Applying $A4$ to three-Higgs doublet model implies alignment

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The A4 Group

Group of of even permutations of four objects comprising of $4!/2 = 12$ elements.

Generated by S and T satisfying $S^2 = T^3 = (ST)^3 = \mathbb{I}$.

Has four inequivalent irreducible representations.

$$\begin{aligned} 1 \qquad S &= 1 \quad T = 1. \\ 1' \qquad S &= 1 \quad T = \omega. \\ 1'' \qquad S &= 1 \quad T = \omega^2. \end{aligned}$$

$$3: \qquad S = diag(1,-1,-1) \quad \text{and} \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Multiplication Rules

$$1' \times 1'' = 1, \qquad 1' \times 1' = 1'', \qquad 1'' \times 1'' = 1',$$

$$\boxed{3 \times 3 = 1 + 1' + 1'' + 3 + \bar{3}.}$$

$$\Downarrow$$

$$1 = a_1b_1 + a_2b_2 + a_3b_3 \equiv \rho_{1ij}a_ib_j, \;\; 1' = a_1b_1 + \omega^2 a_2b_2 + \omega a_3b_3 \equiv \rho_{3ij}a_ib_j,$$

$$1'' = a_1b_1 + \omega a_2b_2 + \omega^2 a_3b_3 \equiv \rho_{2ij}a_ib_j.$$

$$3 \sim (a_2b_3, a_3b_1, a_1b_2), \;\; \bar{3} \sim (a_3b_2, a_1b_3, a_2b_1).$$

$$3_{sym} \equiv \frac{1}{2}(3 + \bar{3}) = \alpha_{ijk}a_jb_k$$

$$\text{and} \;\; 3_{antisym} \equiv \frac{1}{2}(3 - \bar{3}) = \beta_{ijk}a_jb_k$$

Some more on A4 .. .

$A4$ **invariants come from:** a) $1 \times 1 = \boxed{1}$ (trivial)

b) $1' \times 1'' = \boxed{1}$

c) $3 \times 3 = \boxed{1} + 1' + 1'' + 3 + \bar{3}$

- **Consider four $A4$ triplets:** $X_1, \;\; X_2, \;\; X_3$ and X_4

Combine: $X_i X_j$ and $X_k X_l$

$$X_i X_j \Downarrow \boxed{3 \times 3 = 1 + 1' + 1'' + 3 + \bar{3}}$$

$$X_k X_l \Downarrow \boxed{3 \times 3 = 1 + 1' + 1'' + 3 + \bar{3}}$$

First way: $1 \times 1 = \boxed{1}$

Second way: $1' \times 1'' = \boxed{1}$

Third way: $1'' \times 1' = \boxed{1}$

Fourth way:

(a) $3 \times 3 = \boxed{1} + 1' + 1'' + 3 + \bar{3}$

(b) $\bar{3} \times 3 = \boxed{1} + 1' + 1'' + 3 + \bar{3}$

(c) $3 \times \bar{3} = \boxed{1} + 1' + 1'' + 3 + \bar{3}$

(d) $\bar{3} \times \bar{3} = \boxed{1} + 1' + 1'' + 3 + \bar{3}$

Alignment in general

Consider a model with N Higgs doublets: (Φ_i, $i = 1, 2, ..N$) $\rightarrow$ Each Φ_i is a $SU(2)_L$ doublet.

$$\Phi \equiv \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ . \\ . \\ \Phi_N \end{pmatrix} \equiv \begin{pmatrix} \phi_1^+ & \phi_1^0 \\ \phi_2^+ & \phi_2^0 \\ . & . \\ . & . \\ \phi_N^+ & \phi_N^0 \end{pmatrix} \text{ where, } \phi_i^0 = \frac{v_i}{\sqrt{2}} + \frac{1}{\sqrt{2}}(\phi_i + i\chi_i) \Rightarrow \ SU(2)_L \rightarrow \ \text{horizontally.}$$

A general form of the potential: $V = \sum_{i,j=1}^{N} m_{ij}^2 (\Phi_i^\dagger \Phi_j) + \sum_{i,j,k,l=1}^{N} \lambda_{ijkl} (\Phi_i^\dagger \Phi_j)(\Phi_k^\dagger \Phi_l)$

Mass square matrices of physical scalars determined by the couplings $\Rightarrow$ Not diagonal in general$\Rightarrow$ say, a Unitary transformation U_a diagonalizes them.

