Exact solution studies of local and non-local Yang-Mills theories

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SUSY 2021, August 23-28

Introduction

- Strongly coupled field theories are generally hard to solve. Very few techniques are known or, in some lucky cases, we can rely on exact solutions.
- As far as I know, the only exactly solvable quantum field theories being not a toy model, are free theories.
- "Non-perturbative techniques" imply that standard small perturbation theory, with all the machinery of Feynman diagrams, is never used.
- A possible approach can be obtained by a method due to Bender, Savage and Milton (Phys. Rev. D 62, 085001 (2000). [arXiv:hep-th/9907045]) that yields the Dyson-Schwinger set of equations for the correlation functions in form of partial differential equations.
- I will show how this approach permits to solve Yang-Mills theory for the local case and extends naturally to the non-local case for an infinite-derivative theory.

Bender-Milton-Savage technique (1)

Bender-Milton-Savage technique is a method to get the full hierarchy of Dyson-Schwinger equations for a quantum field theory, retaining their form as partial differential equations.

Let us consider the partition function

$$Z[j] = \int [d\phi] e^{i\left[S[\phi] - \int d^4 x j \phi\right]}.$$

being

$$S[\phi] = \int d^4 x \frac{1}{2} (\partial\phi)^2 - \frac{\lambda}{4}\phi^4$$

The first step is to average as follows

$$\left\langle \frac{\delta S}{\delta\phi(x)} \right\rangle_j - j(x) = Z^{-1}[j] \int [d\phi] \frac{\delta}{\delta\phi(x)} e^{iS[\phi] - \int d^4 x j \phi} = 0.$$

We are just averaging the classical equation of motion!

Bender-Milton-Savage technique (2)

This will yield the equation

$$-\Box G_1^{(j)}(x) + \lambda\langle\phi^3(x)\rangle = j(x)$$

being $Z[j]G_1^{(j)}(x) = \langle\phi(x)\rangle$. We derive it two times with respect to $j(x)$. This will yield

$$Z[j][G_1^{(j)}(x)]^3 + 3Z[j]G_1^{(j)}(x)G_2^{(j)}(x,x) + Z[j]G_3^{(j)}(x,x,x) = \langle\phi^3(x)\rangle$$

Inserting into the equation for the 1P-function and setting $j = 0$ we get finally the pde for the 1P-function

$$-\Box G_1(x) + \lambda[G_1(x)]^3 + 3\lambda G_2(0)G_1(x) + \lambda G_3(0,0) = 0$$

Mass gets renormalized and now, we have a mass term!

Bender-Milton-Savage technique (3)

We can derive again, similarly as we did in the classical case, the equation for the 1P function with $j \neq 0$ and, finally, we get the equation for the 2P-function

$$-\Box G_2(x,y) + 3\lambda[G_1(x)]^2 G_2(x,y) +$$
$$3\lambda G_3(0,y)G_1(x) + 3\lambda G_2(0)G_2(x,y) + G_4(0,0,y) = \delta^4(x-y).$$

This equation is linear, solvable and very similar to the one of the classical case whose solution is known.

One can go on in this way to any order one likes.

Once all the correlation functions of a quantum field theory are known, the theory is completely solved and any observable can be obtained by LSZ theorem.

Solution of the quantum equations

We are able to solve these equations. Let us firstly observe that, in the equation for the 1P-function, it is $G_3(0,0) = 0$. Then, we will get an equation that is exactly solvable whose solution is

$$G_1(x) = \sqrt{\frac{2\mu^4}{\delta m^2 + \sqrt{\delta m^4 + 2\lambda\mu^4}}}\,\mathrm{sn}\left(p \cdot x + \theta, \sqrt{\frac{-\delta m^2 + \sqrt{\delta m^4 + 2\lambda\mu^4}}{-\delta m^2 - \sqrt{\delta m^4 + 2\lambda\mu^4}}}\right),$$

provided that

$$p^2 = \delta m^2 + \frac{\lambda\mu^4}{\delta m^2 + \sqrt{\delta m^4 + 2\lambda\mu^4}} = m^2(\delta m^2).$$

Here is

$$\delta m^2 = 3\lambda G_2(0)$$

This is a gap equation: Interaction yields a mass.

