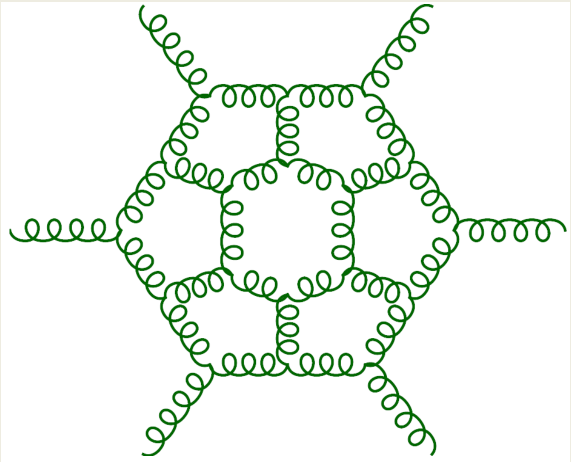
Status: n-point amplitudes in N=4 planar Yang-Mills

- n<6 all loops — Bern, Dixon, Smirnov '05
- n=6 through 7-loops
- n=7 through 4-loops — Caron-Huot, Dixon, Drummond, Dulat, Foster, Gurdogan, von Hippel, McLeod, Papathanasiou, review: 2005.06735
- All n MHV through 2-loops — Caron-Huot '11
- n=8 MHV through 3-loop — Li, Zhang [to appear]
- n=8, 9 NMHV through 2-loops — He, Li, Zhang '19'20

Method: Amplitudes Bootstrap

Write down the answer as linear combo of functions and determine the coefficients by solving a system of linear constraints.

Remaining number of parameters in the ansatz for (MHV, NMHV) n=6 amplitude after each constrain is applied at each loop order:



[Caron-Huot, Dixon, Drummond, Dulat, Foster, Gurdogan, von Hippel, Papathanasiou, review: 2005.06735]

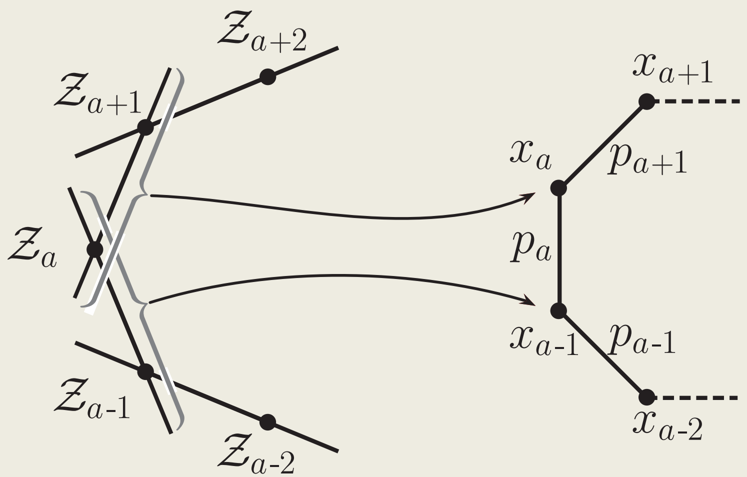
Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. $\mathscr{H}_6$	6	27	105	372	1214	3692?
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795?)
3. Final-entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear	(0,0)	(0,0)	$(0^*,0^*)$	$(0^*,2^*)$	$(1^{*3},5^{*3})$	$(6^{*2},17^{*2})$
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^{*2},2^{*2})$
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^*,0^{*2})$
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	$(1,0^*)$
8. N^3LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. Full MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. T^2 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

Tools for N=4 Yang-Mills Amplitudes

Momentum -> Momentum Twistors

$$\mathbf{Z_i^A = (Z_i^1, Z_i^2, Z_i^3, Z_i^4) \in P^3}$$

$$\langle ijkl \rangle \equiv \langle Z_i Z_j Z_k Z_l \rangle = \det(Z_i Z_j Z_k Z_l)$$



Penrose, Hodges, Arkani-Hamed et al

Tools for N=4 Yang-Mills Amplitudes

MHV and NHMV L-loop amplitudes
can be expressed in terms of
multiple polylogarithms of weight m=2L

$$dF_m = \sum_{\phi_{\alpha_1} \in \Phi} F_{m-1}^{\phi_{\alpha_1}} d\log \phi_{\alpha_1}$$

$$dF_{m-1}^{\phi_{\alpha_1}} = \sum_{\phi_{\alpha_2} \in \Phi} F_{m-2}^{\phi_{\alpha_2},\phi_{\alpha_1}} d\log \phi_{\alpha_2}$$

SYMBOL

$$\mathbf{S}[F_m] = \sum_{\phi_{\alpha_1},\phi_{\alpha_2},\ldots\phi_{\alpha_m} \in \Phi} F_0^{\phi_{\alpha_m},\phi_{\alpha_{m-1}},\ldots,\phi_{\alpha_2}\phi_{\alpha_1}} [\phi_{\alpha_m} \otimes \phi_{\alpha_{m-1}} \otimes \ldots \otimes \phi_{\alpha_2} \otimes \phi_{\alpha_1}]$$

$$dLi_2(z) = -\log(1-z) d\log(z) \to \mathbf{S}[Li_2(z)] = -(1-z) \otimes z$$

SYMBOL ALPHABET

$$\phi_\alpha \in \Phi$$

- encodes singularities
- input in bootstrap

Goncharov, Spradlin, Vergu, AV

n=6

n=6 symbol alphabet is given by 15 letters

all Gr(4,6) Plucker coordinates

<a a+1 b c>

$$R_6^{(2)} = \mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\,\mathrm{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right) + \cdots$$

Del Duca, Duhr, Smirnov;
Goncharov Spradlin Vergu AV

n=7

n=7 symbol alphabet is given by 49 letters

all Gr(4,7) Plucker coordinates <a a+1 b c>

7 cyclic images <1(23)(45)(67)> and <1(27)(34)(56)>

$$R_7^{(2)} = \frac{1}{4}\mathrm{Li}_{2,2}\left(\frac{\langle 1267\rangle\langle 2345\rangle}{\langle 1237\rangle\langle 2456\rangle}, -\frac{\langle 2456\rangle\langle 1(23)(45)(67)\rangle}{\langle 1267\rangle\langle 1456\rangle\langle 2345\rangle}\right) - \frac{1}{2}\mathrm{Li}_{2,2}\left(\frac{\langle 1267\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1567\rangle}, \frac{\langle 1(27)(34)(56)\rangle}{\langle 1267\rangle\langle 1345\rangle}\right) + \cdots$$