- **After spontaneous symmetry breaking (SSB):**

$$\langle \Phi \rangle \equiv \begin{pmatrix} \langle \Phi_1 \rangle \\ \langle \Phi_2 \rangle \\ . \\ . \\ \langle \Phi_N \rangle \end{pmatrix} \equiv \begin{pmatrix} 0 & v_1 \\ 0 & v_2 \\ . & . \\ . & . \\ 0 & v_N \end{pmatrix} \rightarrow \ \text{Apply} \ U_b \rightarrow \ \begin{pmatrix} 0 & \frac{v}{\sqrt{2}} \\ 0 & 0 \\ . & . \\ . & . \\ 0 & 0 \end{pmatrix} \Rightarrow \ \text{Higgs basis}$$

If, Higgs basis = the mass eigenstate basis of the physical scalars $\Rightarrow$ Alignment

Introduction

Three Higgs doublet Model with $A4$ symmetry

$$\Phi \equiv \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \equiv \begin{pmatrix} \phi_1^+ & \phi_1^0 \\ \phi_2^+ & \phi_2^0 \\ \phi_3^+ & \phi_3^0 \end{pmatrix} \text{ where, } \phi_i^0 = \frac{v_i}{\sqrt{2}} + \frac{1}{\sqrt{2}}(\phi_i + i\chi_i)$$

$SU(2)_L \rightarrow$ horizontally, $A4 \rightarrow$ vertically.

Notations

$$\begin{aligned} \mathscr{L}_{mass} &= \frac{1}{2}(\chi_1\ \chi_2\ \chi_3)\, M^2_{\chi_i\chi_j} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} \Rightarrow \text{pseudoscalar masses} \\ &+ \frac{1}{2}(\phi_1\ \phi_2\ \phi_3)\, M^2_{\phi_i\phi_j} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \Rightarrow \text{neutral scalar masses} \\ &+ (\phi_1^-\ \phi_2^-\ \phi_3^-)\, M^2_{\phi_i^\mp\phi_j^\pm} \begin{pmatrix} \phi_1^+ \\ \phi_2^+ \\ \phi_3^+ \end{pmatrix} \Rightarrow \text{charged scalar masses} \end{aligned}$$

The Potential

The $SU(2)_L \times U(1)_Y$ and $A4$ conserving Lagrangian:

$$
\begin{aligned}
V(\Phi_i) &= m^2 \left(\sum_{i=1}^{3} \Phi_i^\dagger \Phi_i\right) + \frac{\lambda_1}{2}\left(\sum_{i=1}^{3} \Phi_i^\dagger \Phi_i\right)^2 \\
&+ \frac{\lambda_2}{2}\left(\Phi_1^\dagger\Phi_1 + \omega^2\Phi_2^\dagger\Phi_2 + \omega\Phi_3^\dagger\Phi_3\right)\left(\Phi_1^\dagger\Phi_1 + \omega\Phi_2^\dagger\Phi_2 + \omega^2\Phi_3^\dagger\Phi_3\right) \\
&+ \frac{\lambda_3}{2}\left[\left(\Phi_1^\dagger\Phi_2\right)\left(\Phi_2^\dagger\Phi_1\right) + \left(\Phi_2^\dagger\Phi_3\right)\left(\Phi_3^\dagger\Phi_2\right) + \left(\Phi_3^\dagger\Phi_1\right)\left(\Phi_1^\dagger\Phi_3\right)\right] \\
&+ \lambda_4\left[\left(\Phi_1^\dagger\Phi_2\right)^2 + \left(\Phi_2^\dagger\Phi_1\right)^2 + \left(\Phi_2^\dagger\Phi_3\right)^2 + \left(\Phi_3^\dagger\Phi_2\right)^2 + \left(\Phi_3^\dagger\Phi_1\right)^2 + \left(\Phi_1^\dagger\Phi_3\right)^2\right] \ .
\end{aligned}
$$

Recall: $3 \times 3 = \boxed{1} + 1' + 1'' + 3 + 3$ and $1' \times 1'' = \boxed{1}$.

