Double Field Theory and Pseudo-Supersymmetry

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Based on work with Falk Hassler and Haoyu Zhang (arXiv:2012.12278), and
Hong Lü and Zhao-Long Wang (arXiv:1105.6114)

Dimensional reduction has played a central role in many of the developments in the understanding of string theory and M-theory. Some dimensional reductions are simple and easily understood, such as reduction on a circle or torus. Some are more subtle, such as the reduction of $D = 11$ supergravity on S^7 to give four-dimensional gauged $\mathcal{N} = 8$ supergravity. It became clear that the consistency of such reductions was intimately associated with supersymmetry, and more recently, with double field theory. But it was conjectured in 1986, and proven in 2015, that non-trivial group manifold consistent reductions of the effective action for the bosonic string in *any dimension* also exist, and for these true supersymmetry obviously cannot play a role. The arbitrary-dimensional bosonic string can nevertheless be *pseudo-supersymmetrised* (i.e. supersymmetry modulo quartic fermion terms), and this can provide a way to understand the consistent reductions. In recent work, we have shown that the $\mathcal{N} = 1$ supersymmetric extension of DFT for the bosonic string extends also to a pseudo-supersymmetric DFT in arbitrary, which draws together the strands of the consistent truncation story.

Dimensional Reductions

In any dimensional reduction, we want the solutions of the lower-dimensional theory to be solutions also of the higher-dimensional theory. This is the requirement that we have a **Consistent Reduction**.

There are three principal kinds of dimensional reduction that have been considered:

- **Reduction on a circle or torus**: These, first considered by Kaluza and Klein, are straightforward; the reduction ansatz is invariant under the circle $U(1)$ isometries, and so the reduction is obviously consistent.

- **Group manifold reduction on a group** G, keeping all the lower-dimensional fields that are invariant under G_L (or under G_R): These were pioneered by Bryce De Witt (1963), and they are also obviously consistent. (Non-linear terms built from the G_L singlets that are retained can never excite the G_L non-singlets that were truncated out.)

- **Reduction on coset spaces, such as spheres**: Pauli (1953) was the first to propose these; he envisioned a reduction of six-dimensional Einstein gravity on S^2, to give a four-dimensional theory with the gauge bosons of $SU(2)$. But he also realised that it wouldn't work; it would be inconsistent. (Because the $SU(2)$ gauge bosons would try to act as sources for massive spin-2 modes that have been truncated. Equivalently stated, the six-dimensional equations do not reduce to purely four-dimensional ones.) He didn't publish, but wrote to Abraham Pais about it.

- It was only with the advent of supergravity that Pauli's idea of getting the gauge bosons of the $SO(n+1)$ isometry group in an S^n reduction was successfully resurrected. It turns out that properties of a particular supergravity theory, and a particular sphere, can conspire to circumvent the inconsistency problem. The first successful consistent sphere reduction was the compactification of $D = 11$ supergravity on S^7 to give four-dimensional $SO(8)$-gauged $\mathcal{N} = 8$ supergravity.

Pauli Reductions; Problem and Resolution

- The reason why Pauli's original idea doesn't work can easily be seen by considering the metric reduction ansatz:

$$d\hat{s}_6^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu + g_{mn}\,(dy^m + K_I^m\, A_\mu^I\, dx^\mu)(dy^n + K_K^n\, A_\nu^J\, dx^\nu)\,, \tag{1}$$

where $K_I = K_I^m\, \partial/\partial y^m$ are the $SO(3)$ Killing vectors on the 2-sphere with metric $g_{mn}\, dy^m\, dy^n$. The four-dimensional components of the six-dimensional Einstein equation $\hat{R}_{MN} = 0$ then imply the "four-dimensional" Einstein equation

$$R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} = \tfrac{1}{2} K_I^m\, K_J^n\, g_{mn}\,(F_{\mu\rho}^I\, F_\nu^{J\rho} - \tfrac{1}{4} F_{\rho\sigma}^I\, F^{J\,\rho\sigma}\, g_{\mu\nu}) + \cdots\,. \tag{2}$$

The problem is that $K_I^m\, K_J^n\, g_{mn}$ are functions on S^2, depending on the y^m coordinates of the 2-sphere (and the RHS is actually trying to source massive spin-2 modes that have been truncated out). (We are being a bit schematic here and omitting scalar fields; but they wouldn't help with this problem.)