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle$$

Caron-Huot;
Golden Goncharov Spradlin Vergu AV

n=8 L=2 NMHV

180 RATIONAL LETTERS

He, Li, Zhang '19: amplitude calculation

- 68 Plücker coordinates of the form $\langle a\; a{+}1\; b\; c\rangle$,
- 8 cyclic images of $\langle 12\bar{4}\cap\bar{7}\rangle$,
- 40 cyclic images of $\langle 1(23)(45)(78)\rangle$, $\langle 1(23)(56)(78)\rangle$, $\langle 1(28)(34)(56)\rangle$, $\langle 1(28)(34)(67)\rangle$ $\langle 1(28)(45)(67)\rangle$,
- 48 dihedral images of $\langle 1(23)(45)(67)\rangle$, $\langle 1(23)(45)(68)\rangle$, $\langle 1(28)(34)(57)\rangle$,
- 8 cyclic images of $\langle \bar{2}\cap(245)\cap\bar{8}\cap(856)\rangle$,
- and 8 distinct dihedral images of $\langle\bar{2}\cap(245)\cap\bar{6}\cap(681)\rangle$.

$$\bar{a}\equiv(a{-}1\; a\; a{+}1)$$

$$\langle ab(cde)\cap(fgh)\rangle=\langle acde\rangle\langle bfgh\rangle-\langle bcde\rangle\langle afgh\rangle$$

$$\langle\bar{x}\cap(abc)\cap\bar{y}\cap(def)\rangle\equiv\langle a,(bc)\cap\bar{x},d,(ef)\cap\bar{y}\rangle$$

$$\langle a,b,c,(de)\cap(fgh)\rangle\equiv\langle abcd\rangle\langle efgh\rangle-\langle abce\rangle\langle dfgh\rangle$$

2 x 9 ALGEBRAIC LETTERS (SQUARE ROOTS)

$$\Delta_{1357}=(\langle1256\rangle\langle3478\rangle-\langle1278\rangle\langle3456\rangle-\langle1234\rangle\langle5678\rangle)^2-4\langle1234\rangle\langle3456\rangle\langle5678\rangle\langle1278\rangle$$

and 1 cyclic

n=8 L=2 NMHV

180 RATIONAL LETTERS

He, Li, Zhang '19

- 68 Plücker coordinates of the form $\langle a\ a{+}1\ b\ c\rangle$,
- 8 cyclic images of $\langle 12\bar{4} \cap \bar{7}\rangle$,
- 40 cyclic images of $\langle 1(23)(45)(78)\rangle$, $\langle 1(23)(56)(78)\rangle$, $\langle 1(28)(34)(56)\rangle$, $\langle 1(28)(34)(67)\rangle$ $\langle 1(28)(45)(67)\rangle$,
- 48 dihedral images of $\langle 1(23)(45)(67)\rangle$, $\langle 1(23)(45)(68)\rangle$, $\langle 1(28)(34)(57)\rangle$,
- 8 cyclic images of $\langle \bar{2} \cap (245) \cap \bar{8} \cap (856)\rangle$,
- and 8 distinct dihedral images of $\langle \bar{2} \cap (245) \cap \bar{6} \cap (681)\rangle$.

Additional 24 letters were very recently found for n=8 L=3 MHV

$$\langle 1(23)(46)(78)\rangle,\ \langle \bar{2} \cap \bar{4} \cap (568) \cap \bar{8}\rangle \text{ and } \langle \bar{2} \cap \bar{4} \cap \bar{6} \cap (681)\rangle$$

2 x 9 ALGEBRAIC LETTERS (SQUARE ROOTS)

Li, Zhang [to appear]

$$\Delta_{1357} = (\langle 1256\rangle\langle 3478\rangle - \langle 1278\rangle\langle 3456\rangle - \langle 1234\rangle\langle 5678\rangle)^2 - 4\langle 1234\rangle\langle 3456\rangle\langle 5678\rangle\langle 1278\rangle$$

and 1 cyclic

n=9

He, Li, Zhang '20: amplitude calculation

59 x 9 RATIONAL LETTERS

- 13 cyclic classes of $\langle 12kl\rangle$ for $3 \leq k < l \leq 8$ but $(k,l) \neq (6,7),(7,8)$;
- 7 cyclic classes of $\langle 12(ijk)\cap(lmn)\rangle$ for $3 \leq i < j < k < l < m < n \leq 9$;
- 8 cyclic classes of $\langle \bar{2}\cap(245)\cap\bar{6}\cap(691)\rangle$, $\langle \bar{2}\cap(346)\cap\bar{6}\cap(892)\rangle$, $\langle \bar{2}\cap(346)\cap\bar{2}\cap(782)\rangle$, $\langle \bar{2}\cap(245)\cap\bar{7}\cap(791)\rangle$, $\langle \bar{2}\cap(245)\cap(568)\cap\bar{8}\rangle$, $\langle \bar{2}\cap(245)\cap(569)\cap\bar{9}\rangle$, $\langle \bar{2}\cap(245)\cap(679)\cap\bar{9}\rangle$, $\langle \bar{2}\cap(256)\cap(679)\cap\bar{9}\rangle$;
- 10 cyclic classes of $\langle 1(i\,i{+}1)(j\,j{+}1)(k\,k{+}1)\rangle$ for $2 \leq i,\ i+1 < j,\ j+1 < k \leq 8$;
- 6 cyclic classes $\langle 1(2i)(j\,j{+}1)(k9)\rangle$ for $3 \leq i < j,\ j+1 < k \leq 8$, but $(i,k) \neq (3,8),(4,7)$;
- 14 cyclic classes of $\langle 1(29)(ij)(k\,k{+}1)\rangle$ for $3 < i < j \leq 8$, $3 \leq k \leq i-2$ or $j+1 \leq k \leq 7$;
- 1 cyclic class of $\langle 1, (56)\cap\bar{3}, (78)\cap\bar{3}, 9\rangle$.

