Effective picture of bubble expansion

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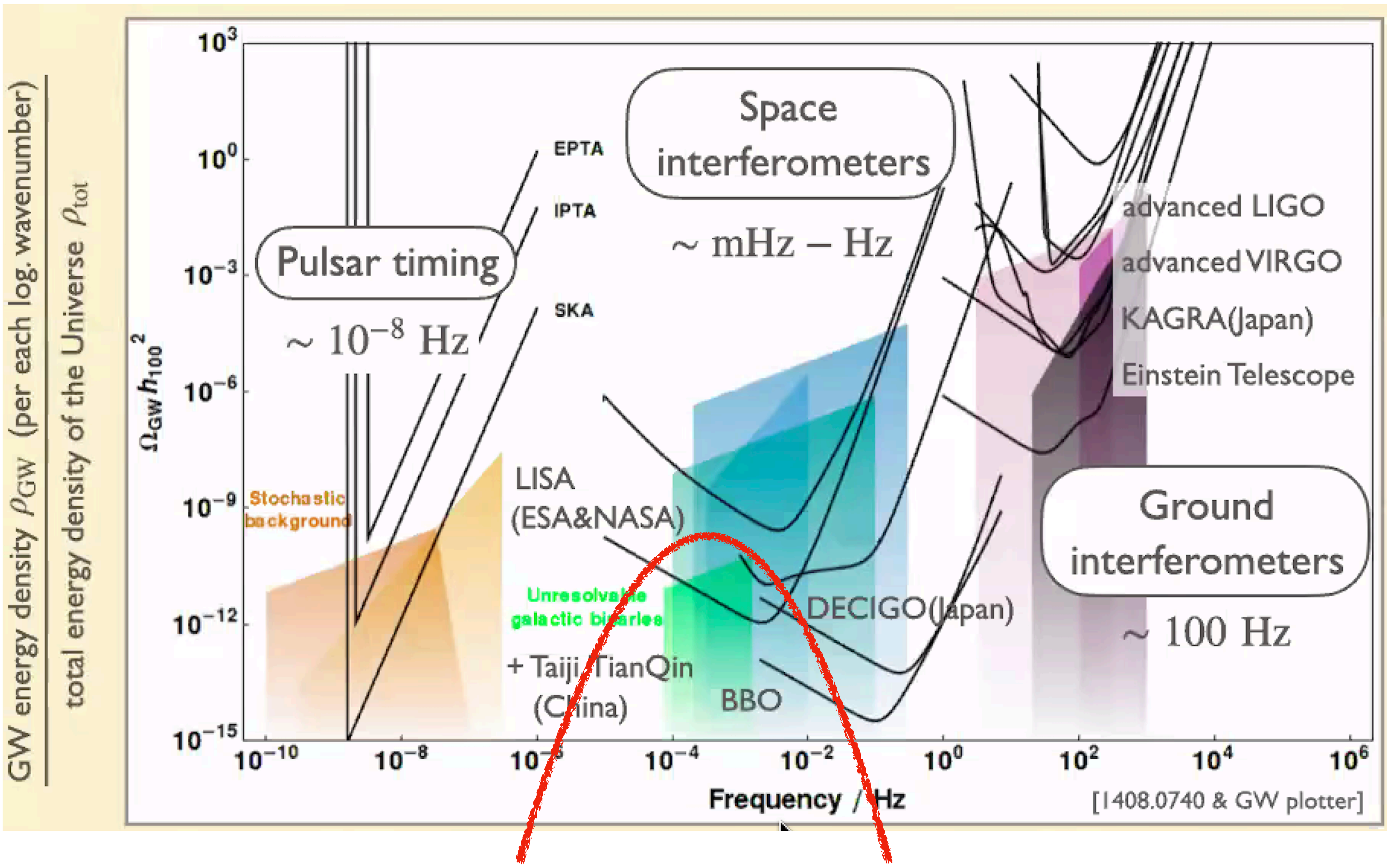
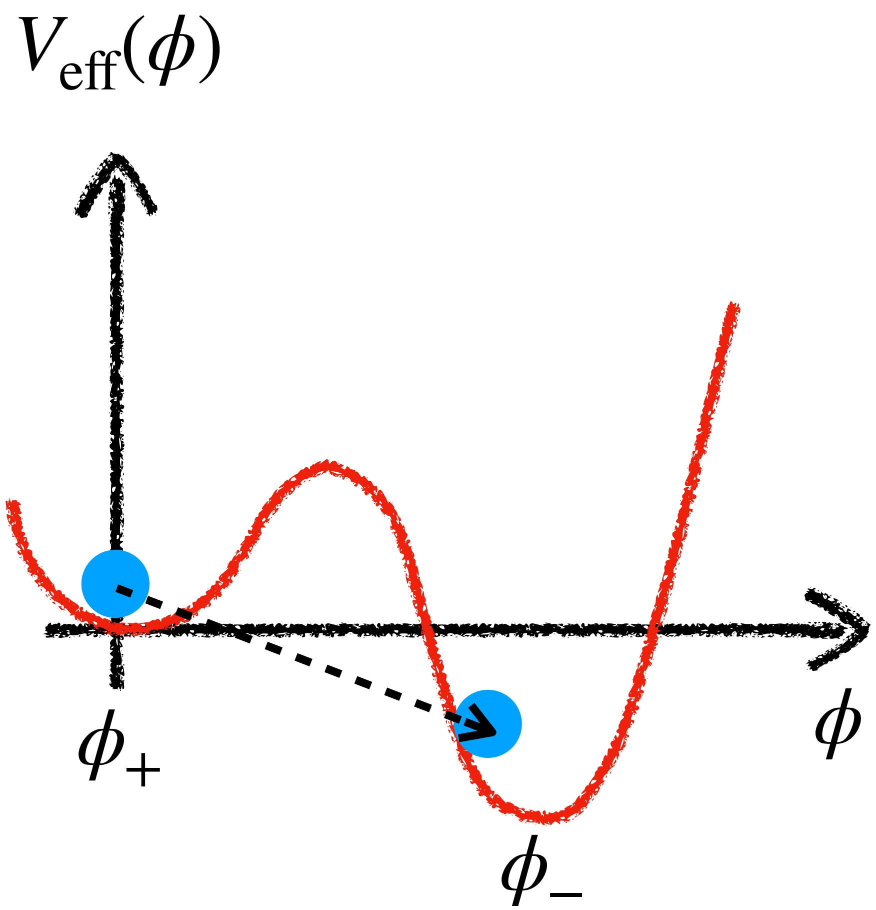
GWs from FOPT