- $D = 11$ supergravity on S^7 evades this problem because the $SO(8)$ gauge bosons A_μ^I enter also in the ansatz for the 4-form field strength. At the linearised level

$$F_{\mu\nu mn} = \epsilon_{\mu\nu\rho\sigma}\, F^{I\,\rho\sigma}\, \nabla_m K_{I\,n} + \cdots\,, \tag{3}$$

and the four-dimensional Einstein equation becomes

$$R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} = \tfrac{1}{2}\Big[\, K_I^m\, K_J^n\, g_{mn} + (\nabla_m K_{I\,n})\,(\nabla^m K_J^n)\,\Big](F_{\mu\rho}^I\, F_\nu^{J\rho} - \tfrac{1}{4} F_{\rho\sigma}^I\, F^{J\,\rho\sigma}\, g_{\mu\nu}) + \cdots \tag{4}$$

and now the $SO(8)$ tensor in the square brackets is actually a constant on S^7 (it is just δ_{IJ}). (Again we are being a bit schematic, omitting scalars and factors, but it all works out.)

- Similar mechanism happens for the other examples of consistent Pauli reductions, such as type IIB on S^5, and $D = 11$ supergravity on S^4.

A Group Theoretic Argument

There is another way of seeing why a generic Pauli reduction would be inconsistent, and how the problem might be evaded in certain special cases (Cvetič, Lü, CNP; hep-th/0003286):

- Suppose in some particular theory a consistent Pauli reduction on S^n does exist. In the lower dimension this will give an $SO(n+1)$ gauge theory, with a gauge coupling g given by the inverse radius of the sphere.

- One could then take the ungauged limit $g \to 0$, corresponding to sending the sphere radius to infinity. This would be equivalent to a reduction on the torus T^n.

- Going the other way, one could construct the $g \neq 0$ gauged theory by gauging an $SO(n+1)$ subgroup of the global symmetry group of the ungauged theory.

- To be able to do this, it would be necessary for the global symmetry group of the T^n-reduced theory to be at least big enough to contain $SO(n+1)$.

- For a generic higher-dimensional theory, the reduction on T^n gives a lower-dimensional theory with $GL(n, R)$ global symmetry. This has maximal compact subgroup $SO(n)$ – which clearly cannot contain $SO(n+1)$!

- So the Pauli S^n reduction of a generic theory could not possibly be consistent. To stand a chance, the higher-dimensional theory must have some special features that result in its T^n reduction having an enhanced global symmetry.

- Precisely such symmetry enhancements are commonly seen in the toroidal reductions of supergravity theories. For example:

Global Symmetries of T^n-reduced Supergravities

- When $D = 11$ supergravity is reduced on T^n, the generic global $GL(n, R)$ global symmetry is enhanced to $E_{n,n}$ (Cremmer-Julia symmetry). In particular

$$T^7: \qquad GL(7, R) \longrightarrow E_{7,7} \supset SU(8) \supset SO(8) = \text{Isometry group of } S^7\,,$$

$$T^4: \qquad GL(4, R) \longrightarrow E_{5,5} = SL(5, R) \supset SO(5) = \text{Isometry group of } S^4\,.$$

- Similarly, type IIB supergravity reduced on T^5:

$$T^5: \qquad GL(5, R) \longrightarrow E_{6,6} \supset USp(8) \supset SO(6) = \text{Isometry group of } S^5\,.$$

- In all these cases, the global symmetry enhancement occurs because of additional form fields in the supergravity theory. The detailed structure of the theory plays an essential role. For example, in $D = 11$ supergravity the 3-form potential A, and its Chern-Simons coupling term $\frac{1}{6}F \wedge F \wedge A$, is responsible. The precise coefficient $\frac{1}{6}$ is essential for the global symmetry enhancement to work. This is also exactly the coefficient needed for the bosonic theory to be supersymmetrisable.

