# Three Notions of Brane Gravity Localisation

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# Braneworld localised gravity in an infinite transverse space

The idea of formulating the cosmology of our universe on a brane embedded in a higher-dimensional spacetime dates was initially considered, among others, by Rubakov and Shaposhnikov.

Phys. Lett. B125 (1983), 136

We shall distinguish three different scenarios of “localisation” of gravitational forces with respect to a brane worldvolume embedded in a *noncompact* transverse space, as revealed by analysis of the gravitational fields arising from sources on the worldvolume:

- Type I: Doubly Ricci-flat brane generalisations $\leftrightarrow$ “black spokes”, not fully localised on the lower dimensional worldvolume.
- Type II: Fully localised point sources in the higher dimension $\leftrightarrow$ default Neumann-Dirichlet boundary conditions, Fierz-Pauli massive intermediate-dimensional gravity.
- Type III: Fully localised point sources in the higher dimension $\leftrightarrow$ Robin-Dirichlet boundary conditions, massless gravity on the lower dimensional worldvolume.

# Type I: Doubly Ricci-flat branes; black spokes

For every supersymmetric flat-worldvolume brane solution of a higher dimensional supergravity it is possible to replace the flat worldvolume by a Ricci-flat manifold. Brecher & Perry hep-th/9908018

If there is surviving supersymmetry in the flat-worldvolume solution, this can be extended to an arbitrary solution of a braneworld supergravity for the surviving supersymmetry and the reduction to this braneworld supergravity is a consistent truncation of the original higher dimensional theory. Lü & Pope, hep-th/0008050

Another type of braneworld generalisation can be seen from a complete reduction on the flat worldvolume, leaving a (bosonic) residual purely Euclidean transverse theory comprising a residual Euclidean metric, form fields and a plethora of scalars. Ordinary brane solutions correspond to harmonic maps from the transverse space onto null geodesics in the scalar-field target space, with the simplest such solutions taking the residual transverse metric to be flat. Such flat residual transverse metrics can also be generalised to Ricci-flat metrics. Neugebaur & Kramer (1969) .

Such constructions have been used to put black holes into dimensionally-reduced worldvolume theories

Chamblin, Hawking & Reall, hep-th/9909205 .

One can also make such flat-to-Ricci-flat worldvolume and transverse metric generalisations simultaneously, giving "doubly Ricci-flat" branes.

However, none of the constructions of this type really correspond to gravity *localised* on the lower-dimensional worldvolume. A consistent Kaluza-Klein reduction employs a standard reduction ansatz for the transverse structure above every point on the worldvolume. From a viewpoint in the full higher-dimensional theory, a solution with a point source in the reduced worldvolume theory has an extended structure in the full spacetime.

Chamblin, Hawking and Reall called such solutions "black strings". For solutions with spherical symmetry in the transverse dimensions, a more appropriate image might be "black spokes".

# An approach to gravity localisation: Salam-Sezgin theory and its embedding

Abdus Salam and Ergin Sezgin constructed in 1984 a version of 6D minimal (chiral, *i.e.* (1,0)) supergravity coupled to a 6D 2-form tensor multiplet and a 6D super-Maxwell multiplet which gauges the U(1) R-symmetry of the theory. Phys.Lett. B147 (1984) 47 This Einstein-tensor-Maxwell system has the bosonic Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{SS}} &= \tfrac{1}{2}R - \tfrac{1}{4g^2}e^{\bar\phi}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{6}e^{-2\bar\phi}G_{\mu\nu\rho}G^{\mu\nu\rho} - \tfrac{1}{2}\partial_\mu\bar\phi\partial^\mu\bar\phi - g^2 e^{-\bar\phi}\\ G_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]} + 3F_{[\mu\nu}A_{\rho]}\end{aligned}$$

Note the *positive* potential term for the scalar field $\bar\phi$. This is a key feature of all R-symmetry gauged models generalising the Salam-Sezgin model, leading to models with noncompact symmetries. For example, upon coupling to yet more vector multiplets, the sigma-model target space can have a structure $SO(p,q)/(SO(p)\times SO(q))$.

