Pion and kaon parton distributions from their light-front wave functions

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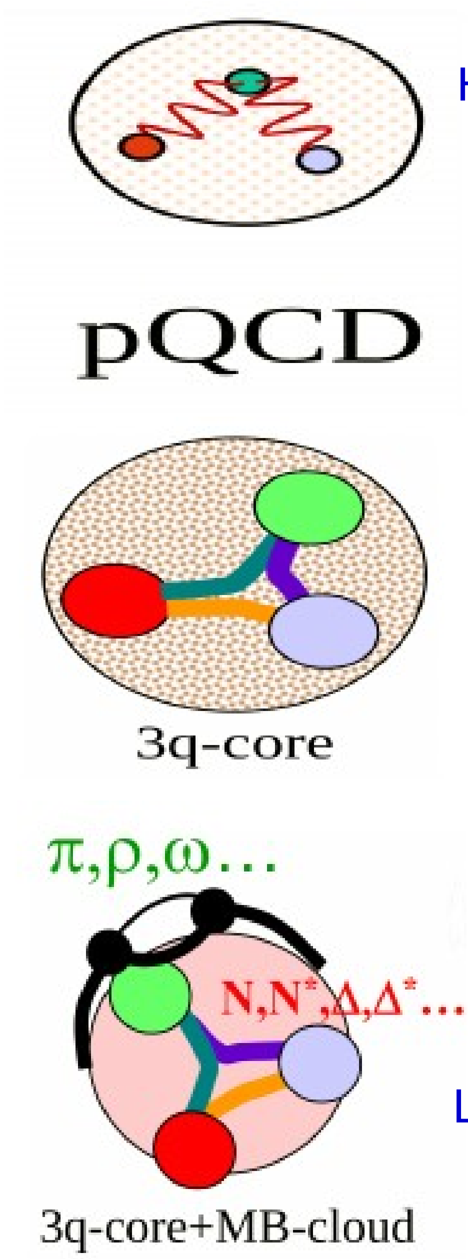
...

Perceiving the Emergence of Hadron Mass through AMBER@CERN: March 30, 2020.

Hadron Physics. General Motivation.

The QCD Holy Grail: the understanding of hadrons in terms of its elementary excitations; namely, quarks and gluons!



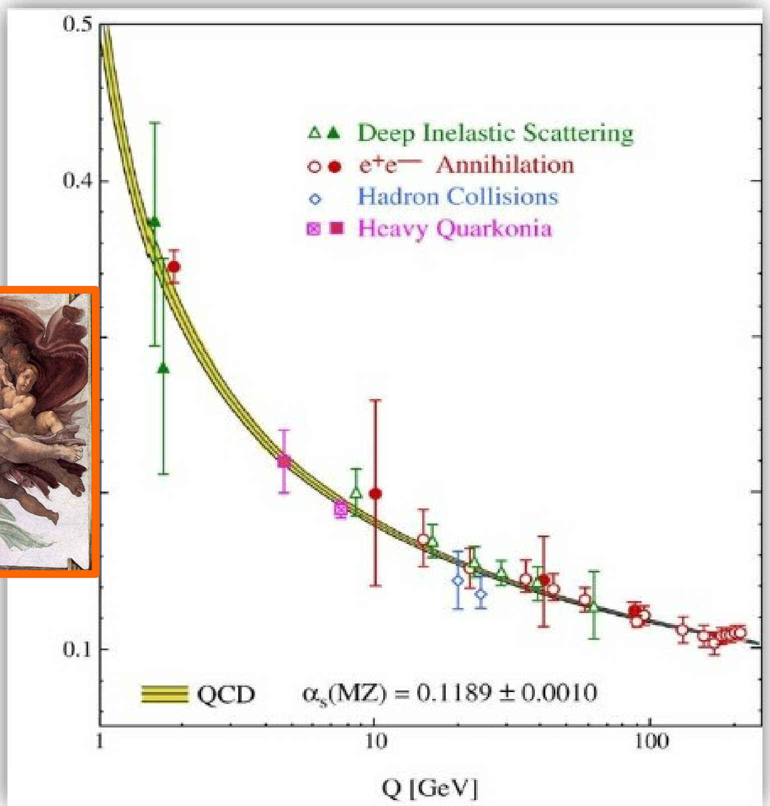
High

Q^2

Low

Confinement

Colored bound states have never been seen to exist as particles in nature



DCSB

Chiral symmetry appears dynamically violated in the Hadron spectrum

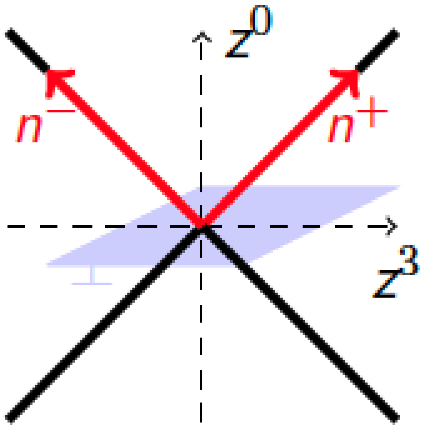
Emergent phenomena playing a dominant role in the real world dominated by the IR dynamics of QCD.

