Muon $g-2$ and Lattice QCD

LHC days Split 2022

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(BMW collaboration)

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BMW collaboration

BERGISCHE UNIVERSITÄT WUPPERTAL

Muon g-2

Free Dirac fermions are have a gyromagnetic factor $g = 2$. Interactions change this values $\rightarrow$ Study the deviation $g - 2$ or $a = (g-2)/2$.

a_μ of the muon can be measured to an extremely high precision [1,2]

$$a_\mu(\text{FNAL}) = 116592040(54) \times 10^{-11}$$
$$a_\mu(\text{BNL}) = 116592080(63) \times 10^{-11}$$

Equally precise theory predictions are necessary.

[1] B. Abi *et al.* [Muon g-2], "Measurement of the Positive Muon Anomalous Magnetic Moment to 0.46 ppm," Phys. Rev. Lett. **126** (2021) no.14, 141801 doi:10.1103/PhysRevLett.126.141801 [arXiv:2104.03281 [hep-ex]].
[2] G. W. Bennett *et al.* [Muon g-2], "Final Report of the Muon E821 Anomalous Magnetic Moment Measurement at BNL," Phys. Rev. D **73** (2006), 072003 doi:10.1103/PhysRevD.73.072003 [arXiv:hep-ex/0602035 [hep-ex]].

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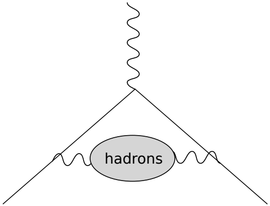

For the muon, electroweak, Higgs and QCD contribution are relevant.

Uncertainty dominated by the Hadron vacuum polarization (HVP) contribution.

HVP contribution can not be determined by perturbation theory. $\rightarrow$ Data-driven approach or lattice calculation.

T. Aoyama, N. Asmussen, M. Benayoun, J. Bijnens, T. Blum, M. Bruno, I. Caprini, C. M. Carloni Calame, M. Cè and G. Colangelo, *et al.* "The anomalous magnetic moment of the muon in the Standard Model," Phys. Rept. **887** (2020), 1-166 doi:10.1016/j.physrep.2020.07.006 [arXiv:2006.04822 [hep-ph]].

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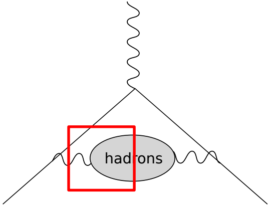

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Data driven approaches rely on dispersion theory: Express the HVP contribution in terms of the hadronic R-ratio.

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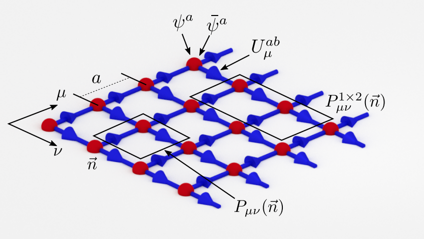
Lattice approach



In lattice QCD, euclidean space time is discretized and quark and gluon fields are put on the sites and links of a space-time lattice.

Use powerfull computers to solve the path integral numerically.

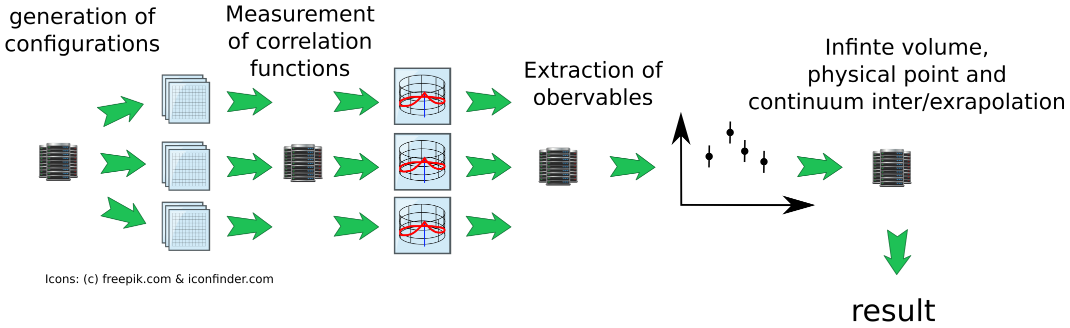
Leading HVP contribution can be written as