The Salam-Sezgin theory does not admit a maximally symmetric 6D solution, but it does admit a $(\text{Minkowski})_4 \times S^2$ "vacuum" solution with the flux for a $U(1)$ monopole turned on in the $S^2$ directions

$$
\begin{aligned}
ds_6^2 &= dx^\mu dx^\nu \eta_{\mu\nu} + \frac{1}{4g^2}\,(d\theta^2 + \sin^2\theta d\varphi^2)\,, \\
A_{(1)} &= -\frac{1}{\sqrt{2}g}\,\cos\theta\, d\varphi\,, \qquad G_{(3)} = 0\,, \qquad \bar{\phi} = 0\,.
\end{aligned}
$$

# $\mathcal{H}^{(2,2)}$ embedding of the Salam-Sezgin theory

A way to obtain the Salam-Sezgin theory from M theory was given by Cvetič, Gibbons & Pope (CGP). Nucl. Phys. B677 (2004) 164 This employed a reduction from 10D type IIA supergravity on the space $\mathcal{H}^{(2,2)}$, or, equivalently, from 11D supergravity on $S^1 \times \mathcal{H}^{(2,2)}$. The $\mathcal{H}^{(2,2)}$ space is a cohomogeneity-one 3D hyperbolic space which can be obtained by embedding into $R^4$ via the condition $\mu_1^2 + \mu_2^2 - \mu_3^2 - \mu_4^2 = 1$.

The transverse-space $\mathcal{H}^{(2,2)}$ admits a natural $SO(2,2)$ group action and the resulting 7D supergravity theory has maximal (32 supercharge) supersymmetry with a gauged $SO(2,2)$ symmetry, linearly realised on $SO(2) \times SO(2)$. This fits neatly into the general scheme of extended Salam-Sezgin gauged models.

# Lifting back to $D = 10$

In the Einstein frame, the CGP lift of the 6D SS vacuum is a 10D nonsingular solution (where $\mu = 0, 1, 2, 3$ correspond to the 4D subspace) that can be written Crampton, Pope & KSS, 1408.7072

$$
\begin{aligned}
d\hat{s}_{10}^2 &= H_{\mathrm{SS}}^{-\frac{1}{4}} \big(dx^\mu dx_\mu + dy^2 + \tfrac{1}{4g^2}\,[d\psi + \operatorname{sech} 2\rho\,(d\chi + \cos\theta\, d\varphi)]^2\big) + H_{\mathrm{SS}}^{\frac{3}{4}}\, d\bar{s}_4^2 \\
e^{\hat{\phi}} &= H_{\mathrm{SS}}^{\frac{1}{2}}\,, \quad \hat{A}_2 = \frac{1}{4g^2}\left[d\chi + \operatorname{sech} 2\rho\, d\psi\right] \wedge (d\chi + \cos\theta\, d\varphi)
\end{aligned}
$$

where

$$
\begin{aligned}
d\bar{s}_4^2 &= \big(\cosh 2\rho d\rho^2 + \tfrac{1}{4}\cosh 2\rho(d\theta^2 + \sin\theta d\phi^2) \\
&\qquad + \tfrac{1}{4}\sinh 2\rho \tanh 2\rho(d\chi + \cos\theta d\phi)^2\big) \\
H_{\mathrm{SS}} &= \operatorname{sech} 2\rho\ .
\end{aligned}
$$

The $d\bar{s}_4^2$ noncompact part of the transverse space metric is a form of the 4-dimensional Eguchi-Hanson metric. The coordinates $y$ and $\psi$ correspond to $S^1$ circles.

# Bound states and mass gaps Crampton, Pope & K.S.S.

An approach to obtaining the localisation of gravity on the 4D subspace is to look for a *normalizable* transverse-space wavefunction $\xi(\rho)$ for $h_{\mu\nu}(x,\rho) = h_{\mu\nu}(x)\xi(\rho)$ with a *mass gap* before the onset of the continuous massive Kaluza-Klein spectrum. This could be viewed as analogous to an effective field theory for electrons confined to a metal by a nonzero work function.