Antecedents:

GPD definition:

$$H^q_\pi(x,\xi,t) \;=\; \frac{1}{2}\int \frac{dz^-}{2\pi}\, e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2}\right| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right)\left|\pi, P-\frac{\Delta}{2}\right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t=\Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



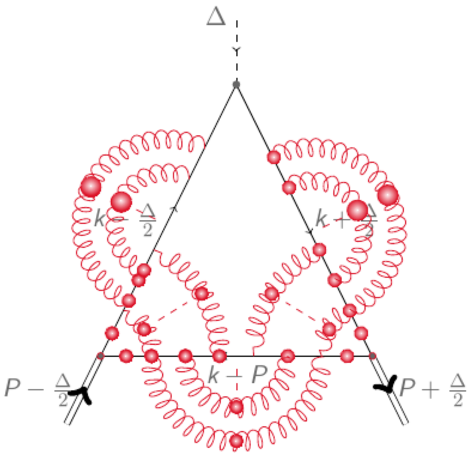
References

Muller et al., Fortchr. Phys. **42**, 101 (1994)
Radyushkin, Phys. Lett. **B380**, 417 (1996)
Ji, Phys. Rev. Lett. **78**, 610 (1997)

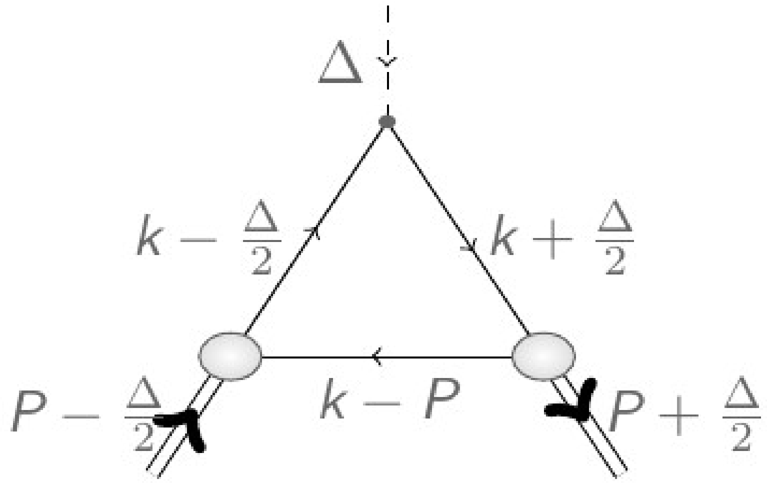
- From **isospin symmetry**, all the information about pion GPD is encoded in $H^u_{\pi^+}$ and $H^d_{\pi^+}$.
- Further constraint from **charge conjugation**: $H^u_{\pi^+}(x,\xi,t) = -H^d_{\pi^+}(-x,\xi,t)$.

Antecedents:

GPDs in the Schwinger-Dyson and Bethe-Salpeter approach

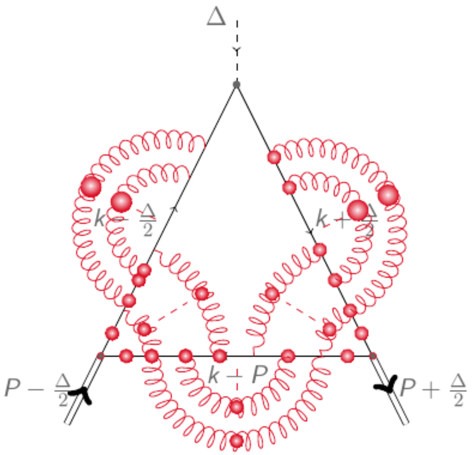


$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+(i\overleftrightarrow{D}^+)^m q(0) \middle| \pi, P - \frac{\Delta}{2} \right\rangle$$



- Compute **Mellin moments** of the pion GPD H.

Antecedents:



GPDs in the Schwinger-Dyson and Bethe-Salpeter approach

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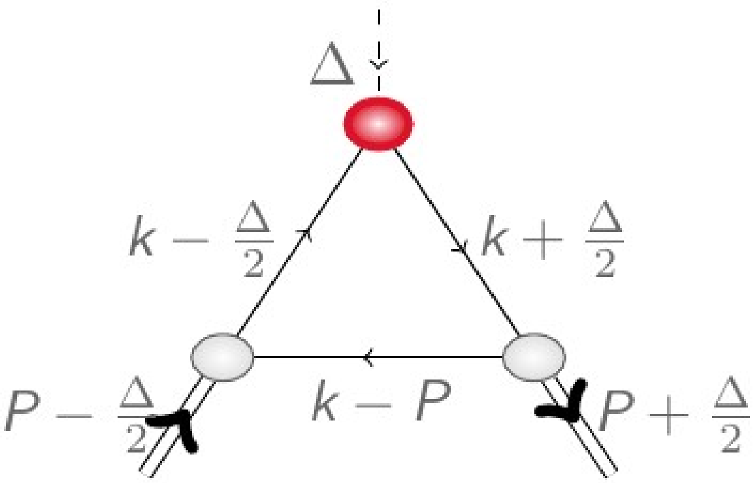
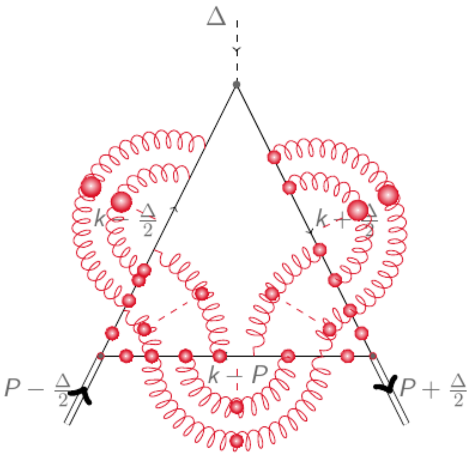


- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.