2P-function and mass spectrum (1)

Using the given G_1 function, we can write down G_2, provided $G_3(0,y)=0$ and $G_4(0,0,y)$ to be checked *a posteriori*.

$$G_2(p)=\sqrt{\delta m^2+\mu^2\sqrt{Ng^2/2}Z_0(\delta m^2,Ng^2)}\frac{2\pi^3}{K^3(k^2(\delta m^2))}\times$$
$$\sum_{n=0}^{\infty}(-1)^n(2n+1)^2\frac{q^{n+1/2}}{1-q^{2n+1}}\frac{1}{p^2-m_n^2(\delta m^2)+i\epsilon},$$

being

$$k^2(\delta m^2)=\frac{\delta m^2-\sqrt{\delta m^4+2Ng^2\mu^4}}{\delta m^2+\sqrt{\delta m^4+2Ng^2\mu^4}},$$

and, for a small mass shift δm,

$$\lim_{\delta m\to 0}\sqrt{\delta m^2+\mu^2\sqrt{Ng^2/2}Z_0(\delta m^2,Ng^2)}=\frac{1}{8}$$

and $q=\exp\left(-\pi K(1+k^2)/K(k^2)\right)$.

2P-function and mass spectrum (2)

The mass spectrum is given by

$$m_n(\delta m^2) = (2n+1)\frac{\pi}{2K(k^2)}\sqrt{\delta m^2 + \frac{Ng^2\mu^4}{\delta m^2 + \sqrt{\delta m^4 + 2Ng^2\mu^4}}}.$$

From the definition of δm, we get the gap equation

$$\delta m^2 = 3\lambda \int \frac{d^4p}{(2\pi)^4} G_2(p)$$

This theory develops a mass gap due to self-interaction.

Mass spectrum is that of a harmonic oscillator![1]

[1] *The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction. (S. Coleman)*

Yang-Mills theory

We consider a gauge theory with no interaction with fermions with a Lagrangian

$$L_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

being

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu.$$

We add a gauge-fixing term

$$L_{gf} = -\frac{1}{2\xi} (\partial \cdot A)^2$$

with the parameter ξ properly chosen on the given gauge.

Dyson-Schwinger hierarchy

- Also for gauge theories we can apply the Bender-Milton-Savage technique to obtain the nP-functions. Knowing the correlation functions of a quantum field theory implies to solve it completely. (see M. F., Eur. Phys. J. Plus (2017) 132: 38; Erratum-ibid. (2017) 132: 242, arXiv:1509.05292 [math-ph])
- Given the Dyson-Schwinger equations, we can apply the mapping with the scalar field as done for the classical case. (M.F., Mod. Phys. Lett. A 24, 2425-2432 (2009), arXiv:0903.2357 [math-ph] & Terence Tao on `https://kvm16.pims.math.ca/DispersiveWiki/index.php?title=Talk:Yang-Mills_equations` and private comm.)
- We obtain the spectrum and its corrections arising from renormalization of the mass.
- The spectrum so obtained agrees stunningly well with lattice computations both in 3 and 4 dimensions (see M.F., Nuclear and Particle Physics Proceedings 294-296 (2018) 124-128, arXiv:1708.06184 [hep-ph]).

Dyson-Schwinger equations for 1P- and 2P-functions (1)

The DS equations can be solved exactly. This can be obtained by the *mapping with the scalar field*

$$A_\mu^a(x) = \eta_\mu^a \phi(x)$$

with the η-symbols having the properties for SU(N) as

$$\eta_\mu^a \eta^{a\mu} = N^2 - 1, \quad \eta_\mu^a \eta^{b\mu} = \delta_{ab}, \quad \eta_\mu^a \eta_\nu^a = \frac{1}{2}\left(g_{\mu\nu} - \delta_{\mu\nu}\right).$$

We also remember that, in Landau-Lorenz gauge, is

$$G_{\mu\nu}^{ab}(x,y) = \delta_{ab}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)\Delta(x,y),$$

Dyson-Schwinger equations for 1P- and 2P-functions (2)

Application of the given methodology will yield the following equations:

- 1P-function:

$$\partial^2\phi(x) + 2Ng^2\Delta(0)\phi(x) + Ng^2\phi^3(x) = 0.$$

- 2P-function:

$$\partial^2\Delta(x-y) + 2Ng^2\Delta(0)\Delta(x-y) + 3Ng^2\phi^2(x)\Delta(x-y) = \delta^4(x-y).$$

- Gap equation:

$$\delta m^2 = 2Ng^2\Delta(0) = 2Ng^2\int\frac{d^4p}{(2\pi)^4}\Delta(p).$$

The 2 factor in the equations, due to re-normalization, arises by the algebraic properties of the gauge group!

String-inspired non-local field theories

We consider the following non-local scalar field theory

$$L = -\phi(x)e^{f(\Box)}\Box\phi(x) - \frac{\lambda}{4}\phi^4(x) + j(x)\phi(x),$$

and gauge theory

$$L = -\frac{1}{4}\,\mathrm{tr}\,F^{a\mu\nu}e^{f(D^2)}F^a_{\mu\nu} + \bar{c}^a D^{ab}_\mu \partial^\mu c^b + \bar{\eta}^a c^a + \bar{c}^a \eta^a + j^a_\mu A^{a\mu}.$$

The exponential factor represents an entire function adding no-poles and so, no ghosts and unitarity preserved.