11 x 9 ALGEBRAIC LETTERS (SQUARE ROOTS)

$$\Delta_{1357} = (\langle 1256\rangle\langle 3478\rangle - \langle 1278\rangle\langle 3456\rangle - \langle 1234\rangle\langle 5678\rangle)^2 - 4\langle 1234\rangle\langle 3456\rangle\langle 5678\rangle\langle 1278\rangle$$ and 8 cyclic

So far I told you about
the results of amplitude calculations.
Is there an independent
mathematical description of
symbol letters?

So far I told you about
the results of amplitude calculations.
Is there an independent
mathematical description of
symbol letters?

Yes: Cluster Algebras.

We observed that symbol alphabets are
given by subsets of cluster coordinates of
Grassmannian Cluster Algebra

$$Gr(4, n)$$

Golden, Goncharov, Spradlin, Vergu, AV

Grassmannian Cluster Algebra

$$Gr(4,n)$$

Initial Quiver

$$\begin{tikzcd}
\boxed{\langle 1234\rangle} \arrow[dr] & & & & & \\
& \langle 1235\rangle \arrow[r] \arrow[d] & \langle 1236\rangle \arrow[d] & \cdots & \langle 123\,n-1\rangle \arrow[r] \arrow[d] & \boxed{\langle 123n\rangle} \\
& \langle 1245\rangle \arrow[r] \arrow[d] & \langle 1256\rangle \arrow[ul] \arrow[d] & \cdots & \langle 12\,n-2\,n-1\rangle \arrow[r] \arrow[d] & \boxed{\langle 12\,n-1\,n\rangle} \arrow[ul] \\
& \langle 1345\rangle \arrow[r] \arrow[d] & \langle 1456\rangle \arrow[ul] \arrow[d] & \cdots & \langle 1\,n-3\,n-2\,n-1\rangle \arrow[r] \arrow[d] & \boxed{\langle 1\,n-2\,n-1\,n\rangle} \arrow[ul] \\
& \boxed{\langle 2345\rangle} & \boxed{\langle 3456\rangle} \arrow[ul] & \cdots & \boxed{\langle n-4\,n-3\,n-2\,n-1\rangle} & \boxed{\langle n-3\,n-2\,n-1\,n\rangle} \arrow[ul]
\end{tikzcd}$$

Mutation Rule

$$a_k \to a'_k = \frac{1}{a_k}\left(\prod_{\text{arrows } i\to k} a_i + \prod_{\text{arrows } k\to j} a_j\right)$$

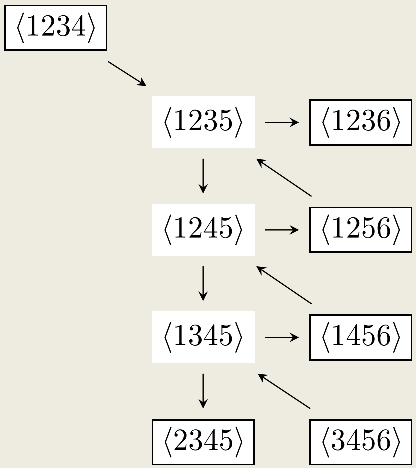
Cluster Coordinates $\{a_k\}$

Fomin, Zelevinsky ‘02; Scott; Gekhtman, Shapiro, Vainshtein

Cluster Algebra Portal: http://www.math.lsa.umich.edu/~fomin/cluster.html

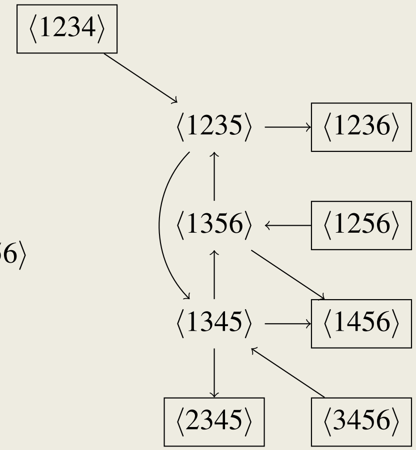
Cluster Coordinates: n=6 and n=7

n=6



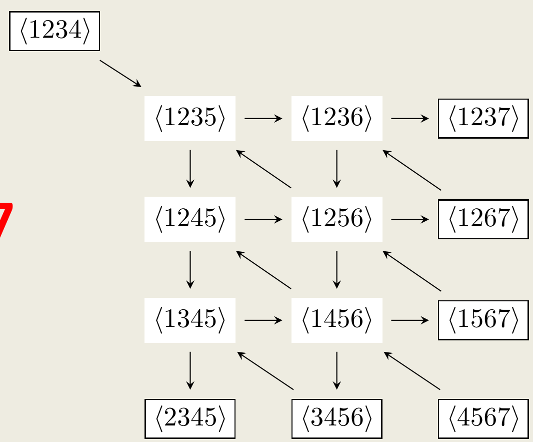
Mutate

$\langle 1245\rangle \to \langle 1356\rangle$



14 quivers give
15 cluster coordinates
<a a+1 b c>

n=7



833 quivers give
49 cluster coordinates <a a+1 b c>,
7 cyclic images <1(23)(45)(67)>, <1(27)(34)(56)>

Matches symbol alphabets for n=6, 7 amplitudes!

Caron-Huot; Golden, Goncharov, Spradlin, Vergu, AV

New Features at n>7

- Gr(4,n) cluster algebra is infinite for n>7

  Fomin, Zelevinsky

- Symbol letters involve square roots

  He, Li, Zhang

New Features at n>7

- Gr(4,n) cluster algebra is infinite for n>7
- Symbol letters involve square roots

Is there a mathematical description?