Antecedents:



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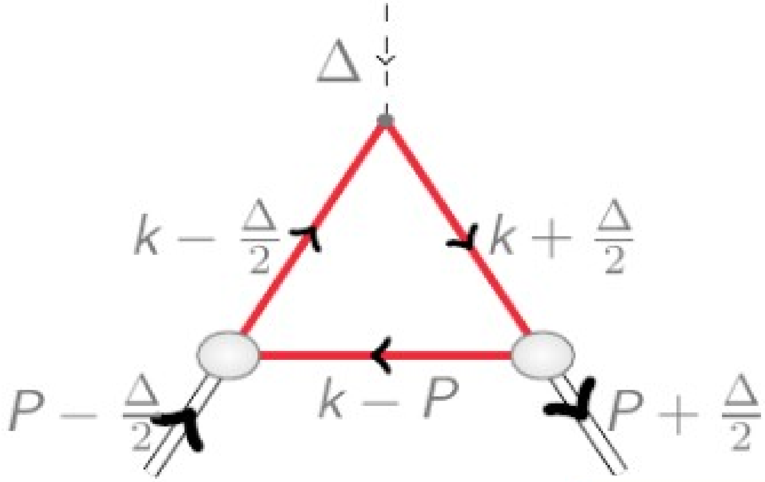
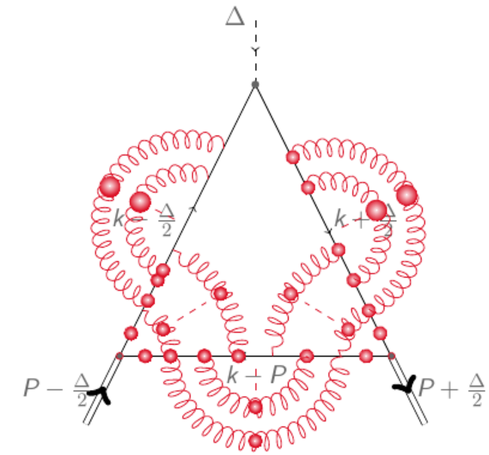


- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.
- Resum **infinitely many** contributions.

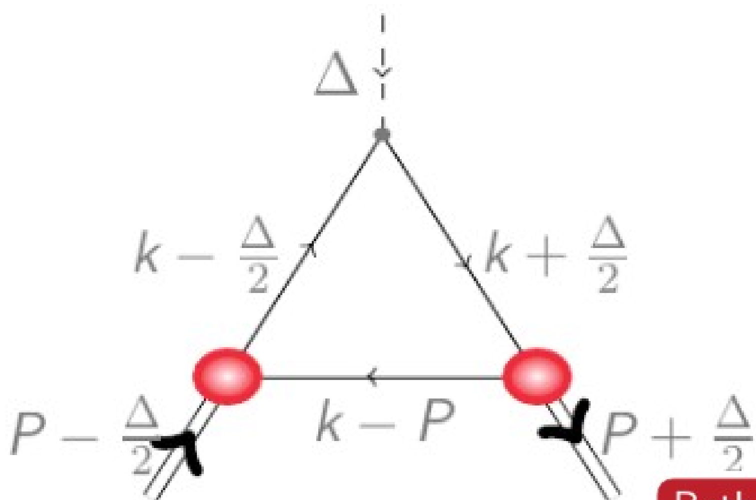
Dyson - Schwinger equation

Antecedents:

GPDs in the Schwinger-Dyson and Bethe-Salpeter approach



$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \middle| \pi, P - \frac{\Delta}{2} \right\rangle$$



- Compute **Mellin moments** of the pion GPD *H*.
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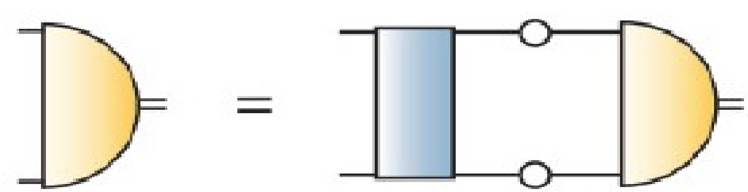
Bethe - Salpeter equation



Antecedents:

GPD asymptotic algebraic model:

- Expressions for vertices and propagators:

$$S(p) = \left[-i\gamma\cdot p + M\right]\Delta_M(p^2)$$
$$\Delta_M(s) = \frac{1}{s+M^2}$$
$$\Gamma_\pi(k,p) = i\gamma_5\frac{M}{f_\pi}M^{2\nu}\int_{-1}^{+1}\mathrm{d}z\,\rho_\nu(z)\,\left[\Delta_M(k_{+z}^2)\right]^\nu$$
$$\rho_\nu(z) = R_\nu(1-z^2)^\nu$$

with R_ν a normalization factor and $k_{+z} = k - p(1-z)/2$.

Chang *et al.*, Phys. Rev. Lett. **110** 132001 (2013)

- Only two parameters:
  - Dimension...
  - Dimensi... asympt...

Antecedents:

GPD asymptotic algebraic model:

- **Analytic expression** in the DGLAP region.

$$H^u_{x\geq\xi}(x,\xi,0) = \frac{48}{5}\left\{\frac{3\left(-2(x-1)^4\left(2x^2-5\xi^2+3\right)\log(1-x)\right)}{20\left(\xi^2-1\right)^3}\right.$$
$$\frac{3\left(+4\xi\left(15x^2(x+3)+(19x+29)\xi^4+5(x(x(x+11)+21)+3)\xi^2\right)\tanh^{-1}\left(\frac{(x-1)}{x-\xi^2}\right.\right.}{20\left(\xi^2-1\right)^3}$$
$$+\frac{3\left(x^3(x(2(x-4)x+15)-30)-15(2x(x+5)+5)\xi^4\right)\log\left(x^2-\xi^2\right)}{20\left(\xi^2-1\right)^3}$$
$$+\frac{3\left(-5x(x(x(x+2)+36)+18)\xi^2-15\xi^6\right)\log\left(x^2-\xi^2\right)}{20\left(\xi^2-1\right)^3}$$
$$+\frac{3\left(2(x-1)\left((23x+58)\xi^4+(x(x(x+67)+112)+6)\xi^2+x(x((5-2x)x+15)+\right.\right.}{20\left(\xi^2-1\right)^3}$$
$$\ldots 5)\xi^4+10x(3x(x+5)+11)\xi^2\right)\log\left(1-\xi^2\right)\right) \ldots -1)^3$$
$$\left.\ldots 3)\log\left(1-\xi^2\right)\right\}$$