$$a_\mu^{\text{LO-HVP}} = \alpha^2 \int_0^\infty \mathrm{d}t K(t) G(t) \quad \text{with} \quad G(t) = \frac{1}{3e^2} \sum_{\vec{x},\mu=1,\dots,3} \langle J_{\mu,t,\vec{x}} J_{\mu,0} \rangle$$

and where $K(t)$ is a known kernel. Divide HVP contribution into parts:

$$a_\mu = a_\mu^{\text{light}} + a_\mu^{\text{strange}} + a_\mu^{\text{charm}} + a_\mu^{\text{disc}} + a_\mu^{\text{pert}}$$

Lattice calculations



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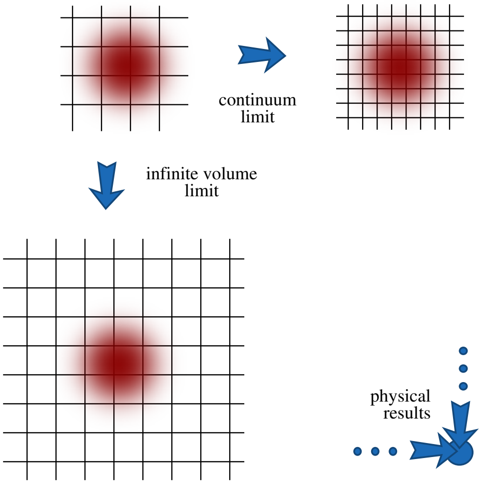
Lattice calculations



27 ensambles at 6 lattice spacing ranging from $a = 0.13\,\mathrm{fm}$ to $a = 0.064\,\mathrm{fm}$.

Configurations are generated 2+1+1 fal-vours of dynamical staggered quarks and with a Symanzik improved gauge action.

Additional 4HEX configurations for finite volume correction.

Ingredients of the calculation

Key challenge: a_μ^{HVP} has to be determined with sub percent precision. $\rightarrow$ All systematics must be controlled to this precision

Important ingredients of the calculation:

- Setting the lattice scale
- Correcting for finite volume effects
- Continuum extrapolation
- Isospin breaking contributions
- Noise reduction and bounding method
- Global fits and systematic error estimation
- ... many more ...

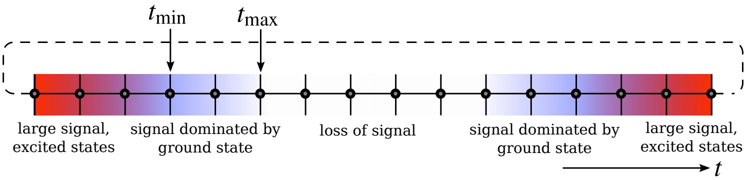
Scale setting

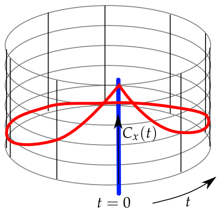
The correlation function of a staggered baryon has the form

$$H(t; A, M) = A_1 h_+(t; M_0) + A_2 h_-(t; M_2) + A_3 h_+(t; M_3) + A_4 h_-(t; M_4) + \cdots$$

with

$$h_+(t, M) = e^{-Mt} + (-1)^{t-1} e^{-M(t-T)}$$
$$h_-(t, M) = -h_+(T - t, M)$$





Scale setting

4 state fit extraction

Fit propagator to expected functional from $H(t; A, M) = A_1 h_+(t; M_0) + A_2 h_-(t; M_2)$ $+A_3 h_+(t; M_3) + A_4 h_-(t; M_4)$

Use priors for the excited state masses:

prior mean	rel. prior width
2012 MeV	0.10
2250 MeV	0.10
2400 MeV	0.15

GEVP based extraction

Construct matrix [1] from folded propagator H_t:

$$\mathcal{H}(t) = \begin{pmatrix} H_{t+0} & H_{t+1} & H_{t+2} & H_{t+3} \\ H_{t+1} & H_{t+2} & H_{t+3} & H_{t+4} \\ H_{t+2} & H_{t+3} & H_{t+4} & H_{t+5} \\ H_{t+3} & H_{t+4} & H_{t+5} & H_{t+6} \end{pmatrix}$$

and solve $\mathcal{H}(t_0)v(t_0, t_1) = \lambda(t_0, t_1)\mathcal{H}(t_1)v(t_0, t_1)$.