The global minima vev configurations for $A4$ triplet scalars:

$$
\langle\Phi\rangle_1 = \frac{v}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 0 & 0\\ 0 & 0\end{pmatrix} \ , \ \langle\Phi\rangle_2 = \frac{v}{2}\begin{pmatrix}0 & 1\\ 0 & e^{i\alpha}\\ 0 & 0\end{pmatrix} \ , \ \langle\Phi\rangle_3 = \frac{v}{\sqrt{6}}\begin{pmatrix}0 & 1\\ 0 & 1\\ 0 & 1\end{pmatrix} \ , \ \langle\Phi\rangle_4 = \frac{v}{\sqrt{6}}\begin{pmatrix}0 & 1\\ 0 & \omega\\ 0 & \omega^2\end{pmatrix}
$$

$\Downarrow$

straight forward

Alignment follows as a consequence of $A4$ for all the above four configurations

$\langle\Phi\rangle_1 = \frac{v}{\sqrt{2}}(1,0,0)^T$ case:

$$\langle\Phi\rangle_1 = \frac{v}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \;, \Rightarrow \;\; \text{Minimization condition}: \; m^2 + \frac{v^2}{2}\left[\lambda_1 + \lambda_2\right] = 0$$

The charged scalar sector the mass-squared matrix is:

$$M^2_{\phi_i^{\mp}\phi_j^{\pm}} = diag(0\;,\; r_+\;,\; r_+) \;\; \text{where} \;\; r_+ = \frac{v^2}{4}\left(-3\lambda_2\right) \;\; .$$

The pseudoscalar mass-squared matrix:

$$M^2_{\chi_i\chi_j} = diag(0\;,\; p\;,\; p) \;\; \text{where} \;\; p = \frac{v^2}{4}\left(-3\lambda_2 + \lambda_3 - 4\lambda_4\right) \;\; .$$

The real neutral scalar mass squared matrix:

$$M^2_{\phi_i\phi_j} = diag(q\;,\; r_0\;,\; r_0) \;\; \text{where} \;\; q = v^2\left(\lambda_1 + \lambda_2\right)\;,\; r_0 = \frac{v^2}{4}\left(-3\lambda_2 + \lambda_3 + 4\lambda_4\right) \;\; .$$

Alignment achieved.

$\langle\Phi\rangle_3 = \frac{v}{\sqrt{6}}(1,1,1)^T$ case:

$$\langle\Phi\rangle_3 = \frac{v}{\sqrt{6}}\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow \text{ Minimization condition : } m^2 + \frac{v^2}{12}\left[3\lambda_1 + \lambda_3 + 4\lambda_4\right] = 0$$

Pseudoscalar Mass square matrix:

$$M^2_{\chi_i\chi_j} = 2\lambda_4\left(\frac{v^2}{6}\right)\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

Neutral scalar Mass square matrix:

$$M^2_{\phi_i\phi_j} = \left(\frac{v^2}{6}\right)\begin{pmatrix} y & z & z \\ z & y & z \\ z & z & y \end{pmatrix} \text{ where, } y = (\lambda_1+\lambda_2) \text{ and } z = (\lambda_1 - \lambda_2/2 + \lambda_3/2 + 2\lambda_4).$$

Charged scalar Mass square matrix:

$$M^2_{\phi_i^{\mp}\phi_j^{\pm}} = \left(\frac{v^2}{6}\right)\left(\lambda_4 + \frac{\lambda_3}{4}\right)\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Diagonalized by unitary transformation:

$$U_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}$$

$\langle\Phi\rangle_3$ case continued .. .

$$D^2_{\chi_i\chi_j} = \lambda_4\, v^2 \mathrm{diag}(0,\ -1,\ -1)\ ,$$

$$D^2_{\phi_i^{\mp}\phi_j^{\pm}} = \left(\lambda_4 + \frac{\lambda_3}{4}\right)\left(\frac{v^2}{2}\right)\mathrm{diag}(0,\ -1,\ -1)\ ,$$

$$D^2_{\phi_i\phi_j} = \left(\frac{v^2}{6}\right)\mathrm{diag}(y+2z,\ y-z,\ y-z)\ .$$

where, $y + 2z = 3\lambda_1 + \lambda_3 + 4\lambda_4$ and $y - z = 3\lambda_2/2 - \lambda_3/2 - 2\lambda_4$

$(3\lambda_1 + \lambda_3 + 4\lambda_4) > 0$, $(3\lambda_2/2 - \lambda_3/2 - 2\lambda_4) > 0$, $\lambda_4 < 0$ and $\left(\lambda_4 + \frac{\lambda_3}{4}\right) < 0$