General study of the fluctuation spectra about brane solutions shows that the mass spectrum of the spin-two fluctuations about a brane background is given by the spectrum of the scalar Laplacian in the transverse embedding space of the brane.

Csáki, Erlich, Hollowood & Shirman, Nucl.Phys. B581 (2000) 309; Bachas & Estes, JHEP 1106 (2011) 005

$$\begin{aligned}
\Box_{(10)}F &= \frac{1}{\sqrt{-\det g_{(10)}}}\partial_M\left(\sqrt{-\det g_{(10)}}g^{MN}_{(10)}\partial_N F\right) \\
&= H_{SS}^{\frac{1}{4}}(\Box_{(4)} + g^2\triangle_{\theta,\phi,y,\psi,\chi} + g^2\triangle) \\
H_{SS} &= (\cosh 2\rho)^{-1} \text{ warp factor;} \quad \triangle = \frac{\partial^2}{\partial\rho^2} + \frac{2}{\tanh(2\rho)}\frac{\partial}{\partial\rho}
\end{aligned}$$

The directions $\theta, \phi, y, \psi$ & $\chi$ are all compact, and one can employ ordinary Kaluza-Klein methods for reduction on them, truncating to the invariant sector for these coordinates, but still allowing dependence on the noncompact coordinate $\rho$.

To handle the noncompact direction $\rho$, one needs to expand all fields in eigenmodes of $\triangle$:

$$\phi(x^\mu, \rho) = \sum_i \phi_{\omega_i}(x^\mu)\xi_{\omega_i}(\rho) + \int_\Lambda^\infty d\omega \phi_\omega(x^\mu)\xi_\omega(\rho)$$

where the $\phi_{\omega_i}$ are discrete eigenmodes and the $\phi_\omega$ are continuous Kaluza-Klein eigenmodes. Their eigenvalues give the Kaluza-Klein masses $g\omega$ in 4D from the wave equation $\Box_{(10)}\phi_\omega = 0$ using $\triangle_{\theta,\phi,y,\psi,\chi}\phi_\omega = 0$

$$\begin{aligned} \triangle\xi_\omega &= -\omega^2\xi_\omega \\ \Box_{(4)}\phi_\omega &= (g^2\omega^2)\phi_\omega \end{aligned}$$

# The Schrödinger equation for $\mathcal{H}^{(2,2)}$ eigenfunctions

One can rewrite the $\triangle$ eigenvalue problem as a Schrödinger equation by making the substitution

$$\Psi_\omega = \sqrt{\sinh(2\rho)}\xi_\omega$$

after which the eigenfunction equation takes the Schrödinger equation form

$$-\frac{d^2\Psi_\omega}{d\rho^2} + V(\rho)\Psi_\omega = \omega^2\Psi_\omega$$

where the potential is

$$V(\rho) = 2 - \frac{1}{\tanh^2(2\rho)}$$

The SS Schrödinger equation potential $V(\rho)$ asymptotes to the value 1 for large $\rho$. In this limit, the Schrödinger equation becomes

$$\frac{d^2\Psi_\omega}{d\rho^2} + (\omega^2 - 1)\Psi_\omega = 0$$

giving "scattering-state" solutions for $\omega^2 > 1$:

$$\Psi_\omega(\rho) \sim \left(A_\omega e^{i\sqrt{\omega^2-1}\rho} + B_\omega e^{-i\sqrt{\omega^2-1}\rho}\right) \qquad \text{for large } \rho$$

while for $\omega^2 < 1$, one can have $L^2$ normalizable bound states. Recalling the $\rho$ dependence of the measure $\sqrt{-g_{(10)}} \sim (\cosh(2\rho))^{\frac{1}{4}} \sinh(2\rho)$, one finds for large $\rho$

$$\int_{\rho_1 \gg 1}^{\infty} |\Psi_\omega(\rho)|^2 d\rho < \infty \Rightarrow \Psi_\omega \sim B_\omega e^{-\sqrt{1-\omega^2}\rho} \text{ for } \omega^2 < 1$$

So for $\omega^2 < 1$ we can have candidate bound states, then a *mass gap* up to the edge of the scattering states' continuum spectrum.