- This seems to suggest that supersymmetry may play an integral role in the consistency of Pauli reductions. Indeed, the complete proof of the consistency in the first example that was fully established, namely the S^7 reduction of $D = 11$ supergravity, made extensive use of supersymmetry in order to construct the reduction ansatz (de Wit and Nicolai).

- But there is a class of examples where supersymmetry cannot possibly play a role:

Pauli Reduction of Bosonic String on Group Manifold

- In a probably ill-fated attempt to derive the heterotic string from the bosonic string, in 1986 Mike Duff, Bengt Nilsson, Nick Warner and I played around with the idea of compactifying the bosonic string in 506 dimensions on the group manifold $E_8 \times E_8$.

- In the course of this work, we discovered that the bosonic string compactified on any group manifold G exhibited a "conspiracy" involving the reduction ansätze for the metric and for the B field, suggesting that the lower-dimensional Einstein equation would be consistent in a Pauli reduction retaining *all* the gauge bosons of the full $G_L \times G_R$ isometry of the group manifold G, and not just the G_L or G_R of a DeWitt reduction. We conjectured that in fact such a Pauli reduction would be consistent.

- This conjecture was further supported by the group theory argument discussed previously. If the bosonic string in any dimension is reduced on T^n, then the global symmetry group of the lower-dimensional theory is enhanced from $GL(n,R)$ to $SO(n,n)$. Its $SO(n) \times SO(n)$ maximal compact subgroup is large enough to contain $G \times G$, for any compact semisimple Lie group G of dimension n.

- More recently, Arnaud Baguet and Henning Samtleben invited me to join with them on a project which proved the conjecture fully, by making use of double field theory techniques (arXiv:1510.08926).

- So where does this leave the idea that supersymmetry might provide an underlying explanation for the consistency of Pauli reductions? After all, there is no supersymmetry in 506 dimensions, or in any dimension above 11!

- Maybe asking for true supersymmetry is too strong...

Pseudo-Supersymmetry

- In 2011, with Hong Lü and Zhao-Long Wang, we investigated the question of whether one could relax the strict requirements for having a truly supersymmetric theory, while still retaining many of the nice features that we like to use in practice:
- Useful properties of supersymmetric theories include: Existence of Killing spinors in special (supersymmetric) bosonic backgrounds; Existence of first-order BPS equations; Use of Killing spinors for constructing consistent Pauli reduction ansätze.
- Features that are necessary for true supersymmetry, but which we often don't directly make use of, include: Full supersymmetry of the Lagrangian including the quartic-fermion terms; full closure of the supersymmetry transformations on a super-algebra.
- We examined what would happen if one followed the steps of trying to supersymmetrise a bosonic Lagrangian, by parameterising possible fermionic terms and possible terms in "supersymmetry" transformation laws, and then requiring invariance of the total Lagrangian...
- ...But, using the rule that anything that involved fermions beyond the quadratic order would be dropped. Never mind that these terms, if one kept them, would fail to work!
- This formed the basis of what we called *pseudo-supersymmetry* (arXiv:1105.6114). We can still have all the benefits of first-order BPS equations, Killing spinors in pseudo-supersymmetric bosonic backgrounds, etc.
- Importantly, this does not lead to a free-for-all where "anything goes." There are still strict conditions that must be satisfied in order to construct pseudo-supersymmetric theories. But there is quite a lot more lattitude than in true supersymmetric theories. For example...