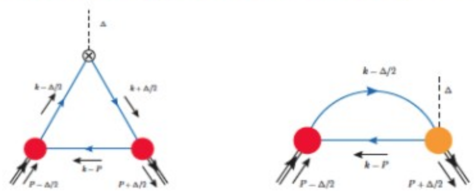
...-ph]

See Cedric's talk!

Antecedents:

GPD asymptotic algebraic model (completion):

The full model:



$$2(P\cdot n)^{m+1}\langle x^m\rangle^u = \mathrm{tr}_{CFD}\int\frac{\mathrm{d}^4k}{(2\pi)^4}(k\cdot n)^m\,\tau_+ i\Gamma_\pi\left(\eta(k-P)+(1-\eta)\left(k-\frac{\Delta}{2}\right),P-\frac{\Delta}{2}\right)$$
$$S\left(k-\frac{\Delta}{2}\right)\,i\gamma\cdot n\;S\left(k+\frac{\Delta}{2}\right)$$
$$\tau_- i\bar{\Gamma}_\pi\left((1-\eta)\left(k+\frac{\Delta}{2}\right)+\eta(k-P),P+\frac{\Delta}{2}\right)S(k-P),$$

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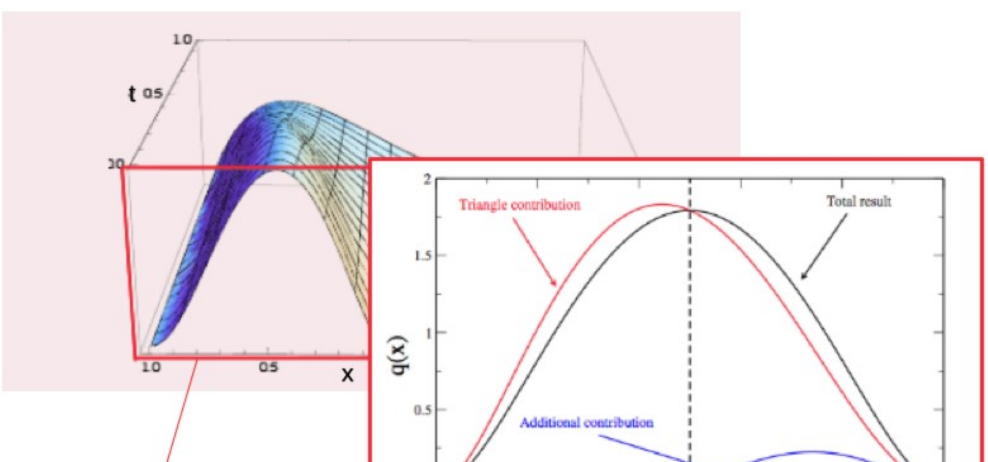
$$S\left(k-\frac{\Delta}{2}\right)\tau_-\frac{\partial}{\partial k}\bar{\Gamma}_\pi\left((1-\eta)\left(k+\frac{\Delta}{2}\right)+\eta(k-P),P+\frac{\Delta}{2}\right)S(k-P)$$

C. Mezrag et al., PLB741(2015)190; ArXiv:1406.7425[hep-ph]

Antecedents:

GPD asymptotic algebraic model (completion):



$q(x) = H^q(x,0,0)$

PDF forward limit

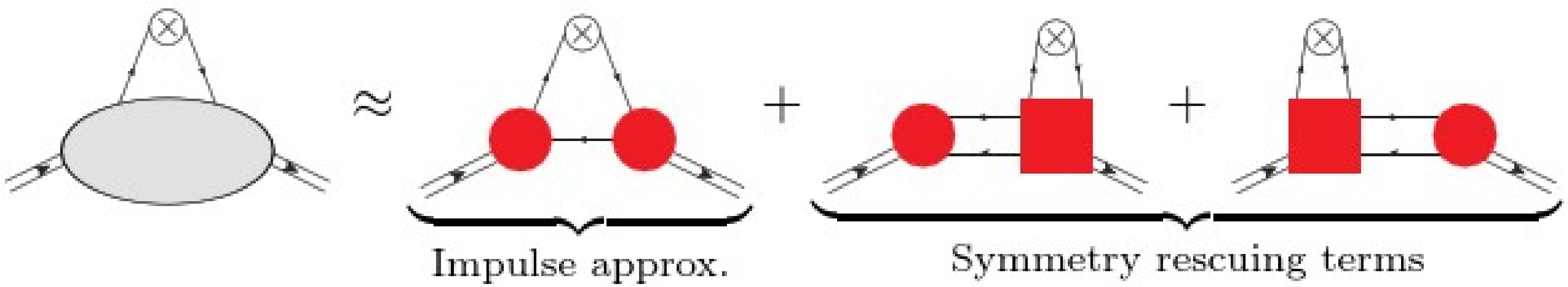
Antecedents:

Antecedents:

GPD asymptotic algebraic mode...