The non-locality introduces a mass-scale factor M that grants an UV-finite theory.

In the limit of the mass-scale factor running to infinity, the local theory is properly recovered.

1P-function for non-local infinite-derivative scalar theory

Bender-Milton-Savage extends naturally to the non-local case (M.F. & A. Ghoshal, Class.Quant.Grav. 38 (2021) 17, 175013, arXiv: 2011.10586 [hep-th]). In this case, Dyson-Schwinger equations cannot be solved exactly but just around the local solution (small non-local effects). So, one has for the 1P-function

$$\phi_{NL}(x) = \phi(x) + \int d^4y G_2(x-y)\delta\phi(y) + \dots$$

where

$$\delta\phi(x) = -\mu^3\left(\frac{2\lambda}{9}\right)^{\frac{1}{4}}\frac{4\pi^3}{3K^3(i)}\left[\sum_{n=1}^{\infty}C_n(x)\right] + \mu^3\left(\frac{2\lambda}{9}\right)^{\frac{1}{4}}\left[1-\frac{4\pi^3}{3K^3(i)}\frac{e^{-\frac{3\pi}{2}}}{(1+e^{-\pi})^3}e^{3f\left(-\frac{\pi^2}{4K^2(i)}p^2\right)}\right]\sin^3\left(\frac{\pi}{2K(i)}(p\cdot x+\theta)\right).$$

$C_n(x)$ are some coefficients obtained through product of series of known terms.

2P-function for non-local infinite-derivative scalar theory

2P-function can be written in the form

$$G_2(k) = G_2^{(c)}(k)\frac{1}{1+\delta m_0^2 e^{f(-k^2)} G_2^{(c)}(k)},$$

where the shift δm_0^2 can then be computed by the gap equation

$$\delta m_0^2 = 3\lambda \int \frac{d^4k}{(2\pi)^4} G_2(k).$$

and

$$G_2^{(c)}(k) = \frac{e^{f(-k^2)}}{k^2 + m_0^2 e^{2f(-k^2)}} \frac{1}{1-\Pi(k)}.$$

Mass gap gets diluted by non-locality and higher order excitations are moved far away in the spectrum making them possibly not observable.

1P- and 2P-function for non-local gauge theory

We apply again the mapping theorem between scalar field theory and gauge theory, taking into account that $\lambda \to Ng^2$, therefore (M.F. & A. Ghoshal, JHEP 21 (2020) 226, arXiv: 2102.10665 [hep-th])

$$G^a_{1\mu}(x) = \eta^a_\mu \phi_{NL}(x)$$

where we introduced the η-symbols. Similarly, in the Landau gauge,

$$G^{ab}_{2\mu\nu}(k) = \delta_{ab}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) G_2(k)$$

provided the gap equation

$$\delta m_0^2 = 2Ng^2 \int \frac{d^4k}{(2\pi)^4} G_2(k).$$

Again, we have a diluted mass gap and higher excited states moved far away in the spectrum.

Confinement

- The beta function for the local theory is obtained with the technique devised in (M.F. & M. Chaichian, Phys. Lett. B 781, 33-39 (2018), arXiv:1801.09873 [hep-th]) based on BRST and Kugo-Ojima confinement criterion.
- This technique extends immediately to the non-local case yielding again a proof of confinement (M.F., A. Ghoshal, N. Okada, arXiv:2106.07629 [hep-th]).
- We aim to apply this technique to quantum gravity to confine the ghost states like in R^2 theories, impeding them to propagate.
- Also, this would have implications for the behavior of cosmological and black hole singularities.

Final considerations

- We have provided some examples of exactly solvable interacting quantum field theory.
- This was possible using a set of exact solutions of the classical equations of motions.
- For the gauge theories, we mapped the gauge fields on the scalar field, this does not select an unique solution but the spectrum of the theory appears in close agreement with lattice computations.
- Confinement can be proven both in the local and non-local case. Presence of quarks makes the theory unsolvable or solvable just through some perturbation techniques.
- Non-local theories get the mass gap diluted at higher energies and, possibly, higher excited states seem to be far detached from the ground state.

Spectrum of the theory (1)

For the ground state 0^{++} of the theory in 3+1 dimensions, as seen on lattice computations, one has

N	Lattice	Theoretical	β	Error
2	3.78(7)	3.550927197	2.4265	6%
3	3.55(7)	3.555252334	6.0625	0.1%
4	3.56(11)	3.556337890	11.085	0.1%
6	3.25(9)	3.557102106	25.452	8.6%
8	3.55(12)	3.557471208	45.70	0.2%

Table: Comparison for the ground state at varying N. The lattice data are obtained from B. Lucini, M. Teper and U. Wenger, JHEP **0406**, 012 (2004) [hep-lat/0404008] for the continuum limit.