# The zero-mode bound state and massless 4D gravitons

The 1-D Schrödinger system with the $V(\rho) = 2 - \coth^2(2\rho)$ potential belongs to a special class of Pöschl-Teller integrable systems. Study of this system, and in particular of its self-adjointness properties, shows that it has a *unique* bound state separated by a mass gap before the onset of a continuum of delta-function-normalizable scattering states.

Happily, for $\omega = 0$ the Schrödinger equation can be solved exactly. The normalised result is

$$\Psi_0(\rho) = \sqrt{\sinh(2\rho)}\xi_0(\rho) = \frac{2\sqrt{3}}{\pi}\sqrt{\sinh(2\rho)}\log(\tanh\rho)$$

Metric excitations $h_{\mu\nu}(x)\xi_0(\rho)$ around the 10D lifted SS background correspond, at the linearised level, to massless 4D gravitons on the 4D worldvolume subspacetime.

# Newtonian behaviour for a genuinely localised source

To understand better the nature of the braneworld localised gravity, consider now a linearised version of a black-hole type solution in the higher dimension and look for long-distance 4D Newtonian behaviour on the worldvolume. For this, the stress-tensor source will be taken to be a point, not a spoke, from the higher-dimensional perspective.

Having said that, the full 10D situation is rather complicated to analyse. For suitably symmetric solutions in the dimensions transverse to the 4D worldvolume and for spherical symmetry on the 4D worldvolume, one may simplify the analysis by reducing on all the $\theta, \phi, y, \psi$ & $\chi$ compact coordinates and considering the gravitational response to such a point source with $h_{\mu\nu}(r, \rho)$ depending only on the transverse and worldvolume "radial" coordinates. The detailed analysis can accordingly begin in $D = 5$.

Note that, for this Newtonian behaviour analysis, one does not assume transverse $h_{\mu\nu}$ structure depending only on $\xi_0(\rho)$.

Now consider how further to simplify the 5D analysis. Unlike the perturbative metric analysis of spin-two gravity waves without a stress-tensor source, for which $h_{\mu\nu}$ can be assumed to be traceless, one does not have such a luxury in the perturbative Newtonian problem with a point source. This removes the automatic reduction of the transverse wavefunction problem to that of a scalar field which one has in the gravity-wave study.

Nonetheless, careful study of the sourced perturbative field equations at leading order still reduces the Newtonian problem to the study of $h_{00}$ alone, and in de Donder gauge this becomes just the study of a scalar Green function on the SS background.

After reduction to $D = 5$, one works with a Lagrangian obtained from Type I supergravity reduced on $T^3 \times S^2$:

$$\mathcal{L}_5 = R * 1 - \frac{1}{2} d\Phi_i \wedge * d\Phi_i - \frac{1}{2} e^{\sqrt{2}\Phi_1} d\sigma \wedge * d\sigma - V * 1 \,,$$

where the scalar potential $V$ is

$$V = 2g^2 e^{\sqrt{\frac{2}{5}}\Phi_2 - \frac{8}{\sqrt{15}}\Phi_3} \Big( e^{-\sqrt{2}\Phi_1} + \sigma^2 + \frac{1}{4} e^{\sqrt{2}\Phi_1} (\sigma^2 - 2)^2 - 4 e^{-\sqrt{\frac{2}{5}}\Phi_2 + \sqrt{\frac{3}{5}}\Phi_3} \Big).$$

The background Salam-Sezgin vacuum solution in this 5D presentation is

$$ds_5^2 = (\sinh 2\rho)^{\frac{2}{3}} \Big( \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{g^2} d\rho^2 \Big) \,, \qquad e^{-\sqrt{2}\Phi_1} = (\tanh 2\rho)^2 \,,$$

$$e^{\sqrt{10}\Phi_2} = e^{\sqrt{15}\Phi_3} = (\sinh 2\rho)^2 \,, \qquad \sigma = \sqrt{2}\, \mathrm{sech}\, 2\rho \,.$$

Letting the 5D metric fluctuations be denoted $H_{MN}$, choosing the de Donder gauge sets $\partial^M H_{Mz} - \frac{1}{2}\partial_z H = 0$, after which the fluctuation of $\Phi_1$ decouples at leading order and can be set to zero.