...ptotic algebraic model:

...region.



L.Chang *et al.*, Phys.Lett. B737 (2014) 23-29

See Cédric's talk!

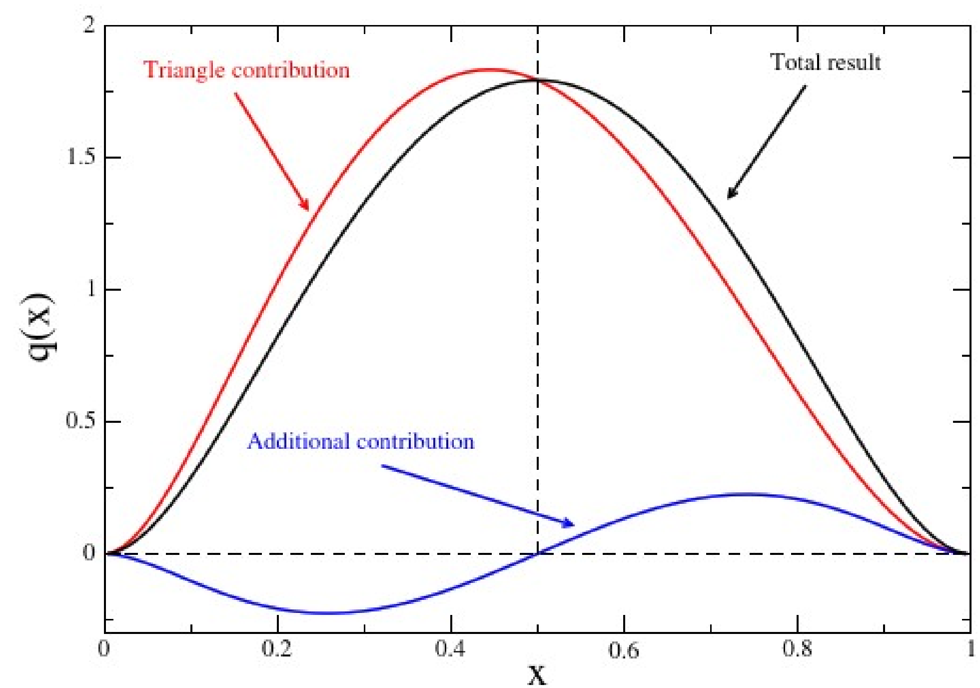
$$q(x) = H^q(x,0,0)$$

$2(P\cdot n)^{m+1}\langle x^m\rangle$

$$2(P\cdot n)^{m+1}\langle x^m\rangle^u = \mathrm{tr}_{CFD}\int\frac{d^4k}{(2\pi)^4}(k\cdot n)^m\,\tau_+ i\Gamma_\pi\left(\eta(k-P)+\ldots\right)$$

$$S(k-\tfrac{\Delta}{2})\tau_-\frac{\partial}{\partial k}\bar{\Gamma}_\pi\left((1-\eta)\left(k+\tfrac{\Delta}{2}\right)+\eta(k-P),P+\tfrac{\Delta}{2}\right)S(k-P)$$

C. Mezrag et al., PLB741(2015)190; ArXiv:1406.7425[hep-ph]



GPD overlap approach:

The overlap quark GPD for a meson in the DGLAP kinematic region reads

$$H^q(x,\xi,t) = \sum_{N,\beta} \sqrt{1-\xi}^{\,2-N} \sqrt{1+\xi}^{\,2-N} \sum_a \delta_{a,q} \int [\mathrm{d}\bar{x}]_N \left[\mathrm{d}^2\bar{\mathbf{k}}_\perp\right]_N \delta(x-\bar{x}_a)$$
$$\times\, \Psi^*_{N,\beta}\left(\hat{x}'_1, \hat{\mathbf{k}}'_{\perp 1}, \ldots, \hat{x}'_a, \hat{\mathbf{k}}'_{\perp a}, \ldots\right) \Psi_{N,\beta}\left(\tilde{x}_1, \tilde{\mathbf{k}}_{\perp 1}, \ldots, \tilde{x}_a, \tilde{\mathbf{k}}_{\perp a}, \ldots\right) ,$$

$$[\mathrm{d}x]_N = \prod_{i=1}^{N} \mathrm{d}x_i \, \delta\left(1 - \sum_{i=1}^{N} x_i\right),$$
$$[\mathrm{d}^2\mathbf{k}_\perp]_N = \frac{1}{(16\pi^3)^{N-1}} \prod_{i=1}^{N} \mathrm{d}^2\mathbf{k}_{\perp i} \, \delta^2\left(\sum_{i=1}^{N} \mathbf{k}_{\perp i} - \mathbf{P}_\perp\right)$$

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$$\times \boxed{\Psi^*_{N,\beta}\left(\hat{x}'_1,\hat{\mathbf{k}}'_{\perp 1},\ldots,\hat{x}'_a,\hat{\mathbf{k}}'_{\perp a},\ldots\right)} \boxed{\Psi_{N,\beta}\left(\tilde{x}_1,\tilde{\mathbf{k}}_{\perp 1},\ldots,\tilde{x}_a,\tilde{\mathbf{k}}_{\perp a},\ldots\right)} ,$$

$$[\mathrm{d}x]_N = \prod_{i=1}^N \mathrm{d}x_i\, \delta\left(1-\sum_{i=1}^N x_i\right),$$

in terms of the meson LFWF

$$[\mathrm{d}^2\mathbf{k}_\perp]_N = \frac{1}{(16\pi^3)^{N-1}} \prod_{i=1}^N \mathrm{d}^2\mathbf{k}_{\perp i}\, \delta^2\left(\sum_{i=1}^N \mathbf{k}_{\perp i} - \mathbf{P}_\perp\right)$$

$$|H;P,\lambda\rangle = \sum_{N,\beta} \int [\mathrm{d}x]_N [\mathrm{d}^2\mathbf{k}_\perp]_N \boxed{\Psi^\lambda_{N,\beta}(x_1,\mathbf{k}_{\perp 1},\ldots,x_N,\mathbf{k}_{\perp N})} |N,\beta;k_1,\ldots,k_N\rangle$$

which are the components in an expansion of the meson on a Fock basis, after light-front quantization.