Spectrum of the theory (2)

A similar situation happens for what is labeled as a 2^{++} resonance in Lucini&al. paper.

N	Lattice	Theoretical	β	Error
2	5.45(11)	4.734569596	2.4265	13%
3	4.78(9)	4.740336446	6.0625	0.8%
4	4.85(16)	4.741783854	11.085	2%
6	4.73(15)	4.742802808	25.452	0.3%
8	4.73(22)	4.743294944	45.70	0.3%

Table: Comparison for the 2^{++} state at varying N. The lattice data are obtained from Lucini&al., ibidem, for the continuum limit.

Equations of motion (1)

Equations of motions for the gauge field will be given by

$$\partial^\mu \partial_\mu A^a_\nu - \xi^{-1}\partial_\nu(\partial^\mu A^a_\mu)$$
$$+gf^{abc}A^{b\mu}(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu) + gf^{abc}\partial^\mu(A^b_\mu A^c_\nu)$$
$$+g^2 f^{abc} f^{cde} A^{b\mu} A^d_\mu A^e_\nu = j^a_\nu.$$

Let us fix the gauge to the *Landau-Lorenz gauge* that yields

$$\partial^\mu A^a_\mu = 0.$$

Our equations simplify to

$$\partial^\mu \partial_\mu A^a_\nu + gf^{abc}A^{b\mu}(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu)$$
$$+gf^{abc}\partial^\mu(A^b_\mu A^c_\nu) + g^2 f^{abc} f^{cde} A^{b\mu} A^d_\mu A^e_\nu = j^a_\nu.$$

Equations of motion (2)

By functional deriving with respect to $j^{h\lambda}(y)$ one has

$$\partial^\mu\partial_\mu G^{ah}_{\nu\lambda}(x,y) + gf^{abc}G^{bh\mu}_{\nu}(x,y)(\partial_\mu A^c_\nu(x) - \partial_\nu A^c_\mu(x))$$
$$+gf^{abc}A^{b\mu}(x)(\partial_\mu G^{ch}_{\nu\lambda}(x,y) - \partial_\nu G^{ch}_{\mu\lambda}(x,y))$$
$$+gf^{abc}\partial^\mu(G^{bh}_{\mu\lambda}(x,y)A^c_\nu)(x)$$
$$+gf^{abc}\partial^\mu(A^b_\mu(x)G^{ch}_{nu\lambda}(x,y))$$
$$+g^2f^{abc}f^{cde}G^{bh\mu}_{\lambda}(x,y)A^d_\mu(x)A^e_\nu(x)$$
$$+g^2f^{abc}f^{cde}A^{b\mu}(x)G^{dh}_{\mu\lambda}(x,y)A^e_\nu(x)$$
$$+g^2f^{abc}f^{cde}A^{b\mu}(x)A^d_\mu(x)G^{eh}_{\nu\lambda}(x,y) = \eta_{\nu\lambda}\delta_{ah}\delta^4(x-y).$$

This is the equation for the Green function of the classical Yang-Mills field and it is linear.

Solving the classical equations (1)

Mapping with the scalar field reduces the equations of the gauge field to

$$\partial^\mu\partial_\mu\eta^a_\nu\phi(x) + gf^{abc}\eta^{b\mu}\phi(x)(\partial_\mu\eta^c_\nu\phi(x) - \partial_\nu\eta^c_\mu\phi(x))$$
$$+gf^{abc}\partial^\mu(\eta^b_\mu\eta^c_\nu\phi^2(x)) + g^2f^{abc}f^{cde}\eta^{b\mu}\eta^d_\mu\eta^e_\nu\phi^3(x) = j^a_\nu.$$

Anti-symmetry of f^{abc} removes the second and the third term giving

$$\partial^\mu\partial_\mu\phi(x) + Ng^2\phi^3(x) = (N^2-1)^{-1}\eta\cdot j$$

This is the scalar field we discussed initially and we know how to treat it!

Solving the classical equations (2)

For the Green function we get

$$\partial^\mu \partial_\mu \Delta(x, y) + 3Ng^2\phi^2(x)\Delta(x, y) = \delta^4(x - y)$$

Again, we have recovered the scalar field equation for the Green function!

- Classically, we get massive solutions to the Yang-Mills equations when the mapping with a scalar field is supported.
- We get also a spectrum that holds, modified by re-normalization effects, for the quantum case.
- *Via the mapping, classical Yang-Mills theory in the Landau-Lorenz gauge is completely solvable by a functional Taylor series.*