1. Tropical Geometry Drummond, Foster, Gurdogan, Kalousios ’19 Henke, Papathanasiou ’19 ‘21
2. Dual Polytopes Arkani-Hamed, Lam, Spradlin ’19
3. Plabic Graphs Mago, Schreiber, Spradlin, Yelleshpur, AV ’20 ’21 He, Li ‘20
4. Tensor Diagrams Ren, Spradlin, AV ’21
5. Scattering Diagrams Herderschee ‘21

1. Tropical Geometry

- Speyer-Williams'03 associated a fan to the positive Grassmanian by solving tropicalized Plucker relations (multiplication->addition, addition->minimum).
- Building on this idea Drummond, Foster, Gurdogan, Kalousios'19 looked at a "smaller" version of Gr(4,8) fan by looking at particular Plucker coordinates.
- This fan has 272 rays that are g-vectors for cluster coordinates that include 180 rational n=8 letters.
- There are 2 exceptional rays from which they reproduced 18 algebraic n=8 letters.
- Henke and Papathanasiou'21 generalized this work and obtained n=9 letters.

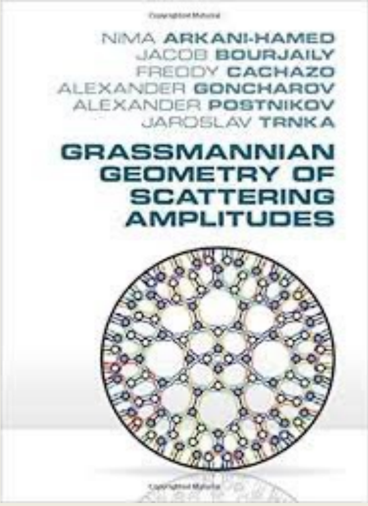
2. Dual Polytopes

- Arkani-Hamed, Lam and Spradlin’19 looked at polytopes dual to these fans.
- To compute variables associated to the exceptional rays they used the method of Chang, Duan, Fraser, Li’19 and found evidence for the expected type of square roots.
- They conjectured these variables come from a generating function of the form

$$\frac{1}{1 - At + Bt^2}$$

$$A = \langle 1\,2\,5\,6\rangle\langle 3\,4\,7\,8\rangle - \langle 1\,2\,7\,8\rangle\langle 3\,4\,5\,6\rangle - \langle 1\,2\,3\,4\rangle\langle 5\,6\,7\,8\rangle$$
$$B = \langle 1\,2\,3\,4\rangle\langle 3\,4\,5\,6\rangle\langle 5\,6\,7\,8\rangle\langle 1\,2\,7\,8\rangle\,.$$

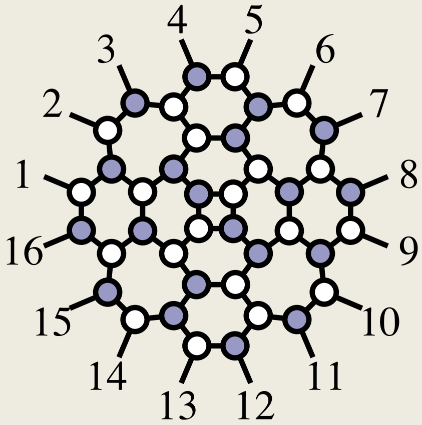
Poles at $A \pm \sqrt{A^2 - 4B}$



3. Plabic Graphs

The building blocks of N=4 SYM amplitudes are Yangian invariants which are given by integrals

$$\mathcal{Y}_{n,k}(\mathcal{Z}) = \frac{1}{\mathrm{vol}[\mathrm{GL}(k)]}\int \frac{d^{k\times n}C_{\alpha a}}{(1\cdots k)(2\cdots k{+}1)\cdots(n\cdots k{-}1)}\prod_{\alpha=1}^{k}\delta^{4|4}(C_{\alpha a}\mathcal{Z}_a)$$

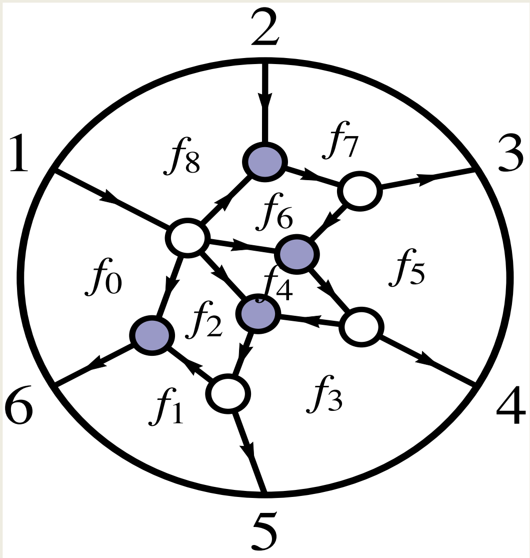


The matrix C parameterizes a cell of the positive Grassmannian; such cells are in correspondence with (equivalence classes) of plabic graphs.

Our Strategy: start with plabic graph, solve C Z=0, compare with known symbol letters.

Mago, Schreiber, Spradlin, Yelleshpur Srikant AV’20 He, Li’20

Example: n=6, k=2



$$C = \begin{pmatrix} 1 & 0 & c_{13} & c_{14} & c_{15} & c_{16} \\ 0 & 1 & c_{23} & c_{24} & c_{25} & c_{26} \end{pmatrix}$$

$$\begin{aligned}
c_{13} &= -f_0 f_1 f_2 f_3 f_4 f_5 f_6\,, & c_{23} &= f_0 f_1 f_2 f_3 f_4 f_5 f_6 f_8\,, \\
c_{14} &= -f_0 f_1 f_2 f_3 f_4 (1+f_6)\,, & c_{24} &= f_0 f_1 f_2 f_3 f_4 f_6 f_8\,, \\
c_{15} &= -f_0 f_1 f_2 (1+f_4+f_4 f_6)\,, & c_{25} &= f_0 f_1 f_2 f_4 f_6 f_8\,, \\
c_{16} &= -f_0 (1+f_2+f_2 f_4+f_2 f_4 f_6)\,, & c_{26} &= f_0 f_2 f_4 f_6 f_8\,.
\end{aligned}$$

