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Superconformal approach to N=4 supergravity.

N=4 Supergravity in four dimensions: A brief review

- N=4 supergravity in four dimensions refers to a theory of gravity that preserves 16 local supersymmetries encoded in 4 Majorana spinors.
- Since 32 local supersymmetries (or N=8) is the maximum number of supersymmetries possible in a theory of gravity in four dimensions, N=4 is often referred to as "half maximal supergravity".
- It is well known that supersymmetry modifies the ultraviolet properties of a theory of gravity.
- However for $N \geq 4$ supergravity theories Electromagnetic duality symmetries also plays a major role in constraining the UV behavior (Cremmer and Julia 1979, Cremmer, Scherk and Ferrara 1982)
- The duality group for N=4 is SU(1,1). It was found that a U(1) subgroup of the duality group is anomalous (Carrasco, Kallosh, Roiban and Tseytlin, 1303.6219).
- This anomaly sources some amplitudes which would otherwise vanish at the tree level due to the duality symmetry. And it is responsible for the 4-loop divergence without affecting the 3-loop finiteness.

N=4 Supergravity in four dimensions: A brief review

- It was found that the anomaly can be removed by the addition of local counter-terms. Some characteristic terms $\sim \int f(\tau) R R^* d^4x$ was found to cancel some of the anomalous amplitudes.
- Since we do not expect supersymmetry to be anomalous, one would expect that the supersymmetric completion of such a counter term exists.
- The focus of our work is to find a supersymmetric completion of such a counter term.
- We will use the superconformal approach for this. We will find a general class of higher derivative action characterized by a holomorphic function of the *coset scalars.* A specific choice of the function would be suitable for the supersymmetric completion of the counter term.

N=4 conformal supergravity

- Conformal supergravity is a theory of supergravity with extra symmetries such as: dilatation, special conformal transformation, R-symmetry, S-supersymmetry
- The extra symmetries allows us to systematically include the degrees of freedom into shorter multiplets and allows us to systematically construct the physical Poincare supergravity theories in a tractable manner.
- In this approach, we need two major ingredients: 1) The Weyl multiplet 2) Compensating multiplets
- Typically or $N \leq 4$ the Weyl multiplet is off-shell (i.e the superconformal algebra is realized on the multiplet without using equations of motion). The compensating multiplets can be off-shell for $N \leq 2$ but not for N>2. Hence the final supergravity that we obtain for N>2 is on-shell.

N=4 conformal supergravity

The N=4 Weyl multiplet: $e_\mu^a\,, V_\mu{}^i{}_j\,, b_\mu\,, E_{ij}\,, T_{ab}{}^{ij}\,, \phi_\alpha\,, fermions$

The field $\phi_\alpha : \alpha = 1{,}2$ satisfies the constraint $\phi^\alpha\phi_\alpha = 1 : \phi^\alpha \equiv \eta^{\alpha\beta}\phi^*_\beta$. There is a rigid SU(1,1) that acts linearly on ϕ_α and there is a spurious local U(1) that acts as $\phi_\alpha = e^{-i\Lambda(x)}\phi_\alpha$. This spurious U(1) acts on the other fields of the Weyl multiplet and the corresponding gauge field is a composite gauge field given as $a_\mu = -\phi^\alpha\partial_\mu\phi_\alpha + fermions$

The physical scalar τ arises upon imposing a suitable gauge fixing condition on ϕ_α and parametrizes the coset $SU(1{,}1)/U(1)$. In the gauge fixed theory the rigid $SU(1{,}1) \sim SL(2{,}\mathbb{R})$ will act non linearly on τ. The $SL(2{,}\mathbb{R})$ will act on all the fields of the Weyl multiplet which was transforming under the local U(1).

