A new approach to t-channel singularities in cosmology

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based on

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t-channel singularities in cosmology and particle physics

arXiv:2108.01757, sent to PRL

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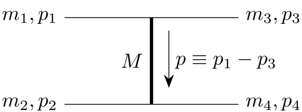
Introduction: the t-channel singularity



context: dark matter scenarios

$$\mathcal{M} \sim \frac{1}{t - M^2}\,, \quad t \equiv p^2 = (p_1 - p_3)^2$$

therm. av. cross section $\quad \langle \sigma v \rangle \to \int ds \int_{t_{\min}(s)}^{t_{\max}(s)} \frac{dt}{(t - M^2)^2}$

$t_{\min}(s) < M^2 < t_{\max}(s) \quad \Rightarrow \quad$ singularity

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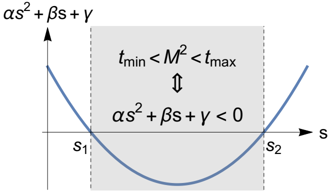
easy to show: $\quad t_{\min}(s) < M^2 < t_{\max}(s)$

$$\Leftrightarrow \quad s_1 < s < s_2$$

$$s_{1,2} \equiv \frac{-\beta \mp \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$



$$\alpha \equiv M^2$$

$$\beta \equiv M^4 - M^2(m_1^2 + m_2^2 + m_3^2 + m_4^2) + (m_1^2 - m_3^2)(m_2^2 - m_4^2)$$

$$\gamma \equiv M^2(m_1^2 - m_2^2)(m_3^2 - m_4^2) + (m_1^2 m_4^2 - m_2^2 m_3^2)(m_1^2 - m_2^2 - m_3^2 + m_4^2)$$

conclusion:

if $s_2 > s_{\min} \equiv \max\{(m_1 + m_2)^2, (m_3 + m_4)^2\}$, singularity in the allowed range

$$\Rightarrow \quad \langle \sigma v \rangle = \infty$$

$$s_{1,2} \equiv \frac{-\beta \mp \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

$$\alpha \equiv M^2$$

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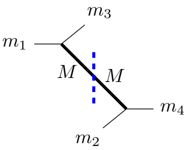
$$\gamma \equiv M^2(m_1^2 - m_2^2)(m_3^2 - m_4^2) + (m_1^2 m_4^2 - m_2^2 m_3^2)(m_1^2 - m_2^2 - m_3^2 + m_4^2)$$

if $\underline{s_2 > s_{\min} \equiv \max\{(m_1+m_2)^2, (m_3+m_4)^2\}}$, singularity in the allowed range

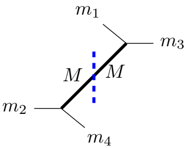
$\Updownarrow$

$m_1 > M + m_3$ and $m_4 > M + m_2$



or

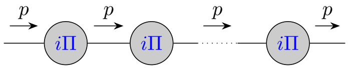
$m_2 > M + m_4$ and $m_3 > M + m_1$



* a special case of the Coleman-Norton theorem
S. Coleman & R. E. Norton, doi:10.1007/BF02750472

Known approaches to the problem

- Breit-Wigner propagator

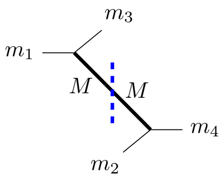


$$\mathcal{M} \sim \frac{i}{p^2 - M^2} \sum_{k=0}^{\infty} \left(i\Pi \frac{i}{p^2 - M^2} \right)^k = \frac{i}{p^2 - M^2 + \Pi}\,, \quad \Im\Pi = M\Gamma$$

→ problem: stable mediator

- complex mass of unstable particles

I. Ginzburg, hep-ph/9601272



at rest:

$$e^{im_1 t} \to e^{im_1 t} e^{-\Gamma_1 t} = e^{i\widetilde{m}_1 t}\,, \quad \widetilde{m}_1 \equiv m_1 \left(1 + i\frac{\Gamma_1}{m_1} \right)$$

after Lorentz boost:

$$p_1 \to \widetilde{p}_1 \equiv p_1 \left(1 + i\frac{\Gamma_1}{m_1} \right)$$

→ problem: $(\widetilde{p}_1 - \widetilde{p}_3)^2 \neq (\widetilde{p}_4 - \widetilde{p}_2)^2 \Rightarrow$ lack of symmetry

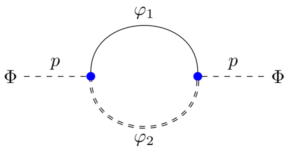
- finite beam width

K. Melnikov & V. G. Serbo, hep-ph/9601290

$$\int \frac{dt}{|t - M^2 + i\epsilon|^2} \to \int \frac{dt}{(t - M^2 + i\epsilon - a)(t - M^2 - i\epsilon + a)}$$

$a \sim [\text{beam width}]^{-1}$ → problem: inapplicable in cosmological context