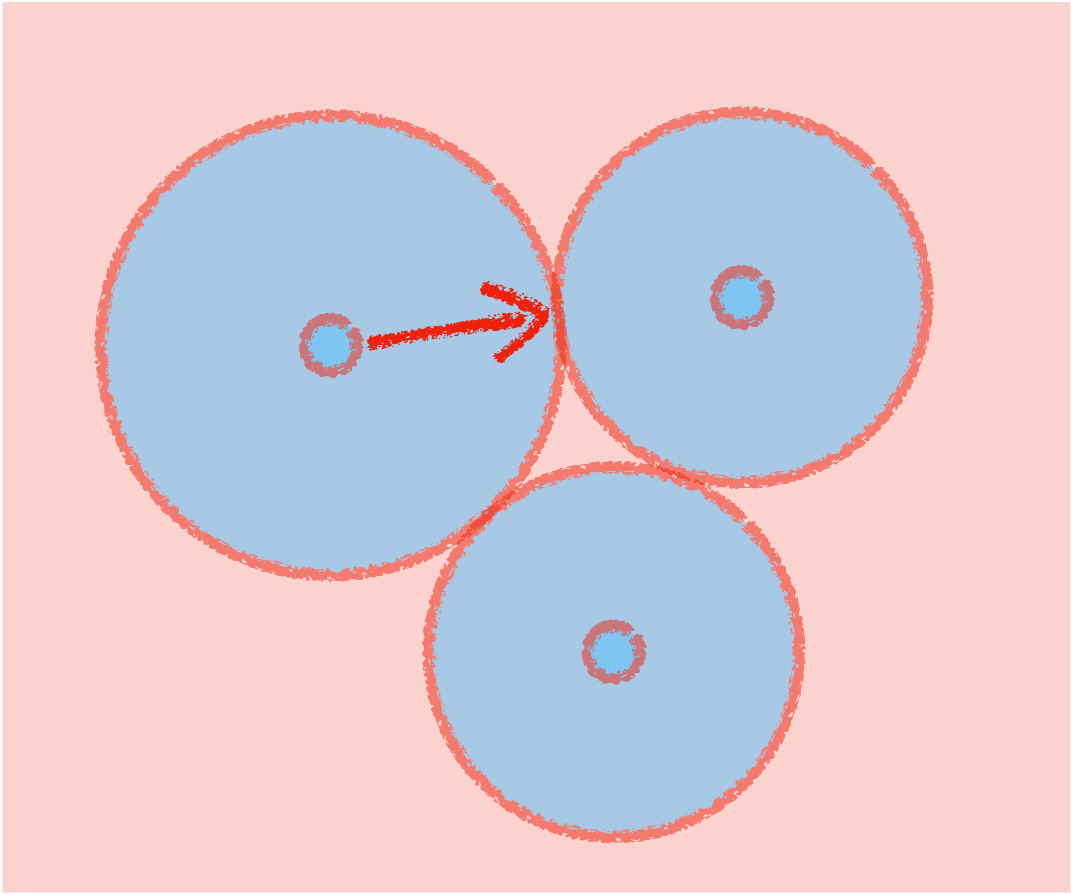


$T_{\rm PT}$ → **BSM**

MeV	**TeV**	**PeV**
QCDPT	**EWPT**	**PQPT**

Phase dynamics





$$\Gamma(t) \sim e^{-S(t)}$$

$$\Gamma(T_*) \sim H(T_*)$$

$$\alpha \sim \frac{\Delta V_{\text{eff}}}{\rho_{\text{rad}}}$$

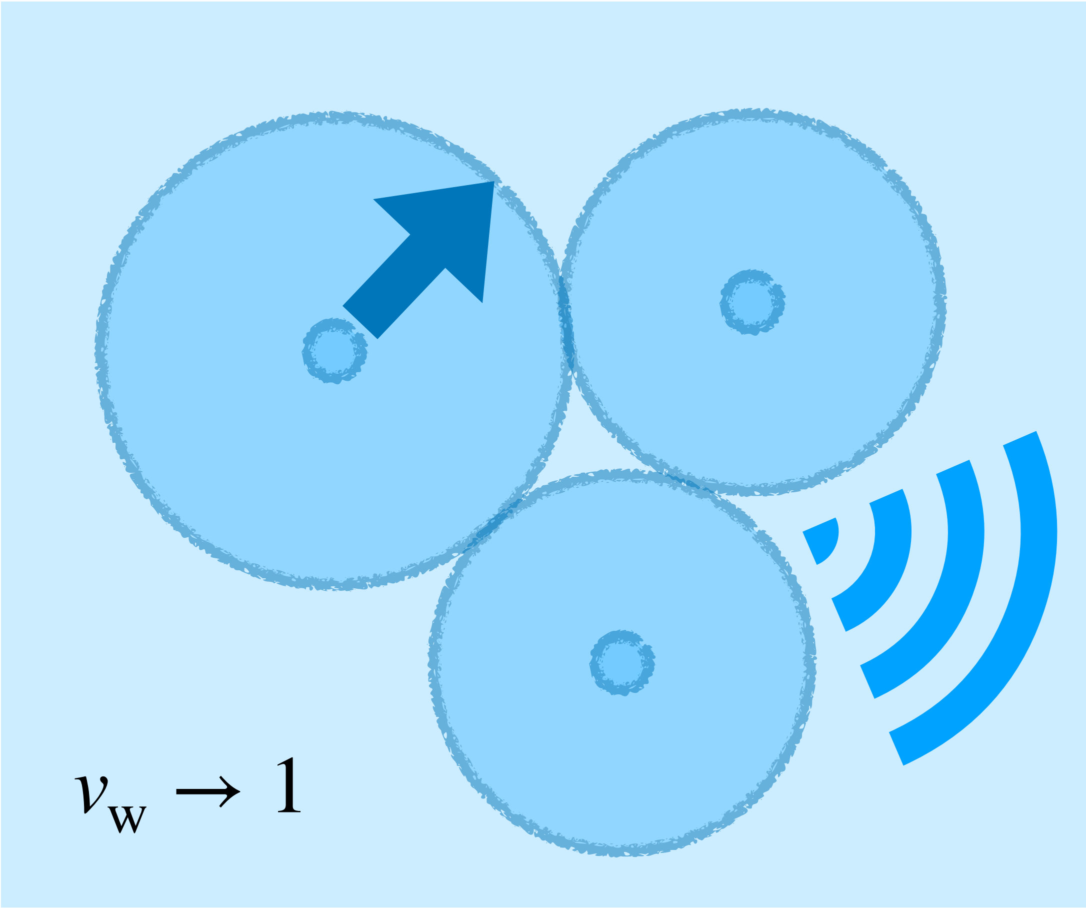
$$R_* \sim \langle R_i(T_*) \rangle$$

$$\Omega_{\text{peak}} \sim \alpha^2$$

$$f_{\text{peak}} \sim \frac{1}{R_*}$$

Bubble dynamics

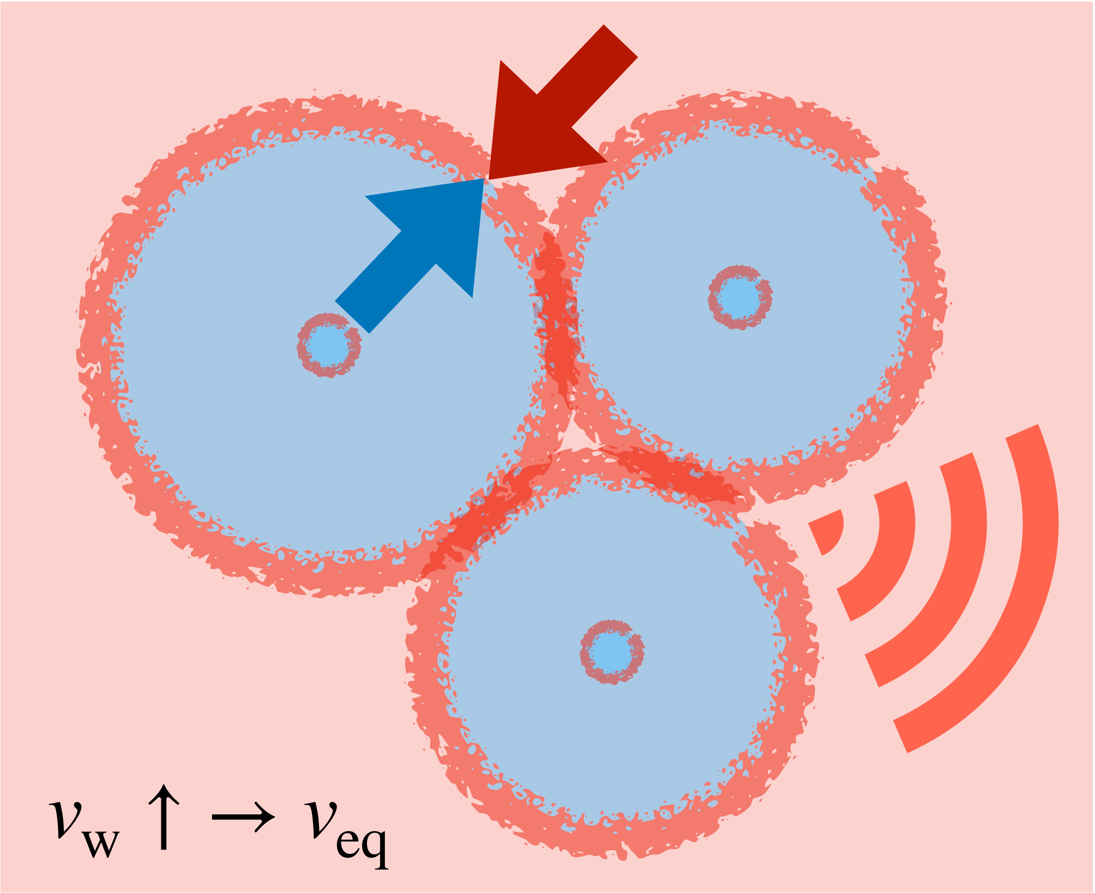
Bubble expansion in vacuum



Bubble walls collide with each other when they are rapidly accelerating

$$\Omega_{GW}^{scalar} \propto \kappa_{collision}^2 \sim \left(\frac{E_{kinetic}^{wall}}{\rho_{vac}} \right)^2$$

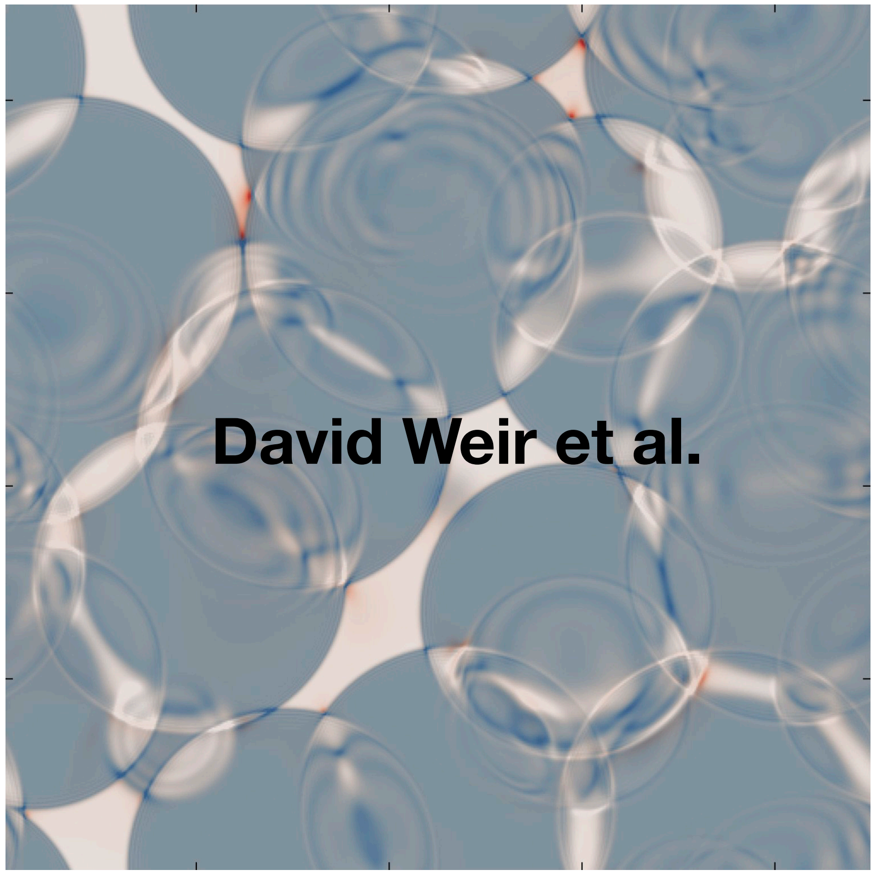
Bubble expansion in plasma



Bubble walls collide with each other when they are steadily expanding

$$\Omega_{GW}^{plasma} \propto \kappa_{soundwave}^2 \sim \left(\frac{E_{kinetic}^{fluid}}{\rho_{vac}} \right)^2$$

Numerical simulation



$$\Box\phi = \frac{\partial V_{\rm eff}}{\partial\phi} + \delta f$$