N=4 conformal supergravity

de Roo, NPB 255, 1985

N=4 Vector Multiplet: A_μ , ϕ_{ij}, ψ_i , where $\phi_{ij} = -\frac{1}{2}\varepsilon_{ijkl}\phi^{kl} : \phi^{kl} \equiv (\phi_{kl})^*$

The rigid SU(1,1) acts on A_μ and its dual as shown below:

$$\begin{pmatrix} F^+_{1\mu\nu} \\ F^+_{2\mu\nu} \end{pmatrix} \to C \begin{pmatrix} F^+_{1\mu\nu} \\ F^+_{2\mu\nu} \end{pmatrix}$$

where , $C \in SU(1,1)$ and $F^+_{1\mu\nu} = \frac{1}{2}\left(G^+_{\mu\nu} - F^+_{\mu\nu}\right)$, $F^+_{2\mu\nu} = \frac{1}{2}\left(G^+_{\mu\nu} + F^+_{\mu\nu}\right)$, $G^+_{\mu\nu} \equiv -\frac{2}{e}\frac{\delta\mathscr{L}}{\delta F^+_{\mu\nu}}$

In order to go from N=4 conformal supergravity to Poincare supergravity a minimum of six vector multiplets are required as compensators.

N=4 conformal supergravity

One of the important step in going from conformal supergravity to leading order Poincare supergravity is the action for the compensators (in this case a minimum of six vector multiplets).

$$
\begin{aligned}
e^{-1}\mathscr{L}_V = &-\frac{\varphi}{4\Phi}F^{+I}_{ab}F^{abJ}\eta_{IJ} + \frac{1}{4}\phi^I_{ij}D^2\phi^{Jij}\eta_{IJ} \\
&+\frac{1}{8}\phi^I_{ij}\phi^{Jkl}D^{ij}{}_{kl}\eta_{IJ} - \frac{1}{48}\phi^I_{ij}\phi^{Jij}\eta_{IJ}\left[E^{kl}E_{kl} + 4\,(D_a\phi^{\alpha})(D^a\phi_{\alpha})\right] \\
&-\frac{\Phi^*}{2\Phi}T^{ab}{}_{ij}T_{ab\,kl}\phi^{Iij}\phi^{Jkl}\eta_{IJ} - \frac{1}{\Phi}F^I_{ab}T^{ab}{}_{ij}\phi^{Jij}\eta_{IJ} + \text{fermions} + \text{h.c}
\end{aligned}
$$

$$\varphi = \phi^1 - \phi^2\,,\Phi = \phi^1 + \phi^2\,,\eta = \text{diag}(-1\cdots - 1)$$

N=4 conformal supergravity

Suitable gauge fixing condition

$$b_\mu = 0, \quad \phi^{Iij}\phi^J_{ij}\,\eta_{IJ} = -\frac{6}{\kappa^2}, \quad \phi^{Iij}\psi^J_j\,\eta_{IJ} = 0$$

Constraints coming from Lagrange multiplier $D^{ij}{}_{kl}$ and $\chi^{ij}{}_k$

$$\left(\phi^{Iij}\phi^J_{kl}\,\eta_{IJ}\right)_{\mathbf{20'}} = 0\,,\ \left(\phi^{Iij}\psi^J_k\,\eta_{IJ}\right)_{\mathbf{20}} = 0$$

A suitable SU(4) gauge fixing condition would fix the scalars ϕ^I_{ij} to some constants. Irrespective of what SU(4) gauge fixing condition one chooses, the SU(4) invariant combination $M^{IJ} \equiv \phi^{Iij}\phi^J_{ij}$ would be set to $-\eta^{IJ}$. The bosonic action depends on this combination.

The local U(1) of the coset scalar sector is gauge fixed by $\mathrm{Im}(\phi_1 - \phi_2) = 0$ and the coset representative is taken to be $\tau = i\dfrac{\phi_1 + \phi_2}{\phi_1 - \phi_2}$

N=4 conformal supergravity

The auxiliary fields T^{ij}_{ab} , E_{ij} , $V_\mu{}^i{}_j$ needs to be eliminated by their e.o.m

After doing all this exercise, the bosonic part of the leading order action for N=4 Poincare supergravity takes the following form

$$e^{-1}\mathscr{L}_0 = -\frac{1}{4}R - \frac{i\tau}{4}F^{+I}\cdot F^{+J}\eta_{IJ} - \frac{1}{8(\mathrm{Im}\tau)^2}\partial_\mu\tau\partial^\mu\bar{\tau} + \mathrm{h.c}$$

In order to obtain the supersymmetric counter term which will be necessary for the cancellation of the U(1) anomaly, we would need to find a higher derivative deformation to the above action. In order to do that, we would first need to obtain N=4 conformal supergravity action for the Weyl multiplet.