Our approach



simple illustrative model:

$$\mathcal{L} = \frac{1}{2}\left[(\partial^\mu \varphi_1)(\partial_\mu \varphi_1) - m_1^2 \varphi_1^2\right] + \frac{1}{2}\left[(\partial^\mu \varphi_2)(\partial_\mu \varphi_2) - m_2^2 \varphi_2^2\right] + \frac{1}{2}\left[(\partial^\mu \Phi)(\partial_\mu \Phi) - M^2 \Phi^2\right] + \mu\, \varphi_1 \varphi_2 \Phi\,, \qquad m_1 > m_2 + M$$

- non-zero imaginary part of self-energy acquired as a result of thermal interactions with the medium of particles (Keldysh-Schwinger formalism)

$$\Pi^+(p,T) = \frac{i}{2}\mu^2 \int \frac{d^4k}{(2\pi)^4}\Big[\Delta_1^+(k+p)\,\Delta_2^{\rm sym}(k,T) + \Delta_1^{\rm sym}(k,T)\,\Delta_2^-(k-p)\Big],$$

$$\Delta^\pm(p) \equiv \frac{1}{p^2 - m^2 \pm i\,{\rm sgn}(p_0)\,\varepsilon},$$

$$\Delta^{\rm sym}(k,T) \equiv -\frac{i\pi}{E_k}\Big(\delta(E_k - k_0) + \delta(E_k + k_0)\Big) \times \big[2f(E_k,T) + 1\big]\,,$$

$$E_k \equiv \sqrt{\vec{k}^{\,2} + m^2}\,, \qquad f(E_k,T) = (e^{E_k/T} - 1)^{-1}\,.$$

the self-energy:

$$\Pi^{+}(p,T) = \frac{i}{2}\mu^2 \int \frac{d^4k}{(2\pi)^4}\Big[\Delta_1^{+}(k+p)\,\Delta_2^{\rm sym}(k,T) + \Delta_1^{\rm sym}(k,T)\,\Delta_2^{-}(k-p)\Big]$$

after manipulations:

$$\Sigma(|\vec{p}|,T) \equiv \Im\Pi^{+}(|\vec{p}|,T) = -\frac{\mu^2}{16\pi}\,\frac{1}{|\vec{p}|/T} \times \left[\ln\frac{e^{(A+C)/T}-1}{e^{A/T}-1} - \ln\frac{e^{(A+B+C)/T}-1}{e^{(A+B)/T}-1}\right],$$

where

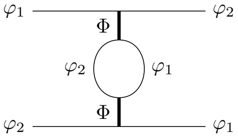
$$A \equiv \frac{(m_1^2 - m_2^2 - M^2)E_p - 2Mk_*\,|\vec{p}|}{2M^2}, \qquad B \equiv E_p, \qquad C \equiv \frac{2\,k_*\,|\vec{p}|}{M}\,,$$

$$E_p \equiv \sqrt{\vec{p}^{\,2} + M^2}\,, \qquad k_* \equiv \frac{\sqrt{\lambda(m_1^2, m_2^2, M^2)}}{2M}\,, \qquad \lambda \text{ – Källén function}$$

effective width!

$$\Gamma_{\rm eff}(|\vec{p}|,T) \equiv M^{-1}\Sigma(|\vec{p}|,T)$$

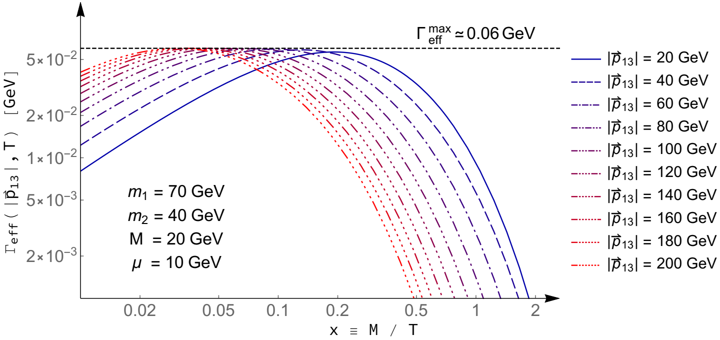
note: $\Gamma_{\rm eff}$ is temperature- and momentum-dependent