$\Downarrow$

consistent with positivity and unitarity

Apply U_3 on $\langle\Phi\rangle_3$:

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}\left[\frac{v}{\sqrt{6}}\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}\right] = \frac{v}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \text{Alignment achieved.}$$

$\langle\Phi\rangle_4 = \frac{v}{\sqrt{6}}(1,\omega,\omega^2)^T$ case:

$$\langle\Phi\rangle_4 = \frac{v}{\sqrt{6}}\begin{pmatrix} 0 & 1 \\ 0 & \omega \\ 0 & \omega^2 \end{pmatrix} \Rightarrow \text{ Minimization condition : } m^2 + \frac{v^2}{12}\left[3\lambda_1+\lambda_3-2\lambda_4\right]=0$$

complex vev $\rightarrow \phi-\chi$ mixing.

The (6×6) mass squared matrix in $(\chi_1,\chi_2,\chi_3,\phi_1,\phi_2,\phi_3)$ basis:

$$M^2_{\Phi^0_i\Phi^0_j} = \frac{v^2}{6}\begin{pmatrix} 2\lambda_4 & -\lambda_4 & -\lambda_4 & 0 & \sqrt{3}\lambda_4 & -\sqrt{3}\lambda_4 \\ -\lambda_4 & f_2 & f_1 & g_1 & g_2 & g_3 \\ -\lambda_4 & f_1 & f_2 & -g_1 & -g_3 & -g_2 \\ 0 & g_1 & -g_1 & (\lambda_1+\lambda_2) & h_1 & h_1 \\ \sqrt{3}\lambda_4 & g_2 & -g_3 & h_1 & h_3 & h_2 \\ -\sqrt{3}\lambda_4 & g_3 & -g_2 & h_1 & h_2 & h_3 \end{pmatrix},$$

where,

$$f_1 = -\frac{3}{4}\left[\lambda_1-\lambda_2/2+\lambda_3/2\right]-\lambda_4,\quad f_2=\frac{3}{4}(\lambda_1+\lambda_2)+\frac{1}{2}\lambda_4,$$

$$g_1 = \frac{\sqrt{3}}{4}\{2\lambda_1-\lambda_2+\lambda_3-4\lambda_4\},\quad g_2=-\frac{\sqrt{3}}{4}\{\lambda_1+\lambda_2-2\lambda_4\},\quad g_3=-\frac{\sqrt{3}}{4}\{\lambda_1-\lambda_2/2+\lambda_3/2-4\lambda_4\},$$

$$h_1 = -\frac{1}{4}\{2\lambda_1-\lambda_2+\lambda_3+4\lambda_4\},\quad h_2=\frac{1}{8}\{2\lambda_1-\lambda_2+\lambda_3-8\lambda_4\},\quad h_3=\frac{1}{4}\{\lambda_1+\lambda_2+6\lambda_4\}.$$

The charged scalar mass squared matrix:

$$M^2_{\phi_i^{\mp}\phi_j^{\pm}} = \frac{v^2}{6}\begin{pmatrix} a & b & b^* \\ b^* & a & b \\ b & b^* & a \end{pmatrix} \rightarrow \text{ diagonalized by } U_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}.$$

where, $a=(2\lambda_4-\lambda_3)/2$ and $b=(\omega^2\lambda_3+4\omega\lambda_4)/4$.

Eigenvalues: $(a+2Re(b)) = -\frac{v^2}{6}(3\lambda_3/4)$, $\boxed{(a-Re(b)-\sqrt{3}Im(b))=0}$ $\Rightarrow$ Goldstone mode

and $(a-Re(b)+\sqrt{3}Im(b)) = -\frac{v^2}{6}(3\lambda_3/4-3\lambda_4)$. $\Rightarrow$ For eigenvalues to be positive: $\boxed{\lambda_3<0 \text{ and } \lambda_4>0}$

$\langle\Phi\rangle_4$ case continued .. .