Pseudo-Supersymmetry of the Bosonic String

- In the Einstein frame, the D-dimensional bosonic string Lagrangian density is

$$e^{-1}\,\mathcal{L} = R - \tfrac{1}{2}(\partial\Phi)^2 - \tfrac{1}{12}e^{a\,\Phi}\,H^2\,,$$

  with the constant $a^2 = 8/(D-2)$. The dilaton coupling a is dictated by the requirement that the global symmetry after T^n reduction is enhanced from $GL(n,R)$ to $SO(n,n)$.

- By parameterising all possible terms in a fermionic Lagrangian and in transformation laws (up to quadratic order in fermions), we solved for the coefficients necessary to obtain invariance (neglecting beyond quadratic in fermions). Gives a unique solution in any dimension D, and fixes a to be as given above. In the string frame (arXiv:1105.6114):

$$\begin{aligned} e^{-1}\,\mathcal{L} &= e^{-2\Phi}\Big[R + 4(\partial\Phi)^2 - \tfrac{1}{12}H^2 - \bar\psi_\mu\,\Gamma^{\mu\nu\rho}\,D_\nu\psi_\rho + \bar\lambda\,\not{D}\lambda - 2i\,\sqrt{\beta}\,\bar\lambda\,\Gamma^{\mu\nu}\,D_\mu\psi_\nu \\ &\quad -2\bar\psi_\mu\,\Gamma^\mu\psi_\rho\,\partial^\rho\Phi + \tfrac{2i}{\sqrt{\beta}}\,\bar\psi_\mu\Gamma^\nu\Gamma^\mu\lambda\,\partial_\nu\Phi \\ &\quad +H_{\nu\rho\sigma}\Big\{\tfrac{1}{24}\bar\psi_\mu\,\Gamma^{\mu\nu\rho\sigma\lambda}\,\psi_\lambda + \tfrac{1}{4}\bar\psi^\nu\,\Gamma^\rho\,\psi^\sigma - \tfrac{1}{24}\bar\lambda\,\Gamma^{\nu\rho\sigma}\,\lambda + \tfrac{i}{12\,\sqrt{\beta}}\,\bar\psi_\mu\,\Gamma^{\mu\nu\rho\sigma}\,\lambda\Big\}\Big]\,, \end{aligned}$$

  with the pseudo-supersymmetry transformation rules

$$\begin{aligned} \delta\psi_\mu &= D_\mu\,\epsilon - \tfrac{1}{8}H_{\mu\nu\rho}\,\Gamma^{\nu\rho}\,\epsilon\,, \qquad \delta\lambda = i\,\sqrt{\beta}\Big(\Gamma^\mu\,\partial_\mu\Phi - \tfrac{1}{12}\,\Gamma^{\mu\nu\rho}\,H_{\mu\nu\rho}\Big)\,\epsilon\,, \\ \delta e^a_\mu &= -\tfrac{1}{2}\bar\psi_\mu\,\Gamma^a\,\epsilon\,, \qquad \delta\Phi = -\tfrac{i}{4\,\sqrt{\beta}}\,\bar\epsilon\,\lambda\,, \qquad \delta B_{\mu\nu} = \bar\epsilon\,\Gamma_{[\mu}\,\psi_{\nu]}\,. \end{aligned}$$

  Here β is either $+1$ or -1, according to dimension, with $\Gamma^T_\mu = \beta\,C\Gamma_\mu\,C^{-1}$.

- By doubling the fermions, can also add a "conformal anomaly" term:

$$\begin{aligned} e^{-1}\,\mathcal{L}_c &= e^{-2\Phi}\Big[-\tfrac{1}{2}m^2 - \tfrac{m}{2\,\sqrt{2\beta}}\Big(\bar\psi_\mu\,\Gamma^{\mu\nu}\,\psi_\nu + 2\,\sqrt{-\beta}\,\bar\psi_\mu\,\Gamma^\mu\lambda - \bar\lambda\,\lambda\Big)\Big]\,, \\ \delta_{\rm extra}\,\psi_\mu &= 0\,, \qquad \delta_{\rm extra}\,\lambda = \tfrac{i}{2\,\sqrt{2}}\,m\,\epsilon\,. \end{aligned}$$