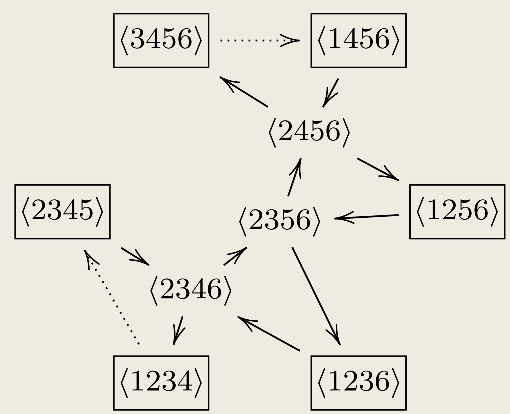
Solution to C Z=0

$$f_0 = -\frac{\langle 1234\rangle}{\langle 2346\rangle}\,, \qquad f_1 = -\frac{\langle 2346\rangle}{\langle 2345\rangle}\,, \qquad f_2 = \frac{\langle 2345\rangle\langle 1236\rangle}{\langle 1234\rangle\langle 2356\rangle}\,,$$

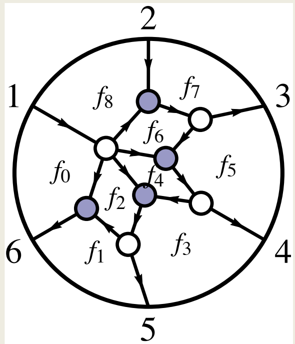
$$f_3 = -\frac{\langle 2356\rangle}{\langle 2346\rangle}\,, \qquad f_4 = \frac{\langle 2346\rangle\langle 1256\rangle}{\langle 2456\rangle\langle 1236\rangle}\,, \qquad f_5 = -\frac{\langle 2456\rangle}{\langle 2356\rangle}\,,$$

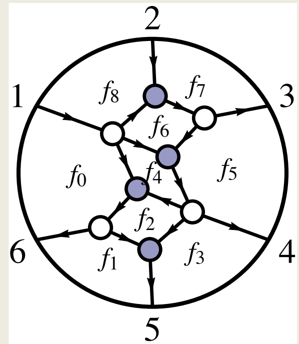
$$f_6 = \frac{\langle 2356\rangle\langle 1456\rangle}{\langle 3456\rangle\langle 1256\rangle}\,, \qquad f_7 = -\frac{\langle 3456\rangle}{\langle 2456\rangle}\,, \qquad f_8 = -\frac{\langle 2456\rangle}{\langle 1456\rangle}\,,$$

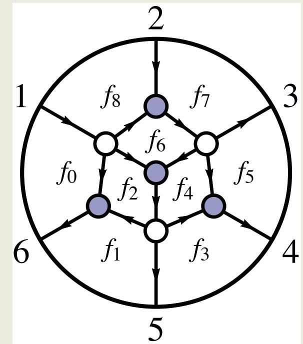
Letters corresponding to this graph can be summarized by quiver:



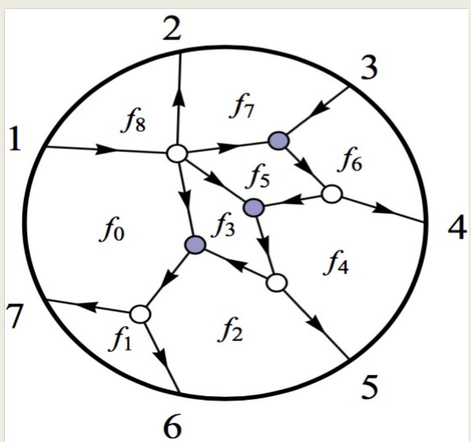
n=6 and n=7

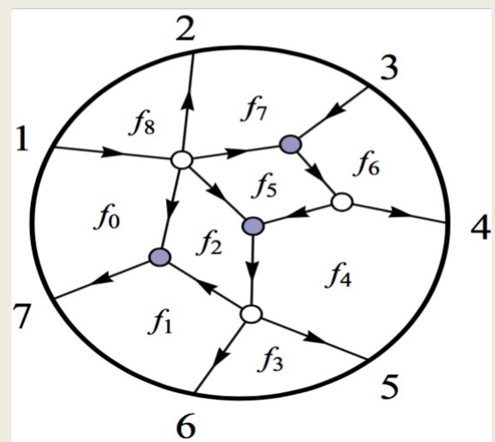






We exactly reproduce n=6 symbol alphabet

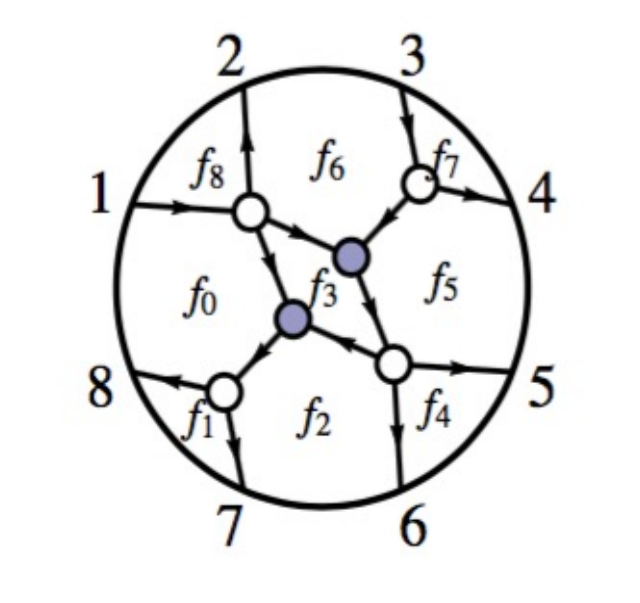




We exactly reproduce n=7 symbol alphabet

Algebraic letters: n=8

This graph gives 8 algebraic letters:



$$f_0 = \sqrt{\frac{\langle 7(12)(34)(56)\rangle \langle 1234\rangle}{a_5 \langle 2(34)(56)(78)\rangle \langle 3478\rangle}}, \quad f_5 = \sqrt{\frac{a_1 a_6 a_9 \langle 3(12)(56)(78)\rangle \langle 5678\rangle}{a_4 a_7 \langle 6(12)(34)(78)\rangle \langle 3478\rangle}},$$