$$\partial_\mu T_{\rm p}^{\mu\nu} + \partial^\nu\phi\frac{\partial V_T}{\partial\phi} = -\,\partial^\nu\phi\cdot\delta f$$

$$\delta f = \sum_{i={\rm B,F}} g_i \frac{{\rm d}m_i^2}{{\rm d}\phi}\int\frac{{\rm d}^3\vec{k}}{(2\pi)^3}\frac{\delta f_i}{2E_i(\vec{k})} \quad ?$$

$$\Box\phi = \frac{\partial V_{\rm eff}}{\partial\phi} + \eta_T u^\mu\partial_\mu\phi$$

? **Parameterization**

Effective EOM $\longrightarrow$

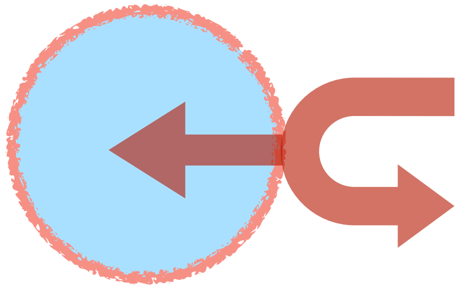
$$\int_{-\infty}^{+\infty}{\rm d}r'\phi'(r')\ \textbf{EOM} = 0$$

$$\sigma\gamma^3\ddot{R} + \frac{2\sigma\gamma}{R} \overset{?}{=} \Delta p_{\rm dr} - \Delta p_{\rm fr}$$

$$\Delta p_{\rm dr} = \Delta V_{\rm eff}$$

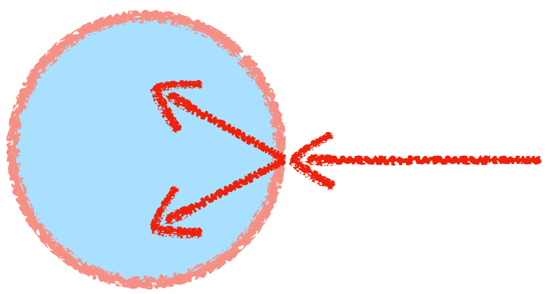
Particle physics $\Delta p_{\rm fr} \propto \gamma$

Thermal friction



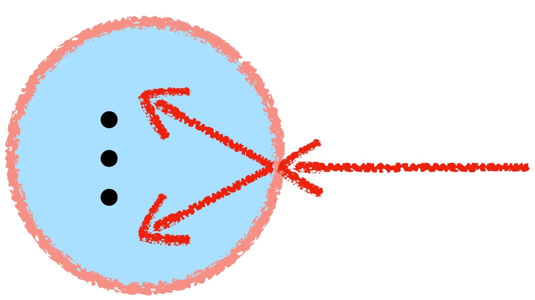
Bodeker & Moore 09 — **Particle transmission and reflection**

$$P_{1\to 1} \approx \frac{\Delta m^2 T^2}{24} \equiv \Delta p_{\rm LO} \longrightarrow \text{Run-away}$$