GPD overlap approach:

The overlap valence-quark GPD for a meson in the DGLAP kinematic region reads

$$H^q(x,\xi,t) = \int \frac{d^2\mathbf{k}_\perp}{16\,\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$

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$$\boxed{2P^+\Psi_{\uparrow\downarrow}(k^+,\mathbf{k}_\perp) = \int \frac{\mathrm{d}k^-}{2\pi}\mathrm{Tr}\left[\gamma^+\gamma_5\chi(k,P)\right]}$$

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$$\boxed{\Gamma_\pi(q,P)} = i\mathcal{N}\gamma_5 \int_0^\infty d\omega \int_{-1}^1 dz \frac{\rho(\omega,z)M^2}{\left(q-\frac{1-z}{2}P\right)^2 + M^2 + \omega}$$

$$\boxed{\chi(q,P)} = S(q)\boxed{\Gamma_\pi(q,P)}S(q-P)$$

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Bethe-Salpeter amplitudes and quark propagators can be obtained from applying continuum functional methods (DSE,BSE)

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$$\boxed{S(q)} = [-i\gamma\cdot q + M]/[q^2 + M^2]$$

Nakanishi weight

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Asymptotic case: $\rho(w,z) = \delta(w)(1-z^2)$

Bethe-Salpeter amplitudes and quark propagators can be obtained from applying continuum functional methods (DSE,BSE) or can be modeled as previously indicated.

GPD overlap approach:

The overlap valence-quark GPD for a meson in the DGLAP kinematic region reads

$$H^q(x,\xi,t) = 30\frac{(1-x)^2(x^2-\xi^2)}{(1-\xi^2)^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)$$

$$z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\xi^2}$$

Encoding the correlation of kinematical variables

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N. Chouika et al., PLB780(2018)287

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$$z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\xi^2}$$

Encoding the correlation of kinematical variables

Forward limit: $\xi = 0, t = 0$

$$\boxed{S(q)} = [-i\gamma\cdot q + M]/[q^2 + M^2]$$

Nakanishi weight

$$\boxed{\Gamma_\pi(q,P)} = i\mathcal{N}\gamma_5\int_0^\infty d\omega\int_{-1}^{1}dz\frac{\rho(\omega,z)M^2}{\left(q-\frac{1-z}{2}P\right)^2+M^2+\omega}$$

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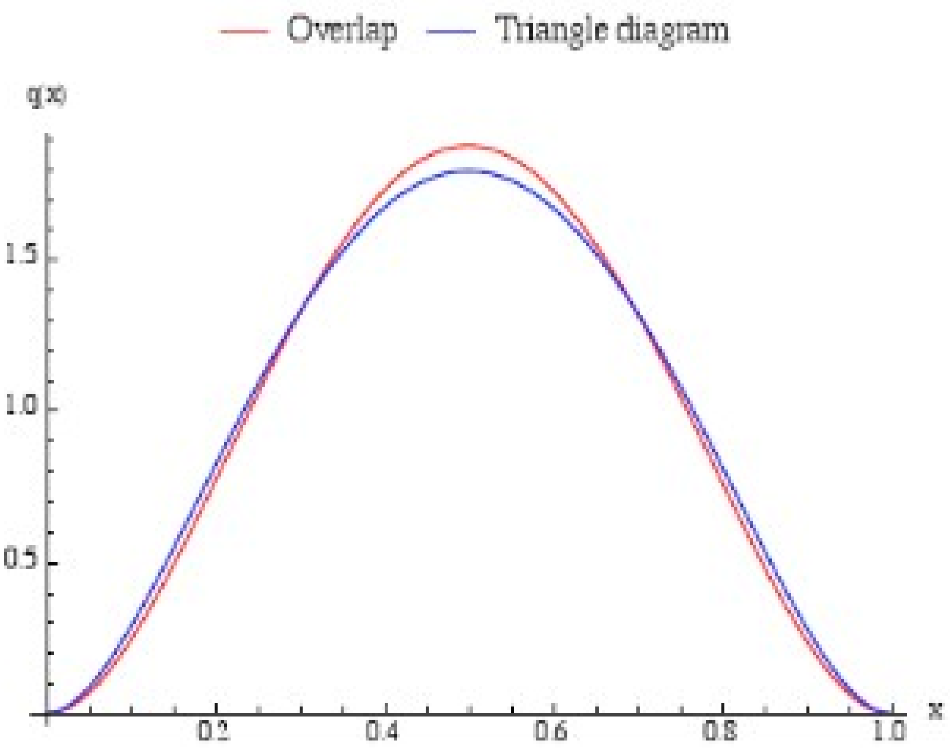