$$f_1 = -\sqrt{\frac{a_7 \langle 8(12)(34)(56)\rangle}{\langle 7(12)(34)(56)\rangle}}, \quad f_6 = -\sqrt{\frac{a_3 \langle 1(34)(56)(78)\rangle \langle 3478\rangle}{a_2 \langle 4(12)(56)(78)\rangle \langle 1278\rangle}},$$

$$f_2 = -\sqrt{\frac{a_4 \langle 5(12)(34)(78)\rangle \langle 3478\rangle}{a_8 \langle 8(12)(34)(56)\rangle \langle 3456\rangle}}, \quad f_7 = -\sqrt{\frac{a_2 \langle 4(12)(56)(78)\rangle}{a_1 \langle 3(12)(56)(78)\rangle}},$$

$$f_3 = \sqrt{\frac{a_8 \langle 1278\rangle \langle 3456\rangle}{a_9 \langle 1234\rangle \langle 5678\rangle}}, \quad f_8 = -\sqrt{\frac{a_5 \langle 2(34)(56)(78)\rangle}{a_3 \langle 1(34)(56)(78)\rangle}},$$

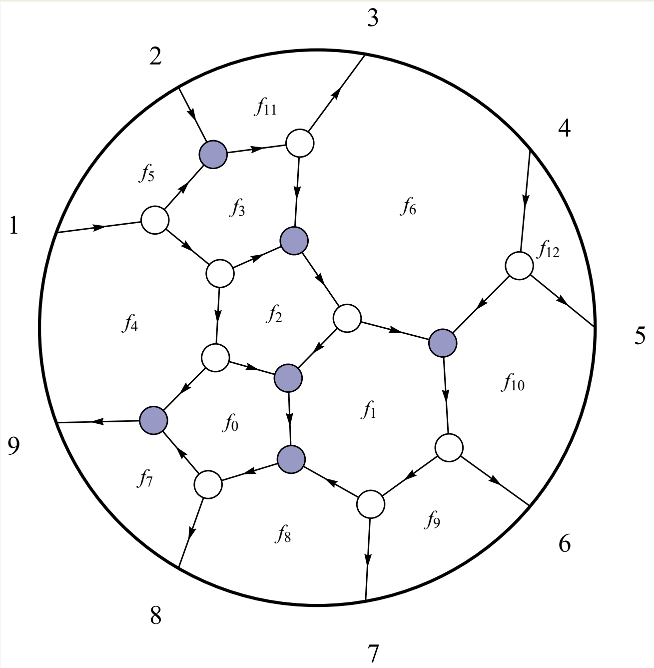
$$f_4 = -\sqrt{\frac{\langle 6(12)(34)(78)\rangle}{a_6 \langle 5(12)(34)(78)\rangle}},$$

$$\sqrt{\Delta_{1357}}$$

To obtain the 9th: square move on f3.

Cycling by one:
we reproduce all n=8 algebraic letters.

Algebraic letters: n=9



Solving C Z =0 we obtain 13 face variables for this graph which can be expressed in terms of a basis of 11 algebraic letters containing

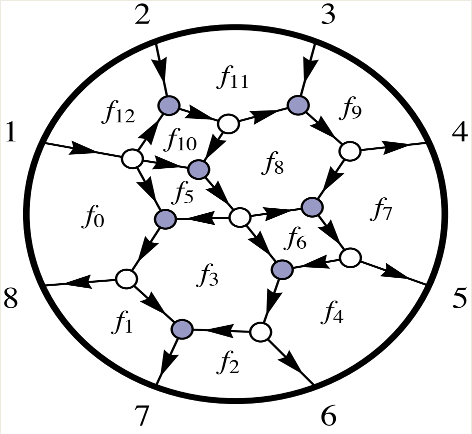
$$\sqrt{\Delta_{3579}}$$

The other 8 square roots can be obtained by cyclic rotations of the external labels.

Performing all possible mutations on the internal faces of this plabic graph we find additional algebraic letters which may appear in higher, not-yet computed n=9 amplitudes.

Rational Letters

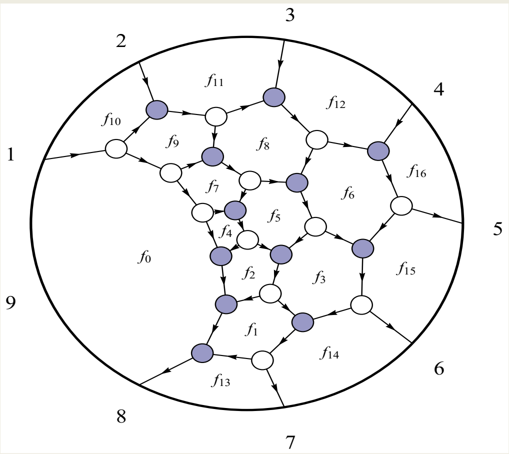
- It is not possible to obtain all rational symbol letters from just plabic graphs.
- We have to consider non-plabic C-matrices.



Mutation of face f_8 gives non-plabic C'

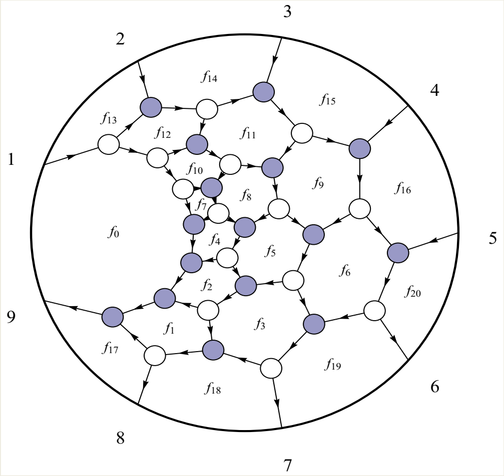
- In some cases, solutions involve non-cluster coordinates.
- We showed that restricting to the top cell (k=n-4) of the Grassmannian but allowing arbitrary non-plabic C-matrices, we will always produce cluster variables.