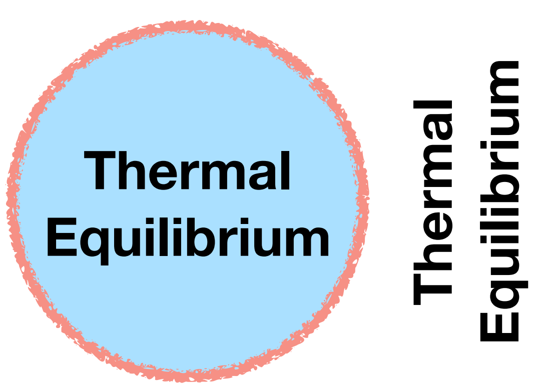
Bodeker & Moore 17 — **Transition splitting of a fermion emitting a soft vector boson**

$$P_{1\to 2} \approx \gamma g^2 \Delta m_V T^3 \equiv \gamma \Delta p_{\rm NLO} \longrightarrow \text{Non-run-away}$$



Hoche et al 2007.10343 — **Re-summing multiple soft gauge bosons scattering to all orders**

$$P_{1\to N} \approx 0.005 \gamma^2 g^2 T^4 \equiv \gamma^2 \Delta p_{NLO} \longrightarrow \text{Non-run-away}$$



Mancha et al 2005.10875

$$\Delta p = (\gamma^2 - 1) T \Delta s$$

$$\delta f = -\tilde{\eta}_T (u^\mu \partial_\mu \phi)^2$$

$$\Delta p_{\rm fr} \propto (\gamma^2 - 1)$$

Effective picture

$$\Delta p_{\rm fr} = \Delta p_{\rm LO} + h(\gamma)\Delta p_{NLO}$$

γ**-independent friction force** γ**-dependent friction force**

$$\cancel{\sigma\gamma^3\ddot{R} + \frac{2\sigma\gamma}{R} = \Delta p_{\rm dr} - \Delta p_{\rm fr}}$$

Kinetic energy of bubble wall

Potential energy of bubble

Work done by friction force on bubble wall

Conservation Law $$E = 4\pi\sigma R^2\gamma - \frac{4}{3}\pi R^3(\Delta p_{\rm dr} - \Delta p_{\rm fr})$$

$$\left(\sigma + \frac{R}{3}\frac{\mathrm{d}\Delta p_{\rm fr}}{\mathrm{d}\gamma}\right)\gamma^3\ddot{R} + \frac{2\sigma\gamma}{R} = \Delta p_{\rm dr} - \Delta p_{\rm fr}$$