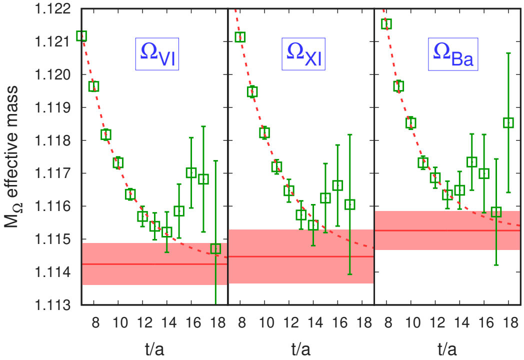
Then, extract mass from $C(t) = v^\dagger(t_0, t_1)\mathcal{H}(t)v(t_0, t_1)$ between t_{start} and t_{stop}.

No assumption on the masses of the excited states.

[1] C. Aubin and K. Orginos, "A new approach for Delta form factors," AIP Conf. Proc. **1374** (2011) no.1, 621-624 doi:10.1063/1.3647217 [arXiv:1010.0202 [hep-lat]].

Comparison of operators

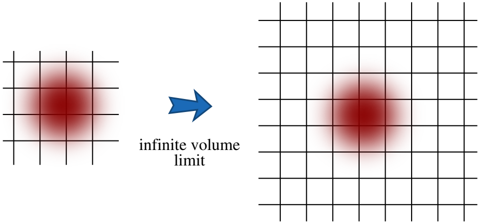
We have investigate the 3 operators `VI`, `XI` and `Ba` on one ensemble with increased statistics:



`VI` and `XI` couple to two tastes, whereas `Ba` is constructed to couple only to one taste.

Difference safely within 0.1% error. → Use `VI` for the main analysis.

Finite volume corrections



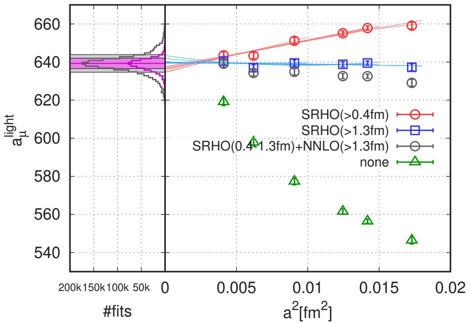
Do continuum extrapolation in a reference box with $L_{\text{ref}} \sim 6.3\,\text{fm}$

Use a large box with $L_{\text{big}} \sim 10.8$ fm.
For the large box, we use a dedicated 4HEX action to reduce the taste breaking.

Using various non-lattice methods [1-5], we find $a_\mu(\infty,\infty) - a_\mu(L_{\text{big}}, T_{\text{big}}) = 0.6(0.3)$

$$a_\mu(\infty,\infty) - a_\mu(L_{\text{ref}}, T_{\text{ref}}) = [a_\mu(L_{\text{big}}, T_{\text{big}}) - a_\mu(L_{\text{ref}}, T_{\text{ref}})]_{\text{4HEX}} + [a_\mu(\infty,\infty) - a_\mu(L_{\text{big}}, T_{\text{big}})]_{\chi\text{PT}}$$