Rational Letters



Starting with top cell, performing the following 13 mutation sequences, we can obtain extended n=8 alphabet:

$$
\begin{aligned}
&\{\{4,7,8,3,6\},\ \{5,7,9,8,2\},\ \{5,8,3,1,2\},\ \{6,8,7,4,2\},\\
&\{7,1,2,5,6\},\ \{7,2,3,6,5\},\ \{7,4,2,3,6\},\ \{7,5,6,2,1\},\\
&\{8,3,5,2,4\},\ \{8,4,5,1,3\},\ \{8,6,3,2,4\},\ \{9,1,2,5,7\},\ \{9,8,5,3,1\}\}
\end{aligned}
$$



Starting with top cell, performing the following 15 mutation sequences, we can obtain n=9 alphabet:

$$
\begin{aligned}
&\{\{1,3,2,5,8,7,11,12\},\ \{1,5,2,10,8,10,12,11\},\ \{1,5,3,9,5,8,11,12\},\\
&\{2,4,6,5,9,8,11,9\},\ \{2,4,6,9,5,8,12,10\},\ \{2,4,7,8,11,8,12,10\},\\
&\{3,1,6,5,8,9,11,12\},\{3,4,2,5,8,4,7,10\},\ \{4,2,8,9,8,12,10,11\},\\
&\{5,6,3,7,11,10,8,12\},\ \{9,4,2,5,1,3,2\},\ \{9,11,6,4,8,7,10\},\\
&\{10,7,5,3,2,4,5\},\ \{11,6,3,2,4,7,10\},\ \{12,10,1,2,4,8,5\}\},
\end{aligned}
$$

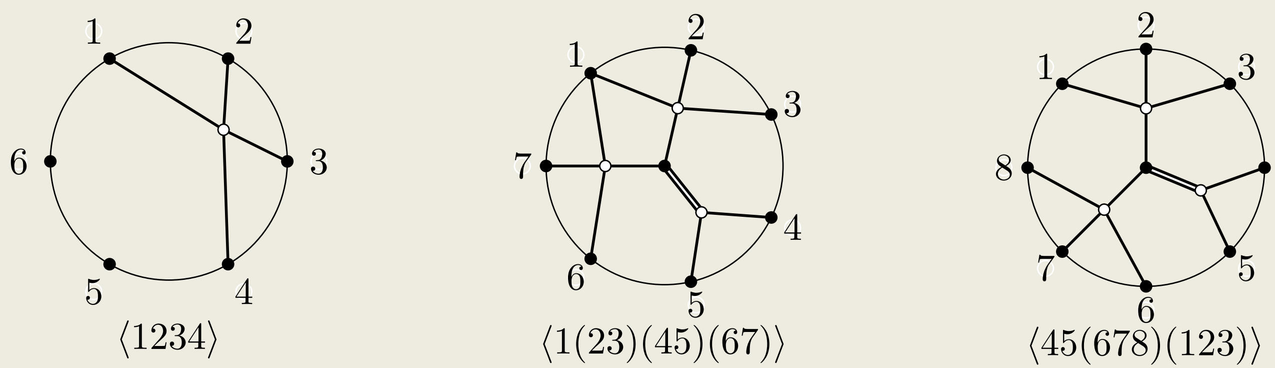
Symbol Alphabet from Plabic Graphs

- We identified set of graphs that reproduced all known n=8 and n=9 symbol alphabets.
- We do not have a theory to explain the pattern of which cells are associated to which symbol letter observed in amplitudes.
- We provided some “phenomenological” data in hope that future work will shed more light on this interesting problem.

Mago, Schreiber, Spradlin, Yelleshput Srikant AV’20 ‘21

4. Tensor Diagrams

Cluster variables can be represented by tensor diagrams Fomin Pilyavsky'16

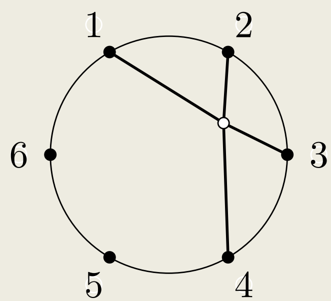


An n-point sl_k tensor diagram is a finite graph drawn inside a circle with n marked points along its boundary, satisfying

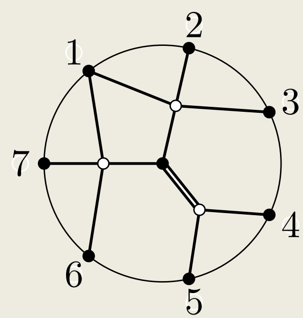
- all boundary vertices are colored black, and can have arbitrary valence
- each internal vertex may be black or white, but must have valence k
- each edge must connect a black and white vertex

4. Tensor Diagrams

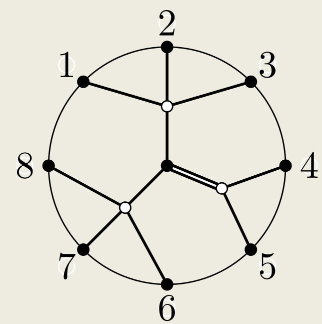
Cluster variables can be represented by tensor diagrams Fomin Pilyavsky'16



$\langle 1234 \rangle$



$\langle 1(23)(45)(67) \rangle$



$\langle 45(678)(123) \rangle$

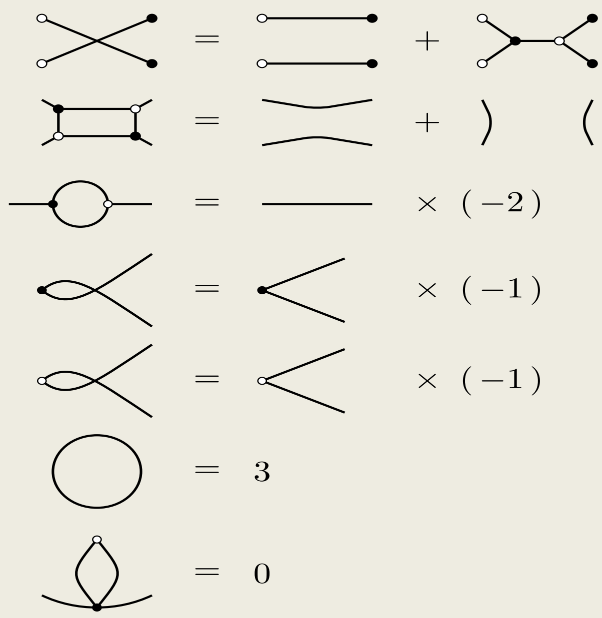
To each diagram D one associates an invariant $[D]$ by assigning