$$R_\sigma = \frac{3}{2}(\gamma_{\rm eq} - 1)R_0$$

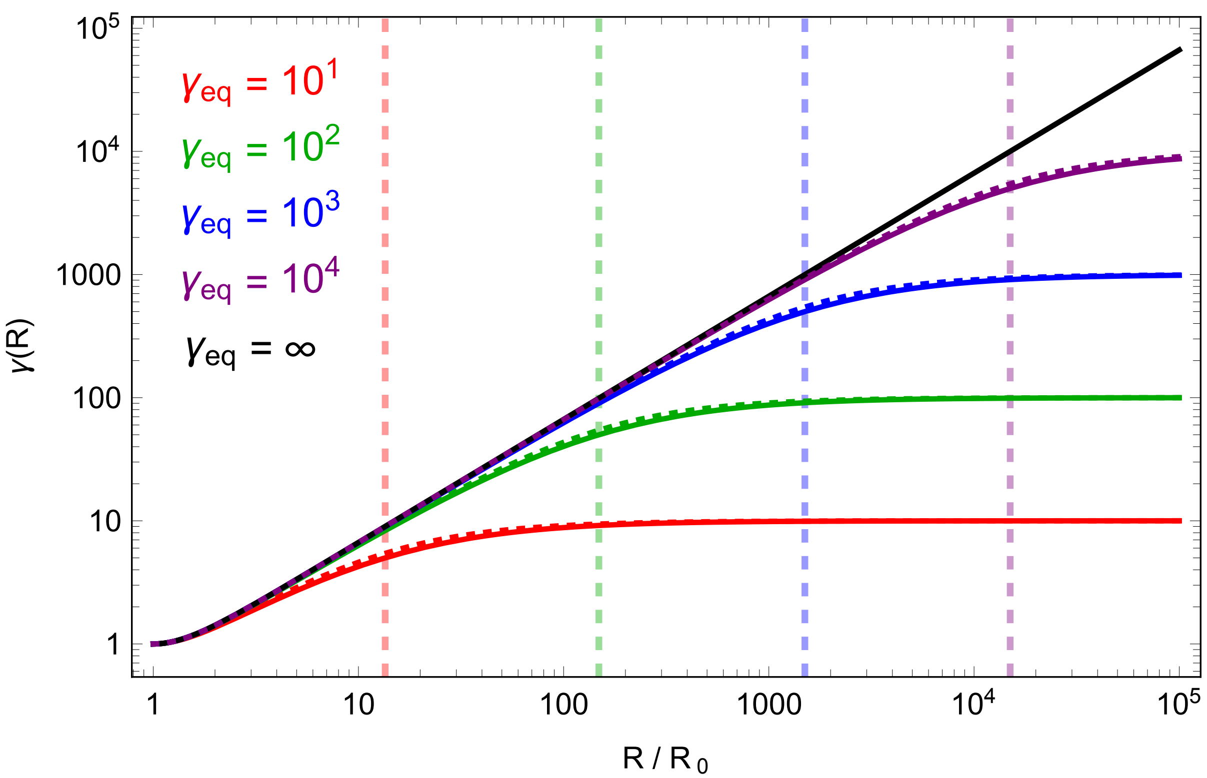
General solution

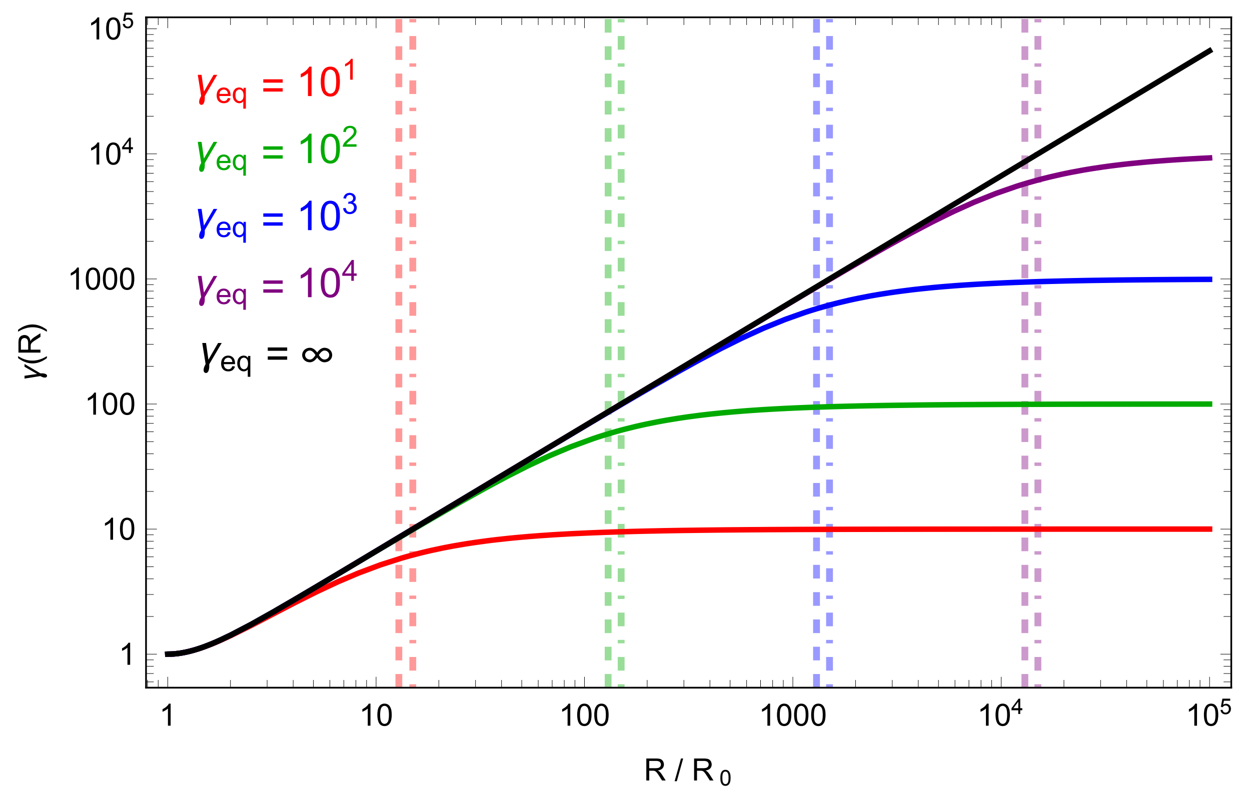
$$\frac{h(\gamma) - h(1)}{h(\gamma_{\rm eq}) - h(1)} + \frac{3\gamma}{2R} = 1 + \frac{1}{2R^3} \qquad h(\gamma_{\rm eq}) \equiv \frac{\Delta p_{\rm dr} - \Delta p_{\rm LO}}{\Delta p_{NLO}}$$

Wall velocity

$$P_{1\to2} \equiv h(\gamma)\Delta p_{\rm NLO}, \quad h(\gamma) = \gamma$$

$$P_{1\to N} \equiv h(\gamma)\Delta p_{\rm NLO}, \quad h(\gamma) = \gamma^2$$





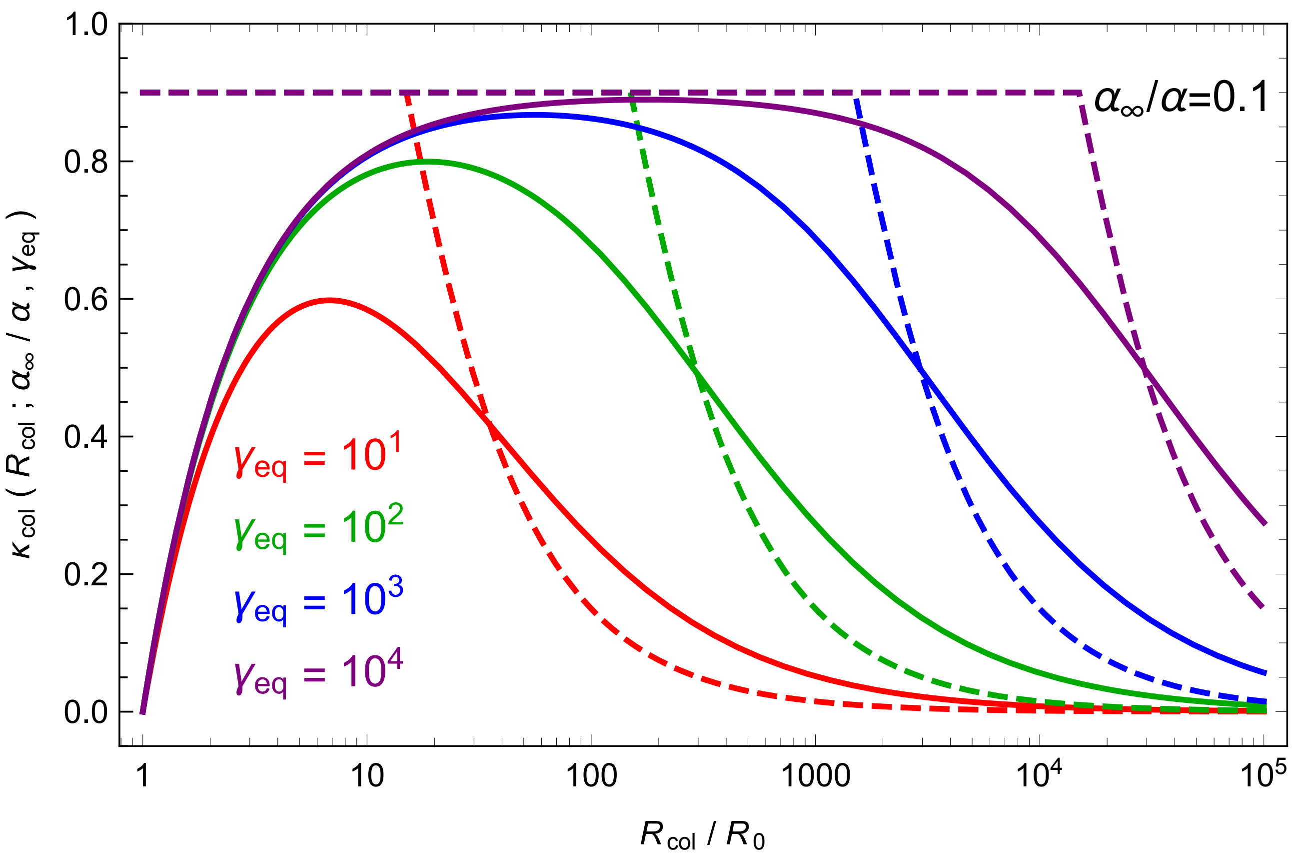
$$\gamma(R) = \frac{2\gamma_{\rm eq}R^3 + \gamma_{\rm eq} - 1}{2R^3 + 3(\gamma_{\rm eq} - 1)R^2}$$