Apply U_3 on $\langle\Phi\rangle_4$:

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega^2 & \omega\\ 1 & \omega & \omega^2\end{pmatrix}\left[\frac{v}{\sqrt{6}}\begin{pmatrix}0 & 1\\ 0 & \omega\\ 0 & \omega^2\end{pmatrix}\right] = \frac{v}{\sqrt{2}}\begin{pmatrix}0 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix} \Rightarrow \text{SM scalar doublet}$$

$M^2_{\Phi_i^0\Phi_j^0}$ **Diagonalization:**

$$\text{Apply}\, U_{6r} = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1 & 0 & 0 & 0\\ 1 & -1/2 & -1/2 & 0 & -\sqrt{3}/2 & \sqrt{3}/2\\ 1 & -1/2 & -1/2 & 0 & \sqrt{3}/2 & -\sqrt{3}/2\\ 0 & 0 & 0 & 1 & 1 & 1\\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 & 1 & -1/2 & -1/2\\ 0 & -\sqrt{3}/2 & \sqrt{3}/2 & 1 & -1/2 & -1/2\end{pmatrix} \text{Recall:}\, \omega = e^{i\frac{2\pi}{3}};\, \cos\frac{2\pi}{3} = -\frac{1}{2}\ \text{and}\ \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

$$U_{6r}\begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\\ \phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \eta_1\\ \eta_2\\ \eta_3\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}\chi_1+\chi_2+\chi_3\\ \chi_1-(\chi_2+\chi_3)/2-\sqrt{3}(\phi_2-\phi_3)/2\\ \chi_1-(\chi_2+\chi_3)/2+\sqrt{3}(\phi_2-\phi_3)/2\\ \phi_1+\phi_2+\phi_3\\ \sqrt{3}(\chi_2-\chi_3)/2+\phi_1-(\phi_2+\phi_3)/2\\ -\sqrt{3}(\chi_2-\chi_3)/2+\phi_1-(\phi_2+\phi_3)/2\end{pmatrix}.$$

Reduces $M^2_{\Phi_i^0\Phi_j^0}$ into two 2×2 block diagonal forms

The decoupled states: $m^2_{\xi_2} = 0$ $\Rightarrow$ Goldstone mode

and $m^2_{\eta_2} = v^2\,(3\lambda_1/2 + \lambda_3/2 - \lambda_4)$

Alignment obtained

$\langle\Phi\rangle_4$ case continued .. .

- The 2×2 block diagonal matrices:

$$M^2_{\xi_1,\xi_3} = \frac{v^2}{6}\begin{pmatrix}\lambda_A & -\lambda_A \\ -\lambda_A & \lambda_A + 18\lambda_4\end{pmatrix}\ ,\ M^2_{\eta_1,\eta_3} = \frac{v^2}{6}\begin{pmatrix}\lambda_A & \lambda_A \\ \lambda_A & \lambda_A + 18\lambda_4\end{pmatrix}$$

where; $\lambda_A = \frac{9}{4}\lambda_2 - \frac{3}{4}\lambda_3 - 3\lambda_4$.

- **The eigenvalues and eigenvectors:**

$$m_1^2 = \frac{v^2}{6}\left[\lambda_A + 9\lambda_4 + \sqrt{\lambda_A^2 + 81\lambda_4^2}\right]\ ,\ \xi_1 = \chi_1\cos\alpha - \chi_3\sin\alpha\ ,\ \eta_1 = \phi_1\cos\alpha + \phi_3\sin\alpha$$

$$m_3^2 = \frac{v^2}{6}\left[\lambda_A + 9\lambda_4 - \sqrt{\lambda_A^2 + 81\lambda_4^2}\right]\ ,\ \xi_3 = \chi_1\sin\alpha + \chi_3\cos\alpha\ ,\ \eta_3 = -\phi_1\sin\alpha + \phi_3\cos\alpha$$

with $\boxed{\tan 2\alpha = \frac{\lambda_A}{9\lambda_4}}$

- U_{6r} acting on real fields (χ_i, ϕ_i) is equivalent to unitary transformation by U_3 on complex fields, $\phi_i^0 = \phi_i + i\chi_i$.
- Positivity and unitarity constraints have no contradiction with positivity of the scalar mass squares.

$\Downarrow$

Alignment is manifested.

Conclusions

- **General discussion:** a) Discrete flavour symmetry $A4$.
  b) Alignment in multi-Higgs models.
- $A4$ **symmetric 3HDM.**
- Identified four *global minima* configurations of $vevs$.
- Alignment follows as a consequence of $A4$ symmetry for all four global minima configurtions.

THANK YOU!!