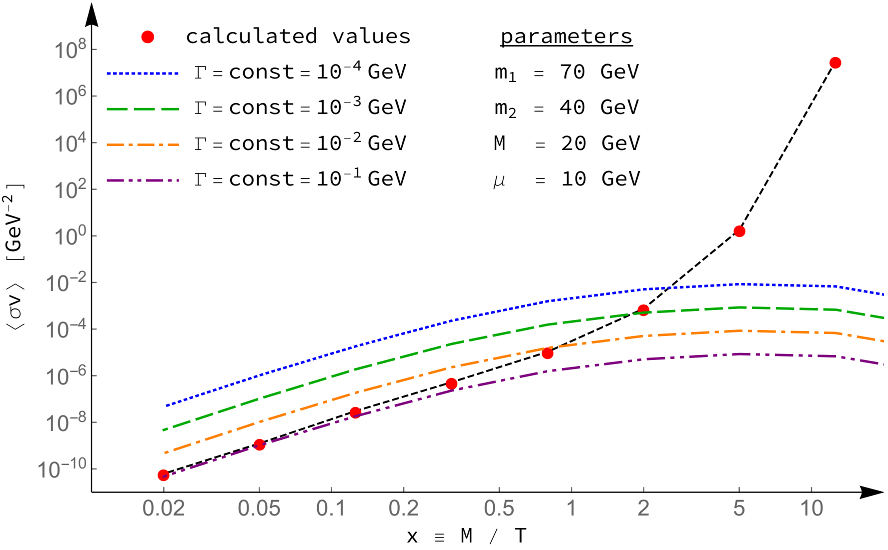
Keldysh-Schwinger formalism

$$\Rightarrow \quad \frac{1}{(t-M^2)^2} \to \frac{1}{(t-M^2)^2 + \Pi(|\vec{p}|,T)^2}$$

$$\Im\Pi = \Sigma\,,\quad M^{-1}\Sigma = \Gamma_{\text{eff}}$$

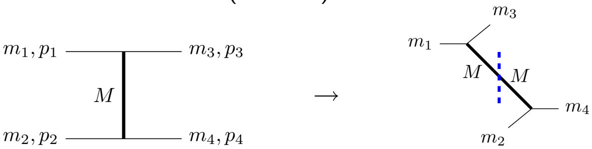


Results: thermally averaged cross section



Summary

- a t-channel singularity of $\langle \sigma v \rangle$ occurs if
  - the process can be seen as a sequence of decay and fusion
  - the mediator has no width (is stable)



- known approaches are either unsatisfactory or inapplicable
- interaction with the medium (thermal bath) results in non-zero effective width (obtained within the Keldysh-Schwinger formalism) that regulates the singularity
- the effective width depends on temperature and mediator's momentum (momentum transfer)

$$\Gamma_{\text{eff}} = \Gamma_{\text{eff}}(T, |\vec{p}|)$$

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- a t-channel singularity of $\langle \sigma v \rangle$ occurs if
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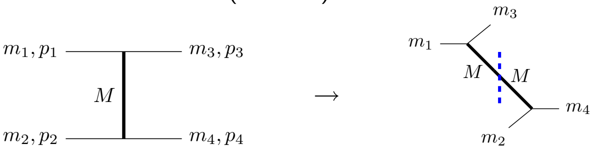

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Thank you!

BACKUP SLIDES

The Keldysh-Schwinger formalism

contour Green function:

$$i\Delta(x,y) \stackrel{\text{def}}{=} \langle \tilde{T}\phi(x)\phi(y)\rangle,$$

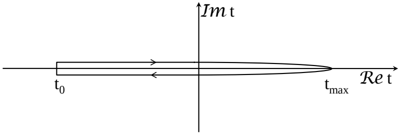
where

$\langle \ldots \rangle$ – ensemble average at time t_0 $(-\infty)$

$\tilde{T}$ – time ordering (along the contour)

$$\tilde{T}\phi(x)\phi(y) \equiv \Theta(x_0,y_0)\phi(x)\phi(y) + \Theta(y_0,x_0)\phi(y)\phi(x)$$

$$\Theta(x_0,y_0) = \begin{cases} 1 & \text{if } x_0 \text{ succeeds } y_0 \\ 0 & \text{otherwise} \end{cases}$$



The time arguments of $i\Delta(x,y)$ are complex with an infinitesimal positive or negative imaginary part which locates them on the upper or lower branch of the contour.

Green functions with real time arguments:

$$i\Delta^{>}(x,y) \equiv \langle \phi(x)\phi(y) \rangle$$
$$i\Delta^{<}(x,y) \equiv \langle \phi(y)\phi(x) \rangle$$

retarded $(+)$, advanced $(-)$ and symmetric Green functions:

$$\begin{aligned}
i\Delta^{+}(x,y) &\equiv \Theta(x_0 - y_0)\langle [\phi(x), \phi(y)] \rangle & &= i\Theta(x_0 - y_0)\big(\Delta^{>}(x,y) - \Delta^{<}(x,y)\big) \\
i\Delta^{-}(x,y) &\equiv -\Theta(y_0 - x_0)\langle [\phi(x), \phi(y)] \rangle & &= i\Theta(y_0 - x_0)\big(\Delta^{<}(x,y) - \Delta^{>}(x,y)\big) \\
i\Delta^{\mathrm{sym}}(x,y) &\equiv \langle \{\phi(x), \phi(y)\} \rangle & &= i\big(\Delta^{>}(x,y) + \Delta^{<}(x,y)\big)
\end{aligned}$$

retarded $(+)$ G. f. $\leftrightarrow$ particles and antiparticles evolving **forwards** in time
advanced $(-)$ G. f. $\leftrightarrow$ particles and antiparticles evolving **backwards** in time

spatial invariance $\quad \Rightarrow \quad \Delta(x,y) = \Delta(y-x)$