[1] G. J. Gounaris and J. J. Sakurai, "Finite width corrections to the vector meson dominance prediction for $\rho \to e^+e^-$," Phys. Rev. Lett. **21** (1968), 244-247 doi:10.1103/PhysRevLett.21.244
[2] L. Lellouch and M. Luscher, "Weak transition matrix elements from finite volume correlation functions," Commun. Math. Phys. **219** (2001), 31-44 doi:10.1007/s002200100410 [arXiv:hep-lat/0003023 [hep-lat]].
[3] D. Bernecker and H. B. Meyer, "Vector Correlators in Lattice QCD: Methods and applications," Eur. Phys. J. A **47** (2011), 148 doi:10.1140/epja/i2011-11148-6 [arXiv:1107.4388 [hep-lat]].
[4] M. T. Hansen and A. Patella, "Finite-volume and thermal effects in the leading-HVP contribution to muonic $(g-2)$," JHEP **10** (2020), 029 doi:10.1007/JHEP10(2020)029 [arXiv:2004.03935 [hep-lat]].
[5] B. Chakraborty, C. T. H. Davies, P. G. de Oliviera, J. Koponen, G. P. Lepage and R. S. Van de Water, "The hadronic vacuum polarization contribution to a_μ from full lattice QCD," Phys. Rev. D **96** (2017) no.3, 034516 doi:10.1103/PhysRevD.96.034516 [arXiv:1601.03071 [hep-lat]].



Dominant effect due to staggered taste breaking.

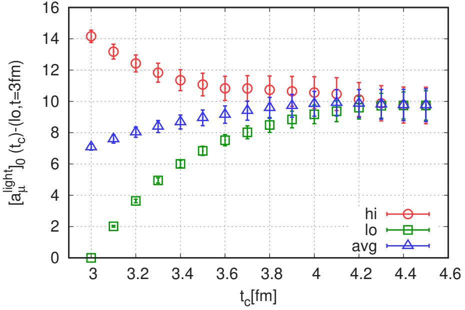
After some time, taste breaking can be described by SRHO [1,2,3] model.

Subtract difference between RHO and SRHO from data in multiple time ranges prior to continuum limit.

Additional error estimate added in quadrature (not entering central value): Difference to NNLO χPT starting from 1.3 fm.

[1] J. J. Sakurai, "Theory of strong interactions," Annals Phys. **11** (1960), 1-48 doi:10.1016/0003-4916(60)90126-3
[2] F. Jegerlehner and R. Szafron, "$\rho^0 - \gamma$ mixing in the neutral channel pion form factor F_π^e and its role in comparing e^+e^- with τ spectral functions," Eur. Phys. J. C **71** (2011), 1632 doi:10.1140/epjc/s10052-011-1632-3 [arXiv:1101.2872 [hep-ph]].
[3] B. Chakraborty, C. T. H. Davies, P. G. de Oliviera, J. Koponen, G. P. Lepage and R. S. Van de Water, "The hadronic vacuum polarization contribution to a_μ from full lattice QCD," Phys. Rev. D **96** (2017) no.3, 034516 doi:10.1103/PhysRevD.96.034516 [arXiv:1601.03071 [hep-lat]].

Noise reduction



Tail of $C(t) = \langle J(t)J(0)\rangle$ dominated by two pion states.

Lower and upper bound can be derived [1,2]:

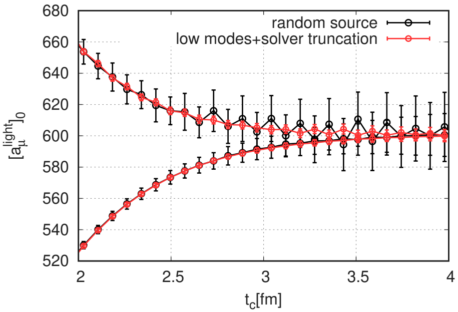
$$0 \leq C^{\mathrm{l}}(t) \leq C^{l}(t_c)\frac{\varphi(t)}{\varphi(t_c)}$$

with

$$\varphi(t) = \frac{\cosh(E_{2\pi}(t - T/2)) + 1}{\cosh(E_{2\pi}T/2) - 1} \approx \exp(-E_{2\pi}t)$$

[1] Lehner, C. *RBRC Workshop on Lattice Gauge Theories (2016)*
[2] S. Borsanyi, Z. Fodor, T. Kawanai, S. Krieg, L. Lellouch, R. Malak, K. Miura, K. K. Szabo, C. Torrero and B. Toth, "Slope and curvature of the hadronic vacuum polarization at vanishing virtuality from lattice QCD," Phys. Rev. D **96** (2017) no.7, 074507 doi:10.1103/PhysRevD.96.074507 [arXiv:1612.02364 [hep-lat]].