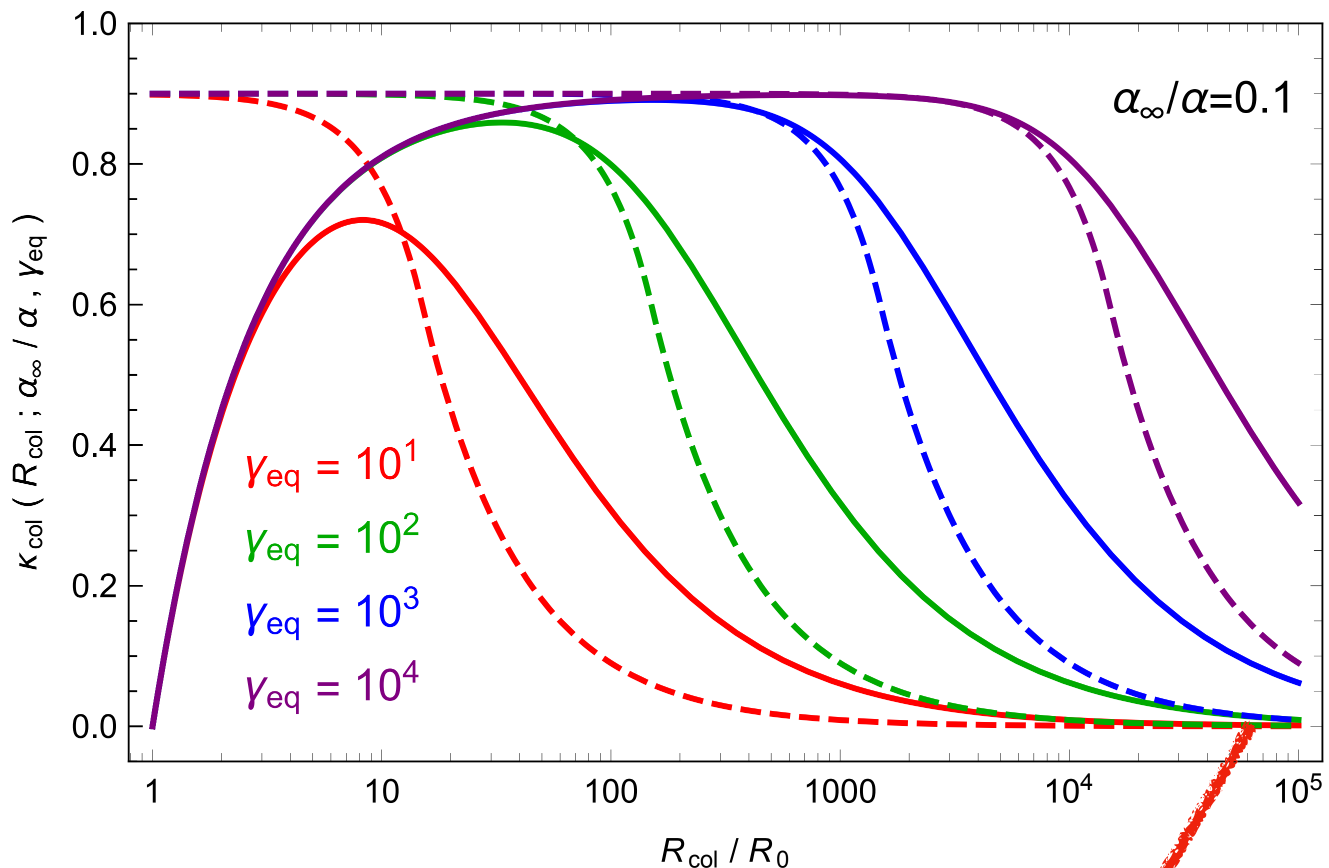
$$\lim_{\gamma_{\rm eq}\to\infty} \gamma(R) = \frac{2}{3}R + \frac{1}{3R^2}$$

$$\gamma(R) = \sqrt{\gamma_{\rm eq}^2 + \frac{9(\gamma_{\rm eq}^2 - 1)^2}{16R^2} + \frac{\gamma_{\rm eq}^2 - 1}{2R^3}} - \frac{3(\gamma_{\rm eq}^2 - 1)}{4R}$$

$$\lim_{R\to\infty} \gamma(R) = \gamma_{\rm eq}$$

Efficiency factor





For dash lines, see Marek Lewicki's talk

1. **Previous estimations are reproduced when bubble walls collide with each other long after they have been approaching the terminal velocity** $R_{\rm col} \gtrsim \mathcal{O}(10^3) R_\sigma$

2. **Important for strong FOPT : bubble walls collide with each other just around the time when they are starting to approach the terminal velocity** $R_{\rm col} \lesssim \mathcal{O}(10^2) R_\sigma$

An example

$\gamma_{\rm eq} = 10$ $\quad v_{\rm eq} \approx 0.995$

$R_\sigma \approx 15 R_0$ $\quad R_{\rm col} = 100 R_0$

$h(\gamma) = \gamma^2$

$\kappa_{\rm col}^{\rm old} \approx 0.1$

$\kappa_{\rm col}^{\rm new} \approx 0.3$

$$\frac{\Omega_{\rm col}^{\rm new}}{\Omega_{\rm col}^{\rm old}} \sim \left(\frac{\kappa_{\rm col}^{\rm new}}{\kappa_{\rm col}^{\rm old}}\right)^2 \approx 9$$

Take-home message

Effective EOM for an expanding bubble wall in thermal plasma out-of equilibrium

$$\left(\sigma + \frac{R}{3}\frac{\mathrm{d}\Delta p_{\mathrm{fr}}}{\mathrm{d}\gamma}\right)\frac{\mathrm{d}\gamma}{\mathrm{d}R} + \frac{2\sigma\gamma}{R} = \Delta p_{\mathrm{dr}} - \Delta p_{\mathrm{fr}}$$

General expansion solution with arbitrary γ-scaling friction $\Delta p_{\mathrm{fr}} \equiv \Delta p_{\mathrm{LO}} + h(\gamma)\Delta p_{NLO}$

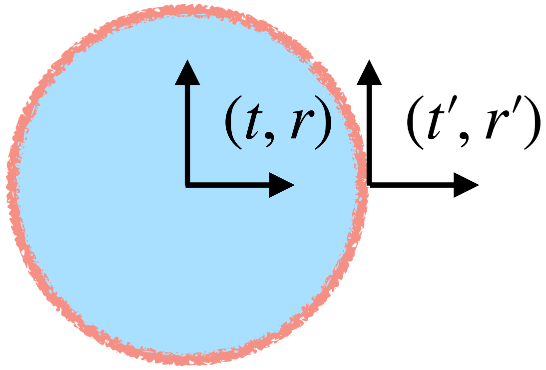
$$\frac{h(\gamma) - h(1)}{h(\gamma_{\mathrm{eq}}) - h(1)} + \frac{3\gamma}{2R} = 1 + \frac{1}{2R^3}$$

General expression for the efficiency factor from bubble wall collisions

$$\kappa_{\mathrm{col}} = \left(1 - \frac{\alpha_\infty}{\alpha}\right)\int_1^{R_{\mathrm{col}}}\frac{\mathrm{d}R}{R_{\mathrm{col}}}\left[1 - \frac{h(\gamma(R))}{h(\gamma_{\mathrm{eq}})}\right]$$

Thank you

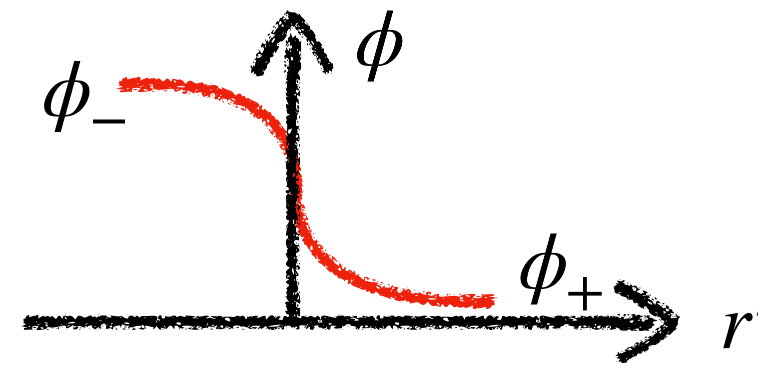
Backup slide



$$\phi(t,r) = \phi(\gamma_{\rm w}(t)[r - r_{\rm w}(t)]) \equiv \phi(r')$$

$$t' = \gamma_{\rm w}(t)[t - v_{\rm w}(t) r] \qquad \phi(r' = -\infty) = \phi_-$$

$$r' = \gamma_{\rm w}(t)[r - r_{\rm w}(t)] \qquad \phi(r' = +\infty) = \phi_+$$



Convert field derivative in bubble wall frame

$$\frac{\partial \phi}{\partial r} = \frac{\mathrm{d}\phi}{\mathrm{d}r'}\frac{\partial r'}{\partial r} = \phi'(r')\gamma_{\rm w}$$

$$\frac{\partial^2 \phi}{\partial r^2} = \frac{\mathrm{d}^2\phi}{\mathrm{d}r'^2}\left(\frac{\partial r'}{\partial r}\right)^2 + \frac{\mathrm{d}\phi}{\mathrm{d}r'}\frac{\partial^2 r'}{\partial r^2} = \phi''(r')\gamma_{\rm w}^2$$