N. Chouika et al., PLB780(2018)287

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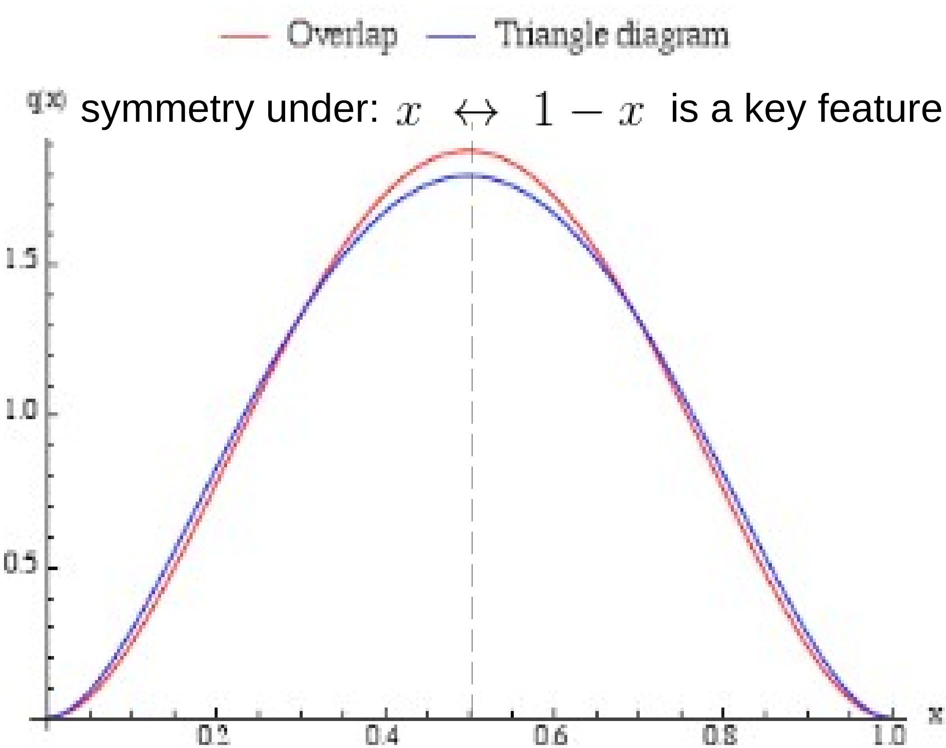
Bethe-Salpeter amplitudes and quark propagators can be obtained from applying continuum functional methods (DSE,BSE) or can be modeled as previously indicated.

Results from the overlap and diagrammatic approaches compare very well (tested at the level of the PDF).



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Integral representation of LFWFs:

See tomorrow Khépani's talk!

- The pseudoscalar LFWF can be written:

$$f_K\psi_K^{\uparrow\downarrow}(x,k_\perp^2) = \mathrm{tr}_{CD}\int_{dk_{||}} \delta(n\cdot k - x n\cdot P_K)\gamma_5\gamma\cdot n\,\chi_K^{(2)}(k_-^K;P_K)\ .$$

- The moments of the distribution are given by:

$$< x^m >_{\psi_K^{\uparrow\downarrow}} = \int_0^1 dx\, x^m \psi_K^{\uparrow\downarrow}(x,k_\perp^2) = \frac{1}{f_K n\cdot P}\int_{dk_{||}}\left[\frac{n\cdot k}{n\cdot P}\right]^m \gamma_5\gamma\cdot n\,\chi_K^{(2)}(k_-^K;P_K)$$

$$\int_0^1 d\alpha\, \alpha^m\left[\frac{12}{f_K}\mathcal{Y}_K(\alpha;\sigma^2)\right]\ , \quad \mathcal{Y}_K(\alpha;\sigma^2) = [M_u(1-\alpha)+M_s\alpha]\mathcal{X}(\alpha;\sigma_\perp^2)\ .$$

Uniqueness of Mellin moments $\Rightarrow$

$$\boxed{\psi_K^{\uparrow\downarrow}(x,k_\perp^2) = \frac{12}{f_K}\mathcal{Y}_K(x;\sigma_\perp^2)}$$

$$\boxed{\chi_K(\alpha;\sigma^3) = \left[\int_{-1}^{1-2\alpha} d\omega\int_{1+\frac{2\alpha}{\omega-1}}^{1} dv + \int_{1-2\alpha}^{1} d\omega\int_{\frac{\omega-1+2\alpha}{\omega+1}}^{1} dv\right]\frac{\rho_K(\omega)}{n_K}\frac{\Lambda_K^2}{\sigma^3}\ .}$$

The Nakanishi weight $\rho_K(z)$ can be modeled...
...Or taken with BSE solutions as an input!

$$\boxed{\Rightarrow \psi_K^{\uparrow\downarrow}(x,k_\perp^2) \sim \int d\omega\ \cdots\ \rho_K(\omega)\cdots}$$

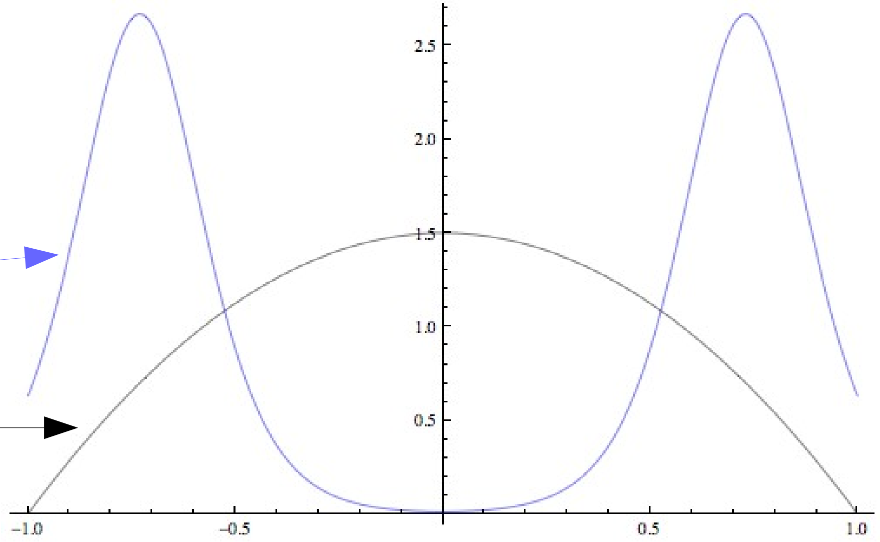
Integral representation of LFWFs: pion case