Pseudo-Supersymmetric Vacuum

- With the conformal anomaly term included, there exist stablised vacuum solutions of the form $(\text{Minkowski})_{(D-\text{dim}G)}\times G$, for any semi-simple compact group G. These solutions are of the form

$$d\hat{s}_D^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu + ds^2(G)\,, \qquad H_{mnp} = c\, f_{mnp}\,, \qquad \Phi = \text{const.}$$

  with $ds^2(G)$ the bi-invariant (Einstein) metric on G, and f_{mnp} the structure constants of G. (The 1986 idea for getting the heterotic string from the bosonic string had $G = E_8\times E_8$ and $D = 506$, yielding a $(\text{Minkowski})_{10}$ lower-dimensional vacuum.)

- The spin connection and the curvature on G is related to the structure constants, with

$$\omega_{ab} = -\tfrac{1}{2}c\, f_{abc}\, e^c\,, \qquad R_{abcd} = \tfrac{1}{4}c^2\, f_{abe}\, f_{cd}{}^e\,, \qquad R_{ab} = \tfrac{1}{4}c^2\, C_A\, \delta_{ab}\,,$$

  with C_A being the quadratic Casimir. Hence from the field equations $\Phi = 0$ and

$$m^2 = \tfrac{1}{3}c^2\, C_A\, \text{dim}\, G\,.$$

- The pseudo-dilatino transformation law becomes

$$\delta\lambda = \frac{c}{12}\, f_{abc}\, \Gamma^{abc}\, \epsilon + \frac{im}{2\sqrt{2}}\, \epsilon\,.$$

  $\frac{1}{6} f_{abc}\, \Gamma^{abc}$ has equal numbers of eigenvalues $\pm i\sqrt{\frac{1}{6}C_A\, \text{dim}G}$, implying $\delta\lambda = 0$ for the half with eigenvalue $-i\sqrt{\frac{1}{6}C_A\, \text{dim}G}$. The pseudo-gravitino transformation law then just gives $\partial_\mu\epsilon = 0$ in Minkowski and $\partial_m\epsilon = 0$ on G (since H_{abc} cancels againt the spin connection on G). Thus the $(\text{Minkowski})_{(D-\text{dim}G)}\times G$ vacuum is pseudo-supersymmetric.

- The full consistent reduction on G yields a pseudo-supersymmetric gauge theory with $G\times G$ gauge group in the lower dimension.

Generalised Geometry and Pseudo-Supersymmetry

- We (Falk Hassler, Haoyu Zhang, CNP) most closely follow the discussion for $\mathcal{N}=1$ ten-dimensional supersymmetric DFT in Hohm and Kwak, arXiv:1111.7293 (and see also Coimbra, Strickland-Constable and Waldram, arXiv:1107.1733). Define the generalised dilaton d and its pseudo-superpartner ρ:

$$d=\Phi-\tfrac{1}{2}\log e\,,\qquad \rho=\Gamma^\mu\,\psi_\mu+\tfrac{i}{\sqrt{\beta}}\,\lambda\,.$$

Define also the generalised frame field with components

$$E_a^{(+)}=\tfrac{1}{\sqrt{2}}\,(e_a^\mu\,\partial_\mu+e_{\mu a}\,dx^\mu-e_a^\mu\,B_{\mu\nu}\,dx^\nu)\,,\qquad E_a^{(-)}=\tfrac{1}{\sqrt{2}}\,(e_a^\mu\,\partial_\mu-e_{\mu a}\,dx^\mu-e_a^\mu\,B_{\mu\nu}\,dx^\nu)\,.\tag{5}$$

These each define the $2D$ components of a vector on the generalised tangent space $TM+T^*M$.