Surface tension $\sigma = \int_{-\infty}^{+\infty} \mathrm{d}r' \phi'(r')^2$

Averaged value $\langle F \rangle = \frac{1}{\sigma}\int_{-\infty}^{+\infty} \mathrm{d}r' \phi'(r')^2 F(r')$

$\langle F \rangle = 0$ **if** $F(r')$ **is odd function**

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= \frac{\mathrm{d}\phi}{\mathrm{d}r'}\frac{\partial r'}{\partial t} \\ &= \phi'(r')[\dot{\gamma}_{\rm w}(r - r_{\rm w}) - \gamma_{\rm w} v_{\rm w}] \\ &= \phi'(r')[(\dot{\gamma}_{\rm w}/\gamma_{\rm w}) r' - \gamma_{\rm w} v_{\rm w}] \\ &= \phi'(r')(\gamma_{\rm w}^2 v_{\rm w} \ddot{r}_{\rm w} r' - \gamma_{\rm w} v_{\rm w}) \\ &= \phi'(r')\gamma_{\rm w} v_{\rm w}(\gamma_{\rm w}\ddot{r}_{\rm w} r' - 1)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \phi}{\partial t^2} &= \frac{\mathrm{d}^2\phi}{\mathrm{d}r'^2}\left(\frac{\partial r'}{\partial t}\right)^2 + \frac{\mathrm{d}\phi}{\mathrm{d}r'}\frac{\partial^2 r'}{\partial t^2} \\ &= \phi''(r')[\dot{\gamma}_{\rm w}(r - r_{\rm w}) - \gamma_{\rm w} v_{\rm w}]^2 \\ &\quad + \phi'(r')[\ddot{\gamma}_{\rm w}(r - r_{\rm w}) - 2\dot{\gamma}_{\rm w} v_{\rm w} - \gamma_{\rm w}\ddot{r}_{\rm w}] \\ &= \phi''(r')\gamma_{\rm w}^2 v_{\rm w}^2(\gamma_{\rm w}\ddot{r}_{\rm w} r' - 1)^2 \\ &\quad + \phi'(r')[(\ddot{\gamma}_{\rm w}/\gamma_{\rm w}) r' - \gamma_{\rm w}(2\gamma_{\rm w}^2 - 1)\ddot{r}_{\rm w}]\end{aligned}$$

Backup slide

Thermal bubble in thermal equilibrium $\nabla_\mu \nabla^\mu \phi \equiv \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} - \frac{\partial^2\phi}{\partial t^2} = \frac{\partial V_{\rm eff}}{\partial \phi}$

$\int_{-\infty}^{+\infty} \mathrm{d}r'\phi'(r')$ **EOM** $= 0$

Effective EOM of bubble wall

$$\int_{-\infty}^{+\infty} \mathrm{d}r'\phi'(r')\frac{\partial V_{\rm eff}}{\partial \phi} = \int_{\phi_-}^{\phi_+} \mathrm{d}\phi \left(\frac{\partial V_0}{\partial \phi} + \frac{\partial V_T}{\partial \phi}\right)$$
$$= \int_{\phi_-}^{\phi_+} \mathrm{d}\phi \frac{\mathrm{d}V_0}{\mathrm{d}\phi} + \sum_{i=\mathrm{B,F}} g_i T^4 \int_{\phi_-}^{\phi_+} \mathrm{d}\phi\, J_i'\left(\frac{m_i^2}{T^2}\right)\frac{1}{T^2}\frac{\mathrm{d}m_i^2}{\mathrm{d}\phi}$$
$$= V_0(\phi)\Big|_{\phi_-}^{\phi_+} + \sum_{i=\mathrm{B,F}} g_i T^4 J_i\left(\frac{m_i^2(\phi)}{T^2}\right)\Big|_{\phi_-}^{\phi_+}$$
$$\equiv -\Delta V_0 - \Delta V_T = -\Delta V_{\rm eff} \equiv V_{\rm eff}(\phi_+) - V_{\rm eff}(\phi_-)$$

$$\int_{-\infty}^{+\infty} \mathrm{d}r'\phi'(r')\frac{\partial^2\phi}{\partial r^2} = \int \mathrm{d}\left(\frac{\phi'^2}{2}\right)\gamma_{\rm w}^2 = \frac{\gamma_{\rm w}^2}{2}\phi'(r')^2\Big|_{-\infty}^{+\infty} = 0$$

$$\int_{-\infty}^{+\infty} \mathrm{d}r'\phi'(r')\frac{2}{r}\frac{\partial\phi}{\partial r} = \int_{-\infty}^{+\infty} \mathrm{d}r'\phi'(r')\frac{2}{r}\phi'(r')\gamma_{\rm w} \simeq \frac{2\sigma\gamma_{\rm w}}{r_{\rm w}}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}r'\phi'(r')\frac{\partial^2\phi}{\partial t^2} = \int_{-\infty}^{+\infty} \mathrm{d}\left(\frac{\phi'^2}{2}\right)\gamma_{\rm w}^2 v_{\rm w}^2(\gamma_{\rm w}\ddot{r}_{\rm w}r' - 1)^2 + \int_{-\infty}^{+\infty} \mathrm{d}r'\phi'(r')^2[(\ddot{\gamma}_{\rm w}/\gamma_{\rm w})r' - \gamma_{\rm w}(2\gamma_{\rm w}^2 - 1)\ddot{r}_{\rm w}]$$
$$= \frac{1}{2}\phi'(r')^2\gamma_{\rm w}^2 v_{\rm w}^2(\gamma_{\rm w}\ddot{r}_{\rm w}r' - 1)^2\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \phi'(r')^2\gamma_{\rm w}^2 v_{\rm w}^2(\gamma_{\rm w}\ddot{r}_{\rm w}r' - 1)\gamma_{\rm w}\ddot{r}_{\rm w}\mathrm{d}r'$$
$$+ \int_{-\infty}^{+\infty} \mathrm{d}r'\phi'(r')^2[(\ddot{\gamma}_{\rm w}/\gamma_{\rm w})r' - \gamma_{\rm w}(2\gamma_{\rm w}^2 - 1)\ddot{r}_{\rm w}]$$
$$= 0 + \sigma\gamma_{\rm w}^3 v_{\rm w}^2\ddot{r}_{\rm w} - \sigma\gamma_{\rm w}(2\gamma_{\rm w}^2 - 1)\ddot{r}_{\rm w} = -\sigma\gamma_{\rm w}^3\ddot{r}_{\rm w},$$

$$\sigma\gamma_{\rm w}^3\ddot{r}_{\rm w} + \frac{2\sigma\gamma_{\rm w}}{r_{\rm w}} = \Delta p_{\rm dr}$$

or

$$\frac{\mathrm{d}\gamma_{\rm w}}{\mathrm{d}r_{\rm w}} + \frac{2\gamma_{\rm w}}{r_{\rm w}} = \frac{\Delta p_{\rm dr}}{\sigma}$$