Backup Slides

Positivity:

$$M_{cop} = \begin{pmatrix} \lambda_P & \lambda_Q & \lambda_Q \\ \lambda_Q & \lambda_P & \lambda_Q \\ \lambda_Q & \lambda_Q & \lambda_P \end{pmatrix} ,$$

$\lambda_P = (\lambda_1 + \lambda_2)/2 \, , \; \lambda_Q = (2\lambda_1 - \lambda_2 + \lambda_3 + 4\lambda_4)/4$

The conditions to be satisfied are:

$$\lambda_P \geq 0 \, , \; \lambda_P + \lambda_Q \geq 0 \, , \; \text{and} \; \sqrt{\lambda_P^3} + (3\lambda_Q) \, \sqrt{\lambda_P} + \sqrt{2(\lambda_P + \lambda_Q)^3} \; \geq 0 \, .$$

For real case: $\lambda_1 + \lambda_2 \geq 0 \, , \; 3\lambda_1 + \lambda_3 + 4\lambda_4 \geq 0$

For complex case: $\lambda_Q \to \lambda_R = (2\lambda_1 - \lambda_2 + \lambda_3 - 2\lambda_4)/4$

Conditions: $\lambda_1 + \lambda_2 \geq 0 \, , \; 3\lambda_1 + \lambda_3 - 2\lambda_4 \geq 0 \, .$

Unitarity:

Quantum numbers		Type	Matrix size	Eigenvalues
$SU(2)_L$	Y			
1	2	Diagonal	3×3	$\lvert(\lambda_1 - 2\lambda_4)\rvert\ ,\ \lvert(\lambda_1 + 4\lambda_4)\rvert$
1	2	Off-diagonal	3×3	$\lvert(\lambda_3 - 3\lambda_2)/2\rvert$
0	2	Off-diagonal	3×3	$\lvert(\lambda_3 + 3\lambda_2)/2\rvert$
1	0	Diagonal	3×3	$\lvert(\lambda_1 - \lambda_3/2)\rvert\ ,\ \lvert(\lambda_1 + \lambda_3)\rvert$
1	0	Off-diagonal	6×6	$\lvert(3\lambda_2 + 4\lambda_4)/2\rvert\ ,\ \lvert(3\lambda_2 - 4\lambda_4)/2\rvert$
0	0	Diagonal	3×3	$\lvert(6\lambda_1 + 6\lambda_2 - \lambda_3)/2\rvert\ ,\ \lvert(3\lambda_1 - 6\lambda_2 + \lambda_3)\rvert$
0	0	Off-diagonal	6×6	$\lvert(-3\lambda_2 + 2\lambda_3 - 12\lambda_4)/2\rvert\ ,\ \lvert(-3\lambda_2 + 2\lambda_3 + 12\lambda_4)/2\rvert$

$$8\pi S(1,2)_{diag} = \begin{pmatrix} \lambda_1 & 2\lambda_4 & 2\lambda_4 \\ 2\lambda_4 & \lambda_1 & 2\lambda_4 \\ 2\lambda_4 & 2\lambda_4 & \lambda_1 \end{pmatrix},$$

$$8\pi S(1,0)_{off-diag} = \begin{pmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X \end{pmatrix}, X = \begin{pmatrix} -3\lambda_2/2 & 2\lambda_4 \\ 2\lambda_4 & -3\lambda_2/2 \end{pmatrix}$$

$\langle\Phi\rangle_1 = \frac{v}{\sqrt{2}}(1,0,0)^T$ case:

$$\langle\Phi\rangle_1 = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} , \Rightarrow \text{ Minimization condition : } m^2 + \frac{v^2}{2}\left[\lambda_1 + \lambda_2\right] = 0$$

The charged scalar sector the mass-squared matrix is:

$$M^2_{\phi_i^{\mp}\phi_j^{\pm}} = diag(0\ ,\ r_+\ ,\ r_+) \text{ where } r_+ = \frac{v^2}{4}(-3\lambda_2)\ .$$

The pseudoscalar mass-squared matrix:

$$M^2_{\chi_i\chi_j} = diag(0\ ,\ p\ ,\ p) \text{ where } p = \frac{v^2}{4}(-3\lambda_2 + \lambda_3 - 4\lambda_4)\ .$$