- The pseudo-supersymmetry transformation rules are now simply written as

$$\begin{aligned}\delta\psi_\mu &= \nabla_\mu^{(-)}\,\epsilon\,,\quad \delta\rho=\Gamma^\mu\,\nabla_\mu^{(+)}\,\epsilon\,,\qquad \delta_{\rm extra}\,\psi_\mu=0\,,\quad \delta_{\rm extra}\,\rho=-\frac{1}{2\sqrt{2\beta}}\,m\,\epsilon\,,\\ \langle E_b^{(-)},\delta E_a^{(+)}\rangle &= -\tfrac{1}{2}\bar\epsilon\,\Gamma_b\,\psi_a\,,\qquad \delta d=-\tfrac{1}{4}\bar\epsilon\,\rho\,.\end{aligned}$$

Here, the $O(D)\times O(D)$ covariant derivatives are defined by

$$\nabla_\mu^{(-)}\,\epsilon=(D_\mu-\tfrac{1}{8}H_{\mu\nu\rho}\,\Gamma^{\nu\rho})\,\epsilon\,,\qquad \Gamma^\mu\,\nabla_\mu^{(+)}\,\epsilon=(\Gamma^\mu\,D_\mu-\tfrac{1}{24}H_{\mu\nu\rho}\,\Gamma^{\mu\nu\rho}-\Gamma^\mu\,\partial_\mu\Phi)\,\epsilon\,.\tag{6}$$

The Lagrangian takes the simple form

$$e^{2d}\,\mathcal{L}_D=R+4(\partial\Phi)^2-\tfrac{1}{12}H^2-\bar\psi^a\,\Gamma^b\nabla_b^{(+)}\psi_a-\beta\,\bar\rho\,\Gamma^a\nabla_a^{(+)}\rho+2\psi^a\nabla_a^{(-)}\rho\,.\tag{7}$$

The additional conformal anomaly is also simplified, becoming

$$e^{2d}\,\mathcal{L}_c=-\frac{m^2}{2}-\frac{m\,\sqrt{\beta}}{2\,\sqrt{2}}\,(\bar\rho\,\rho-\beta\,\bar\psi_\mu\,\psi^\mu)\,.$$

$O(D,D)$ Structure and Covariant Derivatives

- We introduce an $O(D,D)$ invariant metric η_{AB} and also the generalised tangent-frame metric $\mathcal{H}_{AB}$ that breaks this to $O(D)\times O(D)$:

$$\eta_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & -\eta_{\bar{a}\bar{b}} \end{pmatrix}, \qquad \mathcal{H}_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta_{\bar{a}\bar{b}} \end{pmatrix}.$$

  In coordinate indices we have the generalised metric

$$\mathcal{H}^{IJ} = \begin{pmatrix} g_{ij} - B_{ik}\, g^{k\ell}\, B_{\ell j} & -B_{ik}\, g^{kj} \\ g^{ik}\, B_{kj} & g^{ij} \end{pmatrix}.$$

  (Using $i, j, \ldots$ for coordinate indices now, with corresponding $I, J, \ldots$ for the generalised coordinate indices.) The metrics $\mathcal{H}^{AB}$ and $\mathcal{H}^{IJ}$ are related by the generalised vielbein ${E_A}^I$:

$$\mathcal{H}^{IJ} = {E_A}^I\, {E_B}^J\, \mathcal{H}^{AB}, \qquad {E_A}^I = \begin{pmatrix} E^{(+)}_{a\ i} & E^{(+)i}_{a} \\ E^{(-)}_{\bar{a}\ i} & E^{(-)i}_{\bar{a}} \end{pmatrix},$$

  where $E^{(+)}_a$ and $E^{(-)}_{\bar{a}}$ are the generalised frame fields defined previously in eqn (5).