N=4 conformal supergravity action

The N=4 Weyl multiplet was already known in the 80's (Bergshoeff, de Roo and de Wit, NPB 182) but the construction of an invariant action was lacking.

The techniques that was well established for N=1 and N=2 in terms of well known density formulae was not suitable for N=4. The presence of coset scalars also posed challenges.

The bosonic part of the conformal supergravity action was first constructed in 1209.0416 (Buchbinder, Pletnev, Tseytlin) using an indirect method. A direct constructive approach was used in 1510.04999 (Franz Ciceri and BS) to obtain the action up to terms quadratic in fermions.

However the above action that was obtained preserved SU(1,1) and is not suitable in obtaining the higher derivative deformation that is necessary for cancelling the anomaly.

N=4 conformal supergravity action

We devised a new strategy to construct the most general N=4 conformal supergravity action: D. Butter, F. Ciceri, B. de Wit and BS **PRL 118, no. 8, 081602 (2017);** D. Butter, F.Ciceri and BS **JHEP 01,029 (2020)**

The strategy relied on the construction of a "density formula" based on the "superform action principle".

The action principle follows from a simple observation in differential geometry: An integral $S = \int_M J$ of a d-form on a d-dimensional sub manifold (M) of a D-dimensional manifold (N) is invariant under arbitrary diffeomorphism in N if $dJ = 0$

This is because under arbitrary diffeomorphism in N the d-form transforms as $\delta_\xi J = \mathcal{L}_\xi J = d(i_\xi J) + i_\xi dJ$

N=4 conformal supergravity action

For constructing gauge invariant actions, one needs to geometrize the gauge symmetries and impose the closure conditions. In particular for constructing supergravity actions, one need to geometrize and impose the closure condition

One starts with a larger manifold (N) where supersymmetry has been geometrized. The (super)vielbein one forms are $E^A = (e^a, \psi^\alpha)$. The coordinates are $Z^M = (x^\mu, \theta^m)$. The spacetime manifold is defined as $M = N_{|\theta = d\theta = 0}$.

We need to construct a 4-form $J \in \Omega^4(N)$ such that $dJ = 0$. If we can find such a 4-form then the action integral $S = \int_M J$ would be invariant under supersymmetry.

In superconformal gravity, apart from supersymmetry there are a bunch of other gauge symmetries which are easier to realize and we will demand that the 4-form J is manifestly invariant under these symmetries. Then the closure condition on J, for supersymmetry can be replaced by a covariant closure condition $\nabla J = 0$

Ansatz:

$$J = J_{DCBA}E^A E^B E^C E^D = J_{\alpha\beta\gamma\delta}\psi^\alpha\psi^\beta\psi^\gamma\psi^\delta + J_{a\alpha\beta\gamma}e^a\psi^\alpha\psi^\beta\psi^\gamma + J_{ab\alpha\beta}e^a e^b\psi^\alpha\psi^\beta + J_{abc\alpha}e^a e^b e^c\psi^\alpha + J_{abcd}e^a e^b e^c e^d$$

Now $\nabla E^A = T^A$, where T^A is the torsion 2-form and we know it since we know the supersymmetry transformations of the Weyl multiplet

$\nabla J_{DCBA} = e^a \nabla_a J_{DCBA} + \psi^\alpha \nabla_\alpha J_{DCBA}$, ∇_a being the supercovariant derivative and ∇_α being the generator of supersymmetry.

Imposing $\nabla J = 0$ relates the different blocks J_{DCBA} to each other via supersymmetry and also imposes some constraints.

This gives us the invariant action integral $S = \int_M J$ in terms of an abstract multiplet whose components are the different blocks J_{DCBA}. This is what we call as the density formula

In order to get an invariant action for a known multiplet (for eg: Weyl multiplet), we need to find the blocks J_{DCBA} as composite expressions in terms of the components of the known multiplet.