Moreover, for a time-independent solution, one can set $H_{0\rho} = 0$ and then one can further reduce the system to have simply

$$\Delta_5 H_{00} = 0\,, \quad \Delta_5 H_{\rho\rho} = 0$$

where, for time-independent spherically symmetric solutions in the $\mathbb{M}^{1,3}$ worldvolume directions, one has the operator

$$\Delta_5 = {\partial_r}^2 + \frac{2}{r}\partial_r + g^2\big({\partial_\rho}^2 + 2\coth 2\rho\,\partial_\rho\big)$$

It is for this $\Delta_5$ operator that one needs to find the Green function.

# Green functions for the $\Delta_5$ operator

When one includes now a point source of mass $M$ at the origin in the 5D problem, one needs to solve the sourced Green-function equation

$$\Delta_5 G(r,\rho) = \frac{\hat{\kappa}^2 M \delta(r)\delta(\rho)}{4\pi g r^2 \sinh 2\rho}\,,$$

where $\hat{\kappa}^2$ is the higher-dimensional Newton constant.

Near the $(r,\rho) = (0,0)$ origin, and defining $R^2 = g^2 r^2 + \rho^2$, one finds the asymptotic behaviour as $R \to 0$

$$G(r,\rho) = -\frac{g^2 \hat{\kappa}^2 M}{2\pi \left(g^2 r^2 + \rho^2\right)^{\frac{3}{2}}} + \mathcal{O}\left(\frac{1}{R^2}\right)\,,$$

*i.e.* $R^{-3}$ structure, which is the characteristic structure of a Green function in flat six-dimensional spacetime.

Why flat 6D spacetime behaviour near the origin? Two reasons:

- Eguchi-Hansen space has a smooth "nose" near $\rho = 0$, with asymptotic structure $\mathbb{R}^2 \times S^2$
- Although we have been reducing on the angular $\chi$ coordinate for simplicity in obtaining $\mathcal{L}_5$, it is the natural $\mathbb{R}^2$ angular coordinate associated with the radial coordinate $\rho$ near $\rho = 0$.

So near the $(r, \rho) = (0, 0)$ origin, the Green function problem asymptotically limits to a 6D flat spacetime problem, for which a time-independent Green function has $R^{-3}$ structure.

# Type II structure: Default boundary conditions at the worldvolume

Starting out with 6D $R^{-3}$ structure near the $(r, \rho) = (0, 0)$ origin, the Newtonian problem then becomes: What happens on or near the $\rho = 0$ worldvolume at large worldvolume radius $r$? Here is where the choice of boundary conditions to be adopted in solving the sourced Green function equation becomes key.

Away from the $(r, \rho) = (0, 0)$ source, one natural setup from a higher-dimensional point of view (10D really, but 6D in our reduced theory) would be to require nonsingularity of the metric everywhere, including at the $\rho = 0$ worldvolume. Then, for a circularly symmetric solution in the angular $\chi$ coordinate, Green function solutions needing to be non-singular at $\rho = 0$ need to satisfy Neumann boundary conditions $\partial_\rho G = 0$ at $\rho = 0$, as well as standard $G \to 0$ Dirichlet type boundary conditions at infinity in all directions.