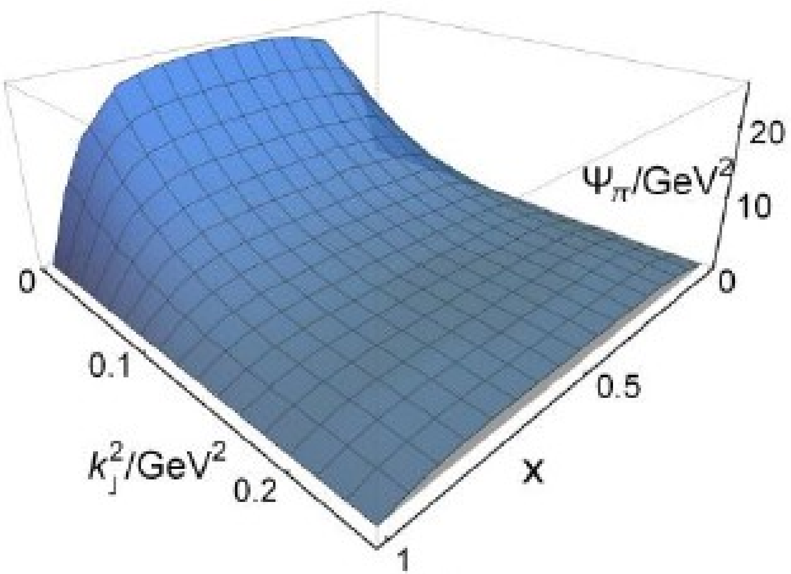
Nakanishi weight parametrization:

$$\rho(z) \;=\; \frac{1}{2b_0}\left[\operatorname{sech}^2\left(\frac{z-z_0}{2b_0}\right)+\operatorname{sech}^2\left(\frac{z+z_0}{2b_0}\right)\right]$$

Phenomelogical model: $b_0^{\pi}=0.1\,, z_0^{\pi}=0.73\,;$

Asymptotic case: $\rho(z) = (1-z^2)$





S-S Xu et al., PRD97(2018)094014

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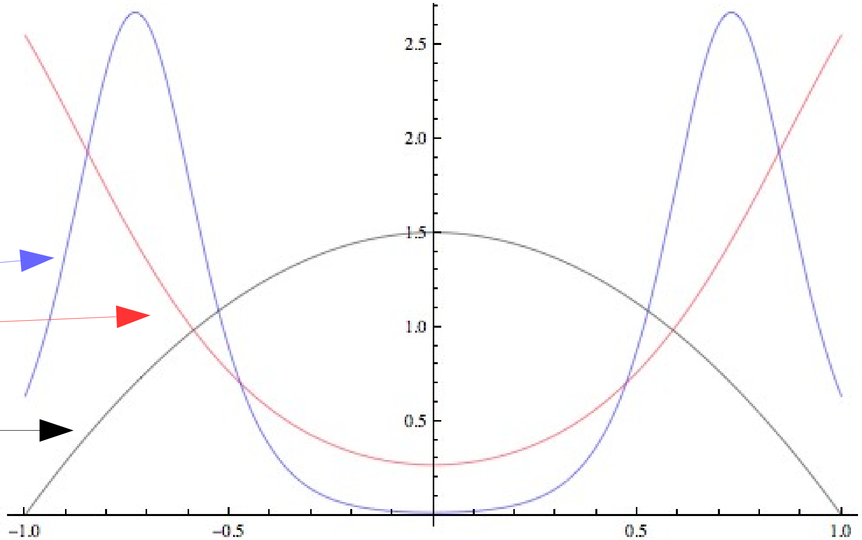
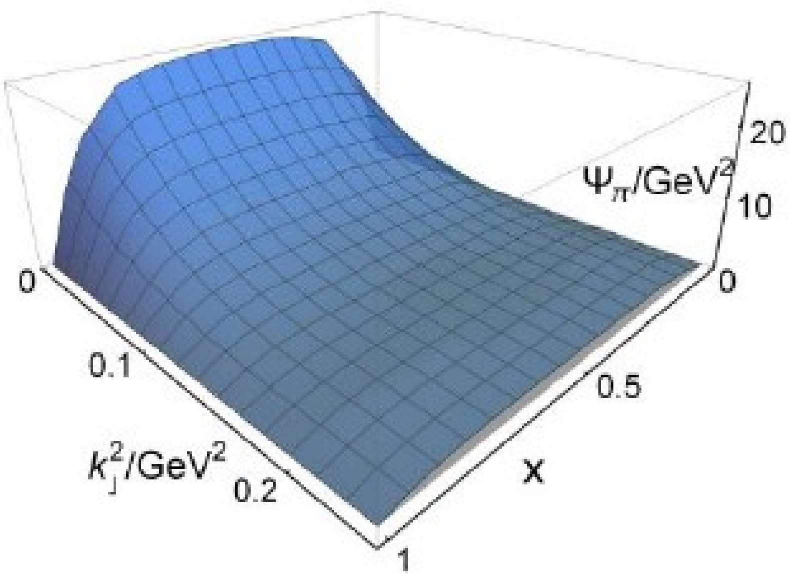