In this way we obtained the most general action for N=4 conformal supergravity.

$$
\begin{aligned}
e^{-1}\mathcal{L}_B = \mathcal{H}\Big[\, & \tfrac{1}{2}\, R(M)^{abcd}\, R(M)^{-}_{abcd} + R(V)^{abi}{}_j\, R(V)^{-}_{ab}{}^j{}_i + \tfrac{1}{8}\, D^{ij}{}_{kl}\, D^{kl}{}_{ij} + \tfrac{1}{4}\, E_{ij}\, D^2 E^{ij} \\
& - 4\, T_{ab}{}^{ij} D^a D_c\, T^{cb}{}_{ij} - \bar{P}^a D_a D_b P^b + P^2 \bar{P}^2 + \tfrac{1}{3}(P^a \bar{P}_a)^2 - \tfrac{1}{6}\, P^a \bar{P}_a\, E_{ij}\, E^{ij} \\
& - 8\, P_a\, \bar{P}^c\, T^{ab}{}_{ij}\, T_{bc}{}^{ij} - \tfrac{1}{16}\, E_{ij}\, E^{jk}\, E_{kl}\, E^{li} + \tfrac{1}{48}\, (E_{ij}\, E^{ij})^2 + T^{ab}{}_{ij}\, T_{ab\,kl}\, T^{cd\,ij}\, T_{cd}{}^{kl} \\
& - T^{ab}{}_{ij}\, T_{cd}{}^{jk}\, T_{ab\,kl}\, T^{cd\,li} - \tfrac{1}{2}\, E^{ij}\, T^{ab\,kl}\, R(V)_{ab}{}^m{}_i\, \varepsilon_{jklm} + \tfrac{1}{2}\, E_{ij}\, T^{ab}{}_{kl}\, R(V)_{ab}{}^i{}_m\, \varepsilon^{jklm} \\
& - \tfrac{1}{16}\, E_{ij} E_{kl}\, T^{ab}{}_{mn}\, T_{ab\,pq}\, \varepsilon^{ikmn}\, \varepsilon^{jlpq} - \tfrac{1}{16}\, E^{ij} E^{kl}\, T^{ab\,mn}\, T_{ab}{}^{pq}\, \varepsilon_{ikmn}\, \varepsilon_{jlpq} \\
& - 2\, T^{ab\,ij} \left(P_{[a} D_{c]} T_b{}^{c\,kl} + \tfrac{1}{6}\, P^c D_c T_{ab}{}^{kl} + \tfrac{1}{3}\, T_{ab}{}^{kl} D_c P^c\right) \varepsilon_{ijkl} \\
& - 2\, T^{ab}{}_{ij} \left(\bar{P}_{[a} D_{c]} T_b{}^c{}_{kl} - \tfrac{1}{2}\, \bar{P}^c D_c T_{ab\,kl}\right) \varepsilon^{ijkl} \Big] \\
+ \mathcal{D}\mathcal{H}\Big[\, & \tfrac{1}{4}\, T_{ab}{}^{ij}\, T_{cd}{}^{kl} R(M)^{abcd}\, \varepsilon_{ijkl} + E_{ij}\, T^{ab\,ik}\, R(V)_{ab}{}^j{}_k + T^{ab\,ij}\, T_a{}^{c\,kl}\, R(V)_{bc}{}^m{}_k\, \varepsilon_{ijlm} \\
& - \tfrac{1}{24}\, E_{ij}\, E^{ij}\, T^{ab\,kl}\, T_{ab}{}^{mn}\, \varepsilon_{klmn} - \tfrac{1}{6}\, E^{ij}\, T_{ab}{}^{kl}\, T^{ac\,mn}\, T^b{}_c{}^{pq}\, \varepsilon_{iklm}\, \varepsilon_{jpqn} \\
& - \tfrac{1}{8}\, D^{ij}{}_{kl} \left(T^{ab\,mn}\, T_{ab}{}^{kl}\, \varepsilon_{ijmn} - \tfrac{1}{2}\, E_{im}\, E_{jn}\, \varepsilon^{klmn} \right) \Big] \\
+ \mathcal{D}^2\mathcal{H}\Big[& \tfrac{1}{32}\, T^{ab\,ij}\, T^{cd\,pq}\, T_{ab}{}^{mn}\, T_{cd}{}^{kl}\, \varepsilon_{ijkl}\, \varepsilon_{mnpq} - \tfrac{1}{64}\, T^{ab\,ij}\, T^{cd\,pq}\, T_{ab}{}^{kl}\, T_{cd}{}^{mn}\, \varepsilon_{ijkl}\, \varepsilon_{mnpq} \\
& + \tfrac{1}{6}\, E_{ij}\, T_{ab}{}^{ik}\, T^{ac\,jl}\, T^b{}_c{}^{mn}\, \varepsilon_{klmn} + \tfrac{1}{384}\, E_{ij}\, E_{kl}\, E_{mn}\, E_{pq}\, \varepsilon^{ikmp}\, \varepsilon^{jlnq} \\
& - \tfrac{1}{8}\, E_{ij}\, E_{kl}\, T_{ab}{}^{ik}\, T^{ab\,jl} \Big] \\
+ 2\, \mathcal{H}\, & e_a{}^\mu f_\mu{}^c\, \eta_{cb} \left[P^a\, \bar{P}^b - P^d\, \bar{P}_d\, \eta^{ab} \right] + \text{h.c.}
\end{aligned} \tag{4.1}
$$