Noise reduction



$\langle JJ \rangle$ correlation function calculated on the lattice by inverting the Dirac operator.

Exact treatment of lowest mode of Dirac operator [1,2]

Use All Mode Averaging / Truncated solver method [3,4] for the orthogonal modes.

[1] H. Neff, N. Eicker, T. Lippert, J. W. Negele and K. Schilling, "On the low fermionic eigenmode dominance in QCD on the lattice," Phys. Rev. D **64** (2001), 114509 doi:10.1103/PhysRevD.64.114509 [arXiv:hep-lat/0106016 [hep-lat]].

[2] A. Li *et al.* [xQCD], "Overlap Valence on 2+1 Flavor Domain Wall Fermion Configurations with Deflation and Low-mode Substitution," Phys. Rev. D **82** (2010), 114501 doi:10.1103/PhysRevD.82.114501 [arXiv:1005.5424 [hep-lat]].

[3] G. S. Bali, S. Collins and A. Schafer, "Effective noise reduction techniques for disconnected loops in Lattice QCD," Comput. Phys. Commun. **181** (2010), 1570-1583 doi:10.1016/j.cpc.2010.05.008 [arXiv:0910.3970 [hep-lat]].

[4] T. Blum, T. Izubuchi and E. Shintani, "New class of variance-reduction techniques using lattice symmetries," Phys. Rev. D **88** (2013) no.9, 094503 doi:10.1103/PhysRevD.88.094503 [arXiv:1208.4349 [hep-lat]].

Isospin breaking corrections

Isospin breaking in QCD+QED calculated by expansion [1] around isospin symmetric QCD:

$$\langle O\rangle = [\langle O\rangle]_0 + e_v^2\langle O\rangle''_{20} + e_v e_s\langle O\rangle''_{11} + e_s^2\langle O\rangle''_{02} + \frac{\delta m}{m}\langle 0\rangle'_m$$

Path integral is

$$Z = \int \mathcal{D}U \exp(-S_g[U]) \int \mathcal{D}A \exp(-S_\gamma[A]) \,\mathrm{dets}[U,A]$$

Express derivatives as

$$\begin{aligned}
[\langle O\rangle]_0 &= \langle O_0\rangle_0 \\
\langle O\rangle''_{20} &= \langle O''_2\rangle_0 \\
\langle O\rangle''_{11} &= \left\langle O'_1 \frac{\mathrm{dets}'_1}{\mathrm{dets}_0}\right\rangle_0 \\
\langle O\rangle''_{02} &= \left\langle O_0 \frac{\mathrm{dets}''_2}{\mathrm{dets}_0}\right\rangle_0 - \langle O_0\rangle \left\langle \frac{\mathrm{dets}''_2}{\mathrm{dets}_0}\right\rangle_0 \\
\langle O\rangle'_m &= \langle O'_m\rangle_0
\end{aligned}$$

[1] G. M. de Divitiis *et al.* [RM123], "Leading isospin breaking effects on the lattice," Phys. Rev. D **87** (2013) no.11, 114505 doi:10.1103/PhysRevD.87.114505 [arXiv:1303.4896 [hep-lat]].

Global fits

Using Type I fit for $Y = a_\mu^{\cdots}$:

$$Y = A + BX_l + CX_s + DX_{\delta m} + Ee_v^2 + Fe_v e_s + Ge_s^2$$

with the shorthand notation

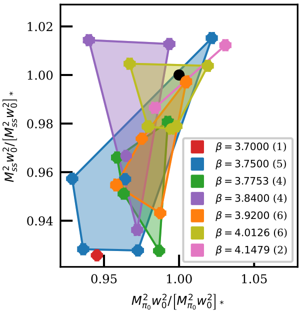
$$X_l = M_{\pi^0}^2 / M_\Omega^2 - \left[M_{\pi^0}^2 / M_\Omega^2\right]_*$$

$$X_s = M_{K_\chi}^2 / M_\Omega^2 - \left[M_{K_\chi}^2 / M_\Omega^2\right]_*$$

$$X_{\delta m} = \frac{(M_{K^0} - M_{K^+})^2}{M_\Omega^2}$$

$$M_{K_\chi}^2 = \frac{1}{2}(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2)$$

The coefficients $A, \dots, G$ themselves depend on the lattice spacing and X_l and X_s.