The sourced $\Delta_5 G(r,\rho)$ equation can then be solved by separation of variables, writing $G(r,\rho) = \int d\omega f_\omega(r)\zeta_\omega(\rho)$, where the transverse eigenmodes $\zeta_\omega$ obey the Neumann-Dirichlet conditions (where the $\sqrt{\sinh 2\rho}$ Dirichlet factor is needed for normalisability)

$$\partial_\rho \zeta_\omega(\rho)\Big|_{\rho=0} = 0\,, \qquad \sqrt{\sinh(2\rho)}\;\zeta_\omega(\rho)\Big|_{\rho=\infty} < \infty$$

and they must also satisfy orthonormality conditions

$$\int_0^\infty \sinh(2\rho)\zeta_i(\rho)\zeta_j(\rho)d\rho = \delta_{i,j}$$

for bound state modes and

$$\int_0^\infty \sinh(2\rho)\zeta_\omega(\rho)\zeta_\tau(\rho)d\rho = \delta(\omega-\tau)$$

for scattering state modes.

The result of the analysis for such Neumann-Dirichlet transverse eigenmodes is that there are no bound states in the expansion of $G(r,\rho)$ and the scattering states have separation eigenvalues $\omega > 1$ and are given by

$$\zeta_\omega(\rho) = \mathcal{N}_\omega \, \mathcal{P}_{-\frac{1}{2}+\frac{\sqrt{1-\omega^2}}{2}} \left(\cosh\left(2\rho\right)\right) \ ,$$

where $\mathcal{P}_\nu$ is a Legendre function of the first kind.

For the corresponding $f_\omega(r)$, one has

$$f_\omega(r) = -\frac{\hat{\kappa}^2 M}{4\pi g} \frac{\exp(-g\omega r)}{r} \ .$$

Putting together the transverse and worldvolume parts, the Green function is

$$G(r,\rho-\eta) = -\int_1^\infty \frac{\hat{\kappa}^2 M \exp(-g\omega r)}{4\pi g r} \zeta_\omega(\rho)\zeta_\omega(\eta) d\omega \ .$$

Indeed, for $\eta << 1$ and $R << 1$, one finds

$$G(r,\rho-\eta) = -\frac{g^2\hat{\kappa}^2 M}{2\pi\left(g^2r^2+(\rho-\eta)^2\right)^{\frac{3}{2}}} + \mathcal{O}\left(\frac{1}{R^2}\right) \ ,$$

behaving as required like $R^{-3}$ as $R \to 0$. For $r >> 1$ near the world volume, however, one finds a structure

$$G(r,\rho-\eta) = \exp(-gr)\left(-\frac{X}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)\right) \ ,$$

where $X$ is a $\rho$-dependent function. This is not 4D Newtonian $1/r$ behaviour on or near the worldvolume – it would be more characteristic of a massive spin-two force in 5D spacetime.

# Type III structure: 4D Newtonian massless gravity

To see what happened with the failure of the "default" higher dimensional boundary conditions at $\rho = 0$, remember how massless worldvolume gravitons come about: they need a $\xi_0(\rho)$ transverse wavefunction, the single $L^2$ normalisable bound state of the transverse Pöschl-Teller Schrödinger system. That wavefunction does not satisfy the "default" Neumann boundary condition at $\rho = 0$.

To see how the boundary condition at $\rho = 0$ needs to be changed in order to include $\xi_0$ into the expansion basis, it is easiest to look at the conditions for self-adjointness of $\triangle = \partial_\rho^2 + 2\coth(2\rho)\partial_\rho$, the transverse part of the $\Delta_5$ operator. For any two functions $f, g$ to be in the self-adjoint domain of $\triangle$, one requires

$$\int_0^\infty \sinh(2\rho)\,(f\Delta g - g\Delta f)\,d\rho = \sinh(2\rho)\,(f\partial_\rho g - g\partial_\rho f)\,\Big|_{\rho=0}^{\rho\to\infty} = 0.$$

Requiring $\xi_0 = \text{const} \times \log\tanh\rho$ to be included in the self-adjoint domain of $\triangle$ leads to changed boundary conditions at $\rho = 0$:

$$(\sinh(2\rho) \log\tanh\rho\, \partial_\rho - 2)\, \xi_\omega(\rho)\Big|_{\rho=0} = 0\ ,\ \sqrt{\sinh(2\rho)}\, \xi_\omega(\rho)\Big|_{\rho\to\infty} < \infty,$$

*i.e.* a generalised Robin boundary condition at $\rho = 0$ together with the Dirichlet condition as $\rho \to \infty$.