Higher derivative deformation

Work in progress with Franz Ciceri

- In order to obtain the higher derivative deformation, we need to add the CSG action for the Weyl multiplet to the compensator action, use the gauge fixing condition and eliminate the auxiliary fields.
- The auxiliary fields (T^{ij}_{ab} , $V_{\mu}{}^{i}{}_{j}$, E^{ij} , χ^{ij}_{k}) comes with their kinetic terms in the CSG action. Hence their elimination will be done in an order by order fashion as an expansion in derivatives. This will give us the deformation to N=4 PSG as an expansion in derivatives.
- If we are solely interested in the four derivative deformation, then the leading order solution of the auxiliary fields are sufficient for the purpose.
- The four derivative deformation that one obtains from this is:

$$\mathscr{H}(\tau)\left(C^{\mu\nu\rho\sigma}C^{-}_{\mu\nu\rho\sigma}+\frac{1}{4(\mathrm{Im}\tau)^2}\left|\nabla^2\tau+\frac{i}{\mathrm{Im}\tau}\nabla_\mu\tau\nabla^\mu\tau\right|^2+\cdots\right)+\frac{\partial\mathscr{H}}{\partial\tau}\left(\frac{1}{4(\mathrm{Im}\tau)^2}\nabla_\mu\tau\nabla^\mu\tau\left(\nabla^2\bar{\tau}-\frac{i}{\mathrm{Im}\tau}\nabla_\mu\bar{\tau}\nabla^\mu\bar{\tau}\right)+\cdots\right)$$

$$+\frac{\partial^2\mathscr{H}}{\partial\tau^2}\left(\frac{\left(\mathrm{Im}\tau\right)^3}{4}F^{-I}\cdot F^{-J}F^{-K}\cdot F^{-L}\eta_{IJ}\eta_{KL}+\cdots\right)+\mathrm{h.c}$$

For a specific choice of $\mathscr{H}(\tau)$ we will obtain the supersymmetrization of the counter-term necessary for cancelling the U(1) anomaly

Future Directions

- As discussed earlier it was found that the anomaly sources some amplitudes which is responsible for the four loop divergence. One interesting question to ask would be the implication of the cancellation of the anomaly on the four loop divergence.
- Dimensional reduction of 6d (2,0) Weyl multiplet to 4d should give rise to a N=4 Weyl multiplet. But instead of the "Standard" Weyl multiplet discussed here, it will give rise to a different version of the Weyl multiplet: the "dilaton" Weyl multiplet (work in progress with Subrabalan, Axel and Franz).
- One can construct a different version of N=4 Poincare supergravity using the dilaton Weyl multiplet where the SL(2,R) duality symmetry will be realized off-shell. It would be interesting to see this construction and what more can it teach us about N=4 supergravity.
- The implication of the higher derivative corrections discussed here on the entropy of N=4 supersymmetric black holes can be studied.


Thank you!

謝謝