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$$A = A_0 + A_2\left[a^2\alpha_s^n(1/a)\right] + A_4\left[a^2\alpha_s^n(1/a)\right]^2 + A_6\left[a^2\alpha_s^n(1/a)\right]^3$$

$$B = B_0 + B_2 a^2$$

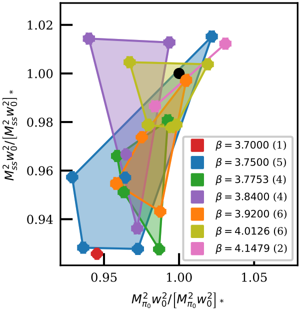
$$C = C_0 + C_2 a^2$$

$$D = D_0 + D_2 a^2 + D_4 a^4 + D_l X_l + D_s X_s$$

$$E = E_0 + E_2 a^2 + E_4 a^4 + E_l X_l + E_s X_s$$

$$F = F_0 + F_2 a^2$$

$$G = G_0 + G_2 a^2$$



Global fits

Using `Type I` fit for $Y = a_\mu^{\cdots}$:

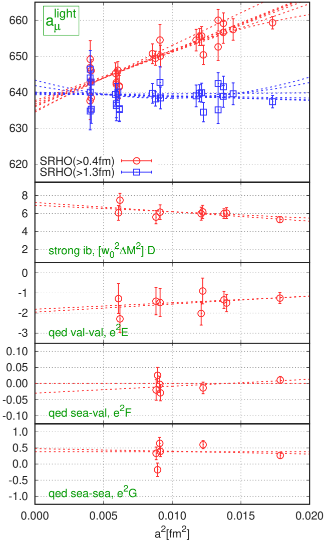
$$Y = A + BX_l + CX_s + DX_{\delta m} + Ee_v^2 + Fe_v e_s + Ge_s^2$$

Write as coupled system of equations

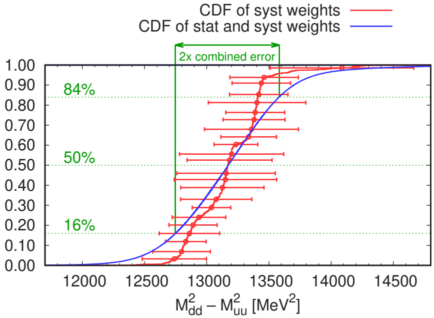
$$\begin{aligned}
[Y]_0 &= [A + BX_l + CX_s]_0 \\
[Y]'_m &= [DX_{\delta m}]'_m \\
[Y]''_{20} &= [A + BX_l + CX_s + DX_{\delta m}]''_{20} + [E]_0 \\
[Y]''_{11} &= [A + BX_l + CX_s + DX_{\delta m}]''_{11} + [F]_0 \\
[Y]''_{02} &= [A + BX_l + CX_s + DX_{\delta m}]''_{02} + [G]_0
\end{aligned}$$

Systematic error by varying the fit parameters and number of lattice spacings.

For the light part: Around $\frac{1}{2}$ million fits.



Error estimation



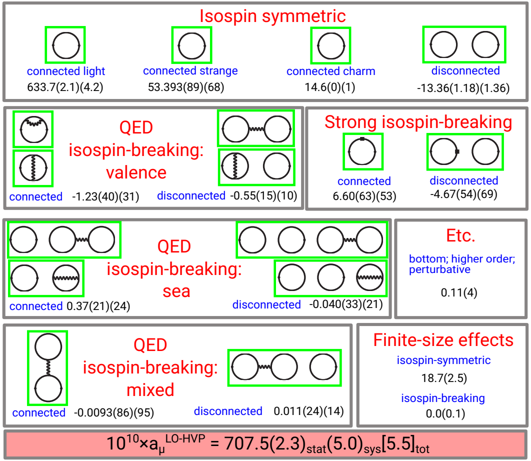
Systematic error estimated using histogram method.