The real neutral scalar mass squared matrix:

$$M^2_{\phi_i\phi_j} = diag(q\ ,\ r_0\ ,\ r_0) \text{ where } q = v^2\ (\lambda_1 + \lambda_2)\ ,\ r_0 = \frac{v^2}{4}(-3\lambda_2 + \lambda_3 + 4\lambda_4)\ .$$

$\langle\Phi\rangle_2 = \frac{v}{2}(1, e^{i\alpha}, 0)$ **case**

$$m^2 + \frac{v^2}{4}\left[\lambda_1 + \frac{1}{4}\lambda_2 + \frac{1}{4}\lambda_3 - |\lambda_4|\right] = 0 \text{ and } \delta + 2\alpha = \pi$$

$$M^2_{\phi_i^{\mp}\phi_j^{\pm}} = \left(\frac{v^2}{4}\right)\begin{pmatrix} -\lambda_3/4 + |\lambda_4| & e^{-i\alpha}(\lambda_3/4 - |\lambda_4|) & 0 \\ e^{i\alpha}(\lambda_3/4 - |\lambda_4|) & -\lambda_3/4 + |\lambda_4| & 0 \\ 0 & 0 & -3\lambda_2/4 - \lambda_3/4 + |\lambda_4| \end{pmatrix}.$$

$\langle \Phi \rangle_2 = \frac{v}{2}(1, e^{i\alpha}, 0)$ **case**

$M^2_{\phi_i^{\mp}\phi_j^{\pm}}$ is diagonalized by:

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = U_{2c} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \quad \text{with } U_{2c} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{-i\alpha} & 0 \\ e^{i\alpha} & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

$$M^2_{\chi_1,\chi_2,\phi_1,\phi_2} = \frac{v^2}{4} 2|\lambda_4| \left[I + \begin{pmatrix} 0 & -\cos\alpha & 0 & \sin\alpha \\ -\cos\alpha & K\sin^2\alpha & J\sin\alpha & K\sin\alpha\cos\alpha \\ 0 & J\sin\alpha & K & J\cos\alpha \\ \sin\alpha & K\sin\alpha\cos\alpha & J\cos\alpha & K\cos^2\alpha \end{pmatrix} \right]$$

where, $K = (\lambda_1 + \lambda_2 - 2|\lambda_4|)/2|\lambda_4|$ and $J = (\lambda_1 - \lambda_2/2 + \lambda_3/2 - 2|\lambda_4|)/2|\lambda_4|$.

$$U^\dagger_{4r}\,[M^2_{\chi_1,\chi_2,\phi_1,\phi_2}]\,U_{4r} = \frac{v^2}{4} 2|\lambda_4| \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 + (-1+K-J)\sin^2\alpha & 0 & (-1+K-J)\sin\alpha\cos\alpha \\ 0 & 0 & (1+K+J) & 0 \\ 0 & (-1+K-J)\sin\alpha\cos\alpha & 0 & 2+(-1+K-J)\cos^2\alpha \end{pmatrix}$$

where,

$$U_{4r} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \cos\alpha & 0 & -\sin\alpha \\ \cos\alpha & -1 & \sin\alpha & 0 \\ 0 & \sin\alpha & 1 & \cos\alpha \\ -\sin\alpha & 0 & \cos\alpha & -1 \end{pmatrix} .$$

$$m_{\xi_1} = 0\ ,\ m_{\eta_1} = \frac{v}{2}\sqrt{2\lambda_1 + \lambda_2/2 + \lambda_3/2 - 2|\lambda_4|} \Rightarrow \text{ Neutral Goldstone and Pseudoscalar}$$

$$m_{\xi_2',\,\eta_2'} = v\sqrt{|\lambda_4|}\ ,\ \tfrac{v}{2}\sqrt{3\lambda_2/2 - \lambda_3/2 + 2|\lambda_4|}\ .$$

$$M^2_{\chi_3,\,\phi_3} = \frac{v^2}{4}\left[(-\frac{3}{4}\lambda_2 + \frac{1}{4}\lambda_3 + |\lambda_4|)I + 2|\lambda_4|\cos(3\alpha) \begin{pmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & -\cos\alpha \end{pmatrix} \right]$$

$$m_{\xi_3',\,\eta_3'} = \tfrac{v}{2}\sqrt{-3\lambda_2/4 + \lambda_3/4 + |\lambda_4|\{1 \mp 2\cos(3\alpha)\}}$$