Given these boundary conditions, the transverse wavefunctions now include $\xi_0$ and the transverse scattering modes, with eigenvalues $\omega > 1$, are

$$\xi_\omega(\rho) = \mathcal{M}_\omega \mathcal{Q}_{-\frac{1}{2}+\frac{\sqrt{1-\omega^2}}{2}}(\cosh(2\rho)) + c.c.\ ,$$

where $\mathcal{Q}_\nu$ is a Legendre function of the second kind.

Constructing anew the full Green function

$$G(r,\rho-\eta)=-\frac{\hat{\kappa}^2M}{4\pi gr}\xi_0(\rho)\xi_0(\eta)-\int_1^\infty\frac{\hat{\kappa}^2M\exp(-g\omega r)}{4\pi gr}\xi_\omega(\rho)\xi_\omega(\eta)d\omega\ ,$$

for the leading behavior in the large worldvolume $r$ regime one now finds

$$G(r,\rho-\eta)=-\frac{\hat{\kappa}^2M}{4\pi gr}\xi_0(\rho)\xi_0(\eta)+\mathcal{O}\left(\exp(-gr)\right)\ ,$$

showing proper 4D $1/r$ dependence of the Green function, and hence of the gravitational potential $H_{00}(r,\rho)$ on the worldvolume radius.

Near the $(r,\rho)=(0,0)$ origin, however, the potential retains its 6D characteristic $1/R^3$ behaviour, the same as in the Type II default boundary condition case.

# Newton's constant

With the modified Robin-Dirichlet boundary conditions, one has achieved a proper 4D localisation of gravity at large distances $r$ away from the mass $\mathcal{M}$ source at the origin. There still is dependence of the potential on the transverse coordinate $\rho$, however, behaving like $\log \rho$ as $\rho \to 0$. How to identify Newton's constant for an effective theory on the worldvolume?

One way to identify Newton's constant in the worldvolume theory is to proceed in the same way as in the effective theory for gravitons Crampton,Pope&KSS . There, the massless spin-two effective theory action was obtained by integrating over the transverse coordinate $\rho$, with all worldvolume $h_{\mu\nu}(r,\rho) = h_{\mu\nu}(r)\xi_0(\rho)$ modes having $\xi_0(\rho)$ transverse wavefunctions, which clearly concentrate near $\rho = 0$.

In the Newtonian problem, this would correspond to a certain "blurriness" about just where the worldvolume is located within the higher-dimensional spacetime. Providing a $\xi_0(\rho)$ profile function and averaging the gravitational potential over the transverse $\rho$ direction then yields exactly the same Newton constant as that obtained from the graviton action:

$$G_4 = \frac{3888\,\zeta(3)^2\,G_{10}g^5}{\pi^8\ell_y}$$

(where $\ell_y$ is the circumference of the compactified $y$ coordinate).

# Overview

- Obtaining lower-dimensional gravitational behaviour requires a normalisable transverse wavefunction zero-mode like $\xi_0(\rho)$. With an infinite transverse space, normalisability rules out a constant transverse wavefunction.
- To allow for a transverse wavefunction like $\xi_0$, boundary conditions as one approaches the worldvolume need to be modified with respect to the default Neumann conditions.
- Implementing such modified transverse boundary conditions near the worldvolume, one can achieve 4D lower-dimensional gravitational behaviour at large worldvolume $r$ distances, while also preserving higher-dimensional "near-field" behaviour as one gets close to a stress-tensor source.
- In the hyperbolic higher-dimensional model studied, the emergence of low-energy structure is related to the existence of a mass gap in the transverse wavefunction eigenvalue spectrum.

This structure clearly has similarities to the detailed study of the Randall-Sundrum II model. Giddings, Katz & Randall, hepth/0002091