- The ${E_A}^I$ generalised vielbein gives a torsion-free connection, with the covariant derivatives $\nabla^{(+)}$ and $\nabla^{(-)}$ we saw earlier in the pseudo-supergravity transformation rules and action; $\nabla_A = \frac{1}{\sqrt{2}}(\nabla^{(+)}_a, \nabla^{(-)}_{\bar{a}})$. This satisfies $\nabla_A\, \eta_{BC} = 0$, $\nabla_A\, \mathcal{H}_{BC} = 0$ and $\int e^{-2d}\, \nabla_A\, V^A = 0$.

Consistent Truncations

- Define generalised Lie derivative $\mathcal{L}_\xi V^I = \xi^J \nabla_J V^I + (\nabla^I \xi_J - \nabla_J \xi^I) V^J + w\,(\nabla_J \xi^J) V^I$ on a weight w vector, etc. Then one has $F_{AB}{}^C$ and F_A defined by

$$\mathcal{L}_{E_A} E_B = F_{AB}{}^C\, E_C\,, \qquad \mathcal{L}_{E_A} e^{-2d} = F_A\, e^{-2d}\,, \qquad (\text{with } F_{ABC} = F_{[ABC]})\,.$$

- Can now describe consistent truncation on a group manifold G. With the $O(D,D)$ structure, where $D = \dim G$, and F_{ABC} the structure constants of the corresponding Lie algebra $\mathfrak{g} \subset O(D,D)$; $[t_A, t_B] = F_{AB}{}^C\, t_C$. Then F_{ABC} and F_A are constants. There is then a covariant derivative $\mathcal{D}_A$ that is flat (no curvature), but with torsion F_{ABC}, and

$$F_{ABC} = 3 E_{[A}{}^I\, \partial_I E_B{}^J\, E_{C]\,J}\,, \qquad \mathcal{D}_I\, F_{ABC} = 0\,, \qquad \partial_I\, F_{ABC} = 0\,. \tag{8}$$

- The invariance conditions on F_{ABC} in eqn (8) do not have a unique solution, but one choice is the one employed in arXiv: 1510.08296, Baguet, Samtleben, CNP,

$$\begin{aligned} \sqrt{2}\, E_a^{(+)} &= K^m_{(L)\,a}\, \partial_m - \eta_{ab}\, (\iota_{K^b_{(L)}} B - K^b_{(L)\,m} dx^m)\,, \\ \sqrt{2}\, E_a^{(-)} &= K^m_{(R)\,a}\, \partial_m - \eta_{ab}\, (\iota_{K^b_{(R)}} B + K^b_{(R)\,m} dx^m)\,, \end{aligned} \tag{9}$$

where $K_{(R)}$ and $K_{(L)}$ denote the Killing vectors of the right-action and the left-action of G on the group G, and $B = -\frac{c}{3!}\, f_{abc}\, K^a_{(R)} \wedge K^b_{(R)} \wedge K^c_{(R)}$. The tensor F_{ABC} has components given by the structure constants of G:

$$F_{abc} = \frac{c}{\sqrt{2}}\, f_{abc}\,, \qquad F_{\bar{a}\bar{b}\bar{c}} = \frac{c}{\sqrt{2}}\, f_{\bar{a}\bar{b}\bar{c}}\,.$$

The dilaton must be constant on G, and F_A vanishes.

- This provides the necessary framework for the consistent Pauli reduction on G, with $G_L \times G_R$ gauge bosons.

Squashed Group Manifold Vacua

- We saw already the $(\text{Minkowski})_{(D-\dim G)}\times G$ pseudo-supersymmetric vacuum of the bosonic string, where G is any semi-simple compact group G equipped with its bi-invariant Einstin metric. All simple compact groups with dimension > 3 admit at least two inequivalent homogeneous Einstein metrics. What about other inequivalent $(\text{Minkowski})_{(D-\dim G)} \times G$ vacua of the bosonic string?