Backup slide

Thermal bubble out-of thermal equilibrium

$$\nabla_\mu \nabla^\mu \phi \equiv \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} - \frac{\partial^2\phi}{\partial t^2} = \frac{\partial V_{\rm eff}}{\partial\phi} + \sum_{i={\rm B,F}} g_i \frac{{\rm d}m_i^2}{{\rm d}\phi}\int \frac{{\rm d}^3\vec{k}}{(2\pi)^3}\frac{\delta f_i}{2E_i(\vec{k})} \qquad \begin{cases} \eta_T u^\mu \partial_\mu \phi \\ -\eta_T (u^\mu\partial_\mu\phi)^2 \end{cases}$$

$$\int_{-\infty}^{+\infty} {\rm d}r'\phi'(r')\ \textbf{EOM} = 0 \xrightarrow[\text{of bubble wall}]{\text{Effective EOM}} \sigma\gamma_{\rm w}^3 \ddot{r}_{\rm w} + \frac{2\sigma\gamma_{\rm w}}{r_{\rm w}} = \Delta p_{\rm dr} + \Delta p_{\rm fr}$$

$$\Delta p_{\rm fr} = \int_{-\infty}^{+\infty} {\rm d}r'\phi'(r')\eta_T u^\mu \partial_\mu \phi$$

$$= \int_{-\infty}^{+\infty} {\rm d}r'\phi'(r')\eta_T \left(\gamma_{\rm p}\frac{\partial\phi}{\partial t} + \gamma_{\rm p} v_{\rm p}\frac{\partial\phi}{\partial r}\right)$$

$$= \int_{-\infty}^{+\infty} {\rm d}r'\phi'(r')^2\eta_T [\gamma_{\rm p}\gamma_{\rm w} v_{\rm w}(\gamma_{\rm w}\ddot{r}_{\rm w} r' - 1) + \gamma_{\rm p} v_{\rm p}\gamma_{\rm w}]$$

$$= -\int_{-\infty}^{+\infty} {\rm d}r'\phi'(r')^2\eta_T \gamma_{\rm p}\gamma_{\rm w}(v_{\rm w} - v_{\rm p})$$

$\downarrow$ **detonation** $\quad v_{\rm p}' = -v_{\rm w}, \quad v_{\rm p} = 0, \quad \gamma_{\rm p} = 1$

$$= -\gamma_{\rm w} v_{\rm w}\sigma\langle\eta_T\rangle \propto -\gamma_{\rm w} v_{\rm w}$$

$$P_{1\to 2} = \gamma_{\rm w}\Delta p_{\rm NLO}$$

$$\Delta p_{\rm fr} = -\int_{-\infty}^{+\infty} {\rm d}r'\phi'(r')^2\eta_T\gamma_{\rm w}^2 v_{\rm w}^2 (\gamma_{\rm w}\ddot{r}_{\rm w} r' - 1)^2$$

$$= -\int_{-\infty}^{+\infty} {\rm d}r'\phi'(r')^2\eta_T\gamma_{\rm w}^2 v_{\rm w}^2 (1 + \gamma_{\rm w}^2\ddot{r}_{\rm w}^2 r'^2)$$

$$= -\gamma_{\rm w}^2 v_{\rm w}^2 \sigma\langle\eta_T\rangle \propto -(\gamma_{\rm w}^2 - 1)$$

late-time $\ddot{r}_{\rm w} \approx 0$

$$P_{1\to N} = \gamma_{\rm w}^2 \Delta p_{NLO}$$

?

Backup slide

Thermal bubble in thermal equilibrium

Effective Lagrangian $L = -4\pi\sigma R^2\sqrt{1-\dot{R}^2} + \frac{4}{3}\pi R^3 \Delta p_{\rm dr}$

Euler-Lagrangian Eq. $\ddot{R} + 2\frac{1-\dot{R}^2}{R} = \frac{\Delta p_{\rm dr}}{\sigma}\left(1-\dot{R}^2\right)^{\frac{3}{2}}$ $\frac{{\rm d}\gamma}{{\rm d}R} = \ddot{R}\gamma^3 = \frac{\Delta p_{\rm dr}}{\sigma} - \frac{2\gamma}{R}$ $\gamma = \frac{1}{\sqrt{1-\dot{R}^2}}$

Solution to effective EOM $\gamma(R) = \frac{\Delta p_{\rm dr}}{3\sigma}R + \frac{C}{R^2}$ C **is fixed by** $\gamma(R_0) = 1$

Bubble energy $E = (\partial L/\partial \dot{R})\dot{R} - L = 4\pi\sigma R^2\gamma - \frac{4}{3}\pi R^3 \Delta p_{\rm dr}$ $\left.\frac{{\rm d}E}{{\rm d}R}\right|_{\gamma(t=0)=1} = 0$ $R_0 = \frac{2\sigma}{\Delta p_{\rm dr}}$

Solution to effective EOM $\gamma(R) = \frac{R_0^2}{R^2} + \frac{\Delta p_{\rm dr}}{\sigma}\frac{R^3 - R_0^3}{3R^2} = \frac{2}{3}R + \frac{1}{3R^2}$ $R_0 \equiv 1$

Bubble wall velocity $\dot{R}(t) = \sqrt{1 - \frac{9R^4}{(1+2R^3)^2}} \to 1, \quad R \to \infty$

Backup slide

Thermal bubble out-of thermal equilibrium

$$L = -4\pi\sigma R^2\sqrt{1-\dot{R}^2} + \frac{4}{3}\pi R^3\left(\Delta p_{\rm dr} - \Delta p_{\rm LO} + f(R)g(\dot{R})\right)$$

$$\left(1 + \frac{f}{3\sigma}\frac{g''}{\gamma^3}R\right)\frac{\mathrm{d}\gamma}{\mathrm{d}R} + \frac{2\gamma}{R} = \frac{\Delta p_{\rm dr} - \Delta p_{\rm LO}}{\sigma} - \frac{3f + Rf'}{3\sigma}(\dot{R}g' - g)$$

$$\eta \equiv \frac{\Delta p_{NLO}}{\sigma}, \quad h(\gamma_{\rm eq}) \equiv \frac{\Delta p_{\rm dr} - \Delta p_{\rm LO}}{\Delta p_{NLO}}$$

$$\left(1 + \frac{\eta R}{3}h'(\gamma)\right)\frac{\mathrm{d}\gamma}{\mathrm{d}R} + \frac{2\gamma}{R} = \eta(h(\gamma_{\rm eq}) - h(\gamma))$$

Same solution

Our proposal of effective description

$$f(R) = \Delta p_{NLO}$$

$$3f(R) + Rf'(R) = 3\Delta p_{NLO} \longrightarrow \frac{h(\gamma)\Delta p_{NLO}}{\sigma}$$

$$\dot{R}g'(\dot{R}) - g(\dot{R}) = h(\gamma(\dot{R}))$$

$$g''(\dot{R}) = \gamma^3 h'(\gamma)$$

$$L = -4\pi\sigma R^2\sqrt{1-\dot{R}^2} + \frac{4}{3}\pi R^3\sigma\eta\left(h(\gamma_{\rm eq}) + \dot{R}g'(\dot{R}) - h(\gamma(\dot{R}))\right)$$

$$E = \frac{\partial L}{\partial \dot{R}}\dot{R} - L = 4\pi\sigma R^2\gamma + \frac{4}{3}\pi R^3\sigma\eta(h(\gamma) - h(\gamma_{\rm eq})) + \frac{4}{3}\pi R^3\sigma\eta\left(g''(\dot{R}) - \gamma^3 h'(\gamma)\right)\dot{R}^2$$

$$= 4\pi\sigma R^2\gamma + \frac{4}{3}\pi R^3\sigma\eta(h(\gamma) - h(\gamma_{\rm eq})) \longrightarrow \left.\frac{\mathrm{d}E}{\mathrm{d}R}\right|_{\gamma(R_0)=1} = 0 \Rightarrow R_0 = \frac{2}{\eta(h(\gamma_{\rm eq}) - h(1))}$$