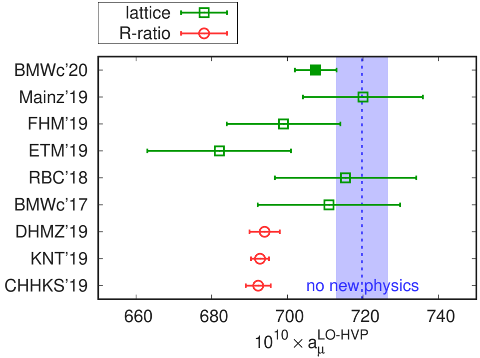
Quantiles of CDF of combined systematic and statistical errors.

Scaling the statistical error allows disentangling of systematic and statistical error.

Statistical errors via Jacknive procedure.

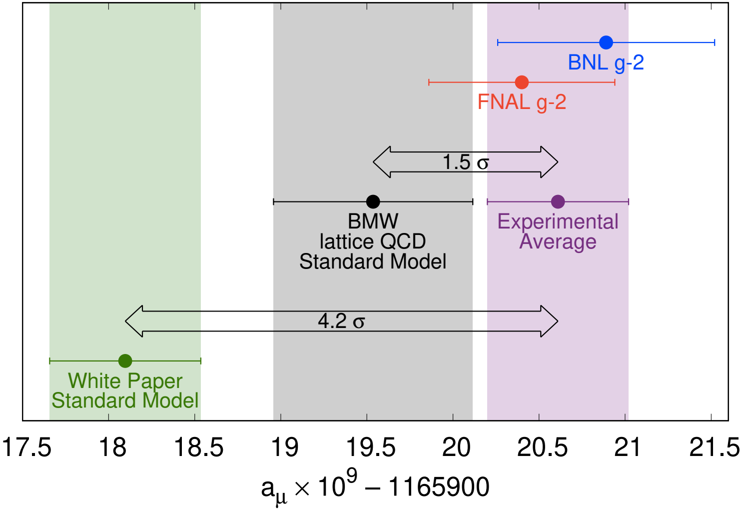
Lattice results for a_μ





S. Borsanyi, Z. Fodor, J. N. Guenther, C. Hoelbling, S. D. Katz, L. Lellouch, T. Lippert, K. Miura, L. Parato and K. K. Szabo, *et al.* "Leading hadronic contribution to the muon magnetic moment from lattice QCD," Nature **593** (2021) no.7857, 51-55 doi:10.1038/s41586-021-03418-1 [arXiv:2002.12347 [hep-lat]].

Comparison of Results

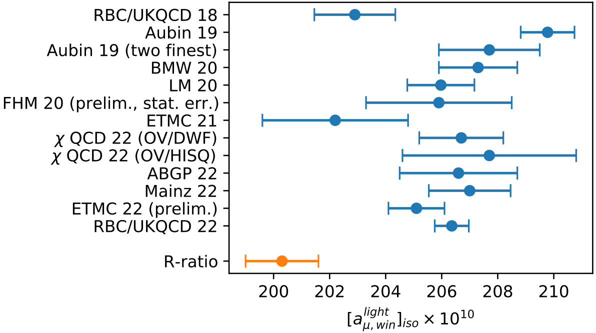


Window observable

One can define a window observable where the lattice correlation function is multiplied by a window function

$$W(t;t_1,t_2) = \Theta(t;t_1,\Delta) - \theta(t,t_2,\Delta) \quad \text{with} \quad \theta(t;t',\Delta) = \frac{1}{2} + \frac{1}{2}\tanh\left[\frac{t-t'}{\Delta}\right]$$

$$(t_1, t_2, \Delta) = (0.4, 1.0, 0.15)\,\mathrm{fm}$$



The light, isospin symmetric part of the window quantity is much easier to calculate on the lattice and is a good consistency check.

Many lattice collaborations have calculated that value.

thank you!