- Consider $SO(5)$ as an example. Define left-invariant 1-forms L_{IJ}, with $L_{IJ} = L_{[IJ]}$ and $dL_{IJ} = L_{IK} \wedge L_{KJ}$ and $I, J = 1, \ldots 5$. Any metric $ds^2 = x_{IJ,KL}\, L_{IJ} \otimes L_{KL}$ with constants $x_{IJ,KL} = x_{KL,IJ}$ is homogeneous and left-invariant. Solving the equations of motion becomes tractable if one restricts the coefficients so that some subgroup of G_R is preserved. For example, consider $SO(5)_L \times SO(3)_R$-invariant metrics

$$d\tilde{s}^2 = x_1\, L_{1i} \otimes L_{1i} + x_2\, L_{2i} \otimes L_{2i} + x_3\, L_{ij} \otimes L_{ij} + x_4\, L_{12} \otimes L_{12}\,,$$

where $i = 3, 4, 5$. There are three such inequivalent (up to scale) Einstein metrics, for $(x_1, x_2, x_3, x_4) = (1, 1, 1, 1),\ (1, 2, 1, 2)$ or $(14, 14, 4, 19)$ (arXiv:0903.2493.)

- The 3-form $G = L_{IJ} \wedge L_{JK} \wedge L_{KI}$ is closed, $dG = 0$. This is the 3-form $f_{abc}\, e^a \wedge e^b \wedge e^c$ where f_{abc} are the structure constants. One can obtain a solution of the bosonic string equations of the form

$$d\hat{s}_D^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu + d\tilde{s}^2\,, \qquad H = c\, G\,, \qquad \Phi = 0\,,$$

if $d\tilde{*}G = 0$ in the squashed group-manifold metric $d\tilde{s}^2$. We find two solutions (up to scale):

$$(x_1, x_2, x_3, x_4) = (1, 1, 1, 1)\,, \qquad \text{or} \qquad (x_1, x_2, x_3, x_4) = (1, 1, 3, -3)\,.$$

The first is the solution with the bi-invariant metric that we saw previously; it is pseudo-supersymmetric. The second solution, which is $SO(5) \times SO(3)$-invariant has no pseudo-supersymmetry. Since x_4 is negative, this metric on $SO(5)$ is Lorentzian, with signature $(9, 1)$.

Summary

- Supersymmetry is a powerful tool for constructing consistent Pauli dimensional reductions, such as the S^7 reduction of $D = 11$ supergravity. But not all consistent Pauli reductions can be understood using supersymmetry; notably, the reduction of the D-dimensional bosonic string on a group manifold G, in a consistent truncation that retains all the $G_L \times G_R$ gauge bosons.

- Double field theory provides another powerful tool for constructing consistent Pauli reductions; including the bosonic string reduction on G. The $\mathcal{N} = 1$ super-extension of DFT cannot directly be generalised to the DFT of the bosonic string in general dimensions D, because strict supersymmetry does not exist.

- For many purposes, the full closure of the supersymmetry transformations is not necessary. Invariance of the Lagrangian modulo higher-order fermion terms suffices for finding Killing spinors in bosonic backgrounds, first-order BPS equations, etc. Such a pseudo-supersymmetric extension of the bosonic string does exist in arbitrary dimension D.

- We have shown how one can extend the DFT of the arbitrary-dimensional bosonic string to incorporate pseudo-supersymmetry. This can be used in order to obtain consistent Pauli reductions, etc. It seems plausible that all known consistent Pauli reductions occur in (pseudo)-supersymmetric theories.

- We saw that the pseudo-supersymmetrised bosonic string admits pseudo-supersymmetric Minkowski $\times\, G$ vacua, where G has its bi-invariant metric. Solutions exist also with squashed metrics; examples we have studied have no pseudo-supersymmetry. Might examples preserving some fraction of pseudo-supersymetry exist?

- Relaxing the strict requirement of invariance at the full quartic order in fermions opens up possibilities for broader classes of pseudo-supersymmetric theories, in higher dimensions where interesting geometric structures can arise. It is not an anarchic free-for-all though; e.g. there is no $\mathcal{N} = 2$ pseudo-supersymmetric extension.