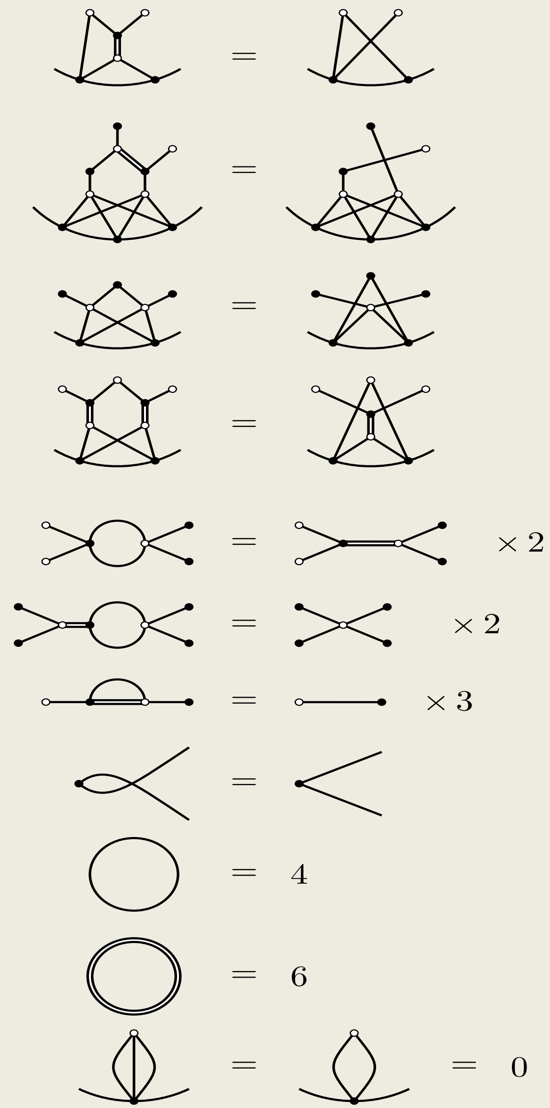
- a momentum twistor Z_i
- $\epsilon^{i_1 \cdots i_k}$ to each white vertex
- $\epsilon_{i_1 \cdots i_k}$ to each black vertex

and then contract the indices together as indicated by the edges.

Skein Relations

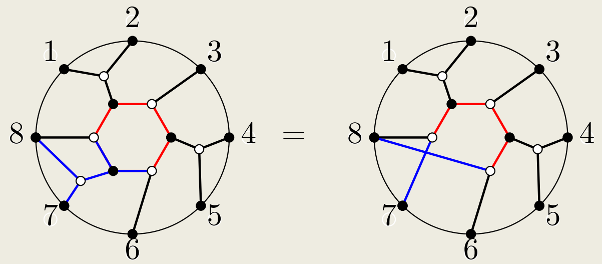
Tensor invariants [D] are invariant under graphical moves called skein relations.

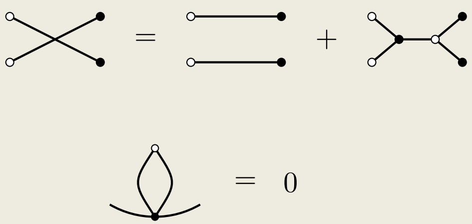




Fomin-Pylyavsky Conjecture

- A web is a planar tensor diagram.
- An aborizable web is a web that can be turned into a tree diagram using skein relations.





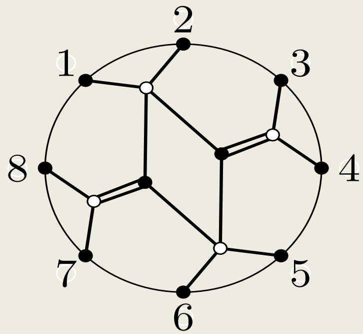
- Fomin-Pilyavsky ‘16 conjecture:

tensor invariants for an arborizable web are in one-to-one correspondence with cluster variables.

[Proven by Fraser ‘17 for Gr(3,9) and Gr(4,8).]

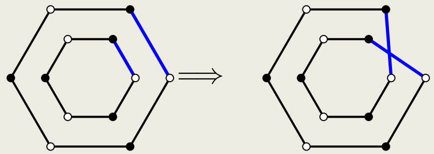
Algebraic Letters from Tensor Diagrams

- We proposed to look at almost aborizable webs (that can be reduced to having one inner loop), and assign to them a "web series"



$$\mathcal{W} = 1 + \sum_{m=1}^{\infty} t^m W_m$$

the coefficients can be derived graphically by twisting the inner loop



- We showed that the series takes the form:

$$\frac{1 - B\,t^2}{1 - A\,t + B\,t^2}$$

$$A = \langle 1256\rangle\langle 3478\rangle - \langle 1278\rangle\langle 3456\rangle - \langle 1234\rangle\langle 5678\rangle$$
$$B = \langle 1234\rangle\langle 3456\rangle\langle 5678\rangle\langle 1278\rangle\,.$$

- We observe square roots in the poles: $A \pm \sqrt{A^2 - 4B}$
- We reproduce square roots up to n=9.

Ren, Spradlin, AV'21

Conclusions

- Symbol Alphabet of N=4 Yang-Mills amplitudes is described by Gr(4,n) cluster algebras for n=6, 7.
- Starting with n=8 one needs a mechanism producing finite subsets in Gr(4,n) and square roots.
- We studied candidate mechanisms coming from plabic graphs and tensor diagrams.
- Future: more systematics, more examples, cluster adjacency, cluster functions, non-N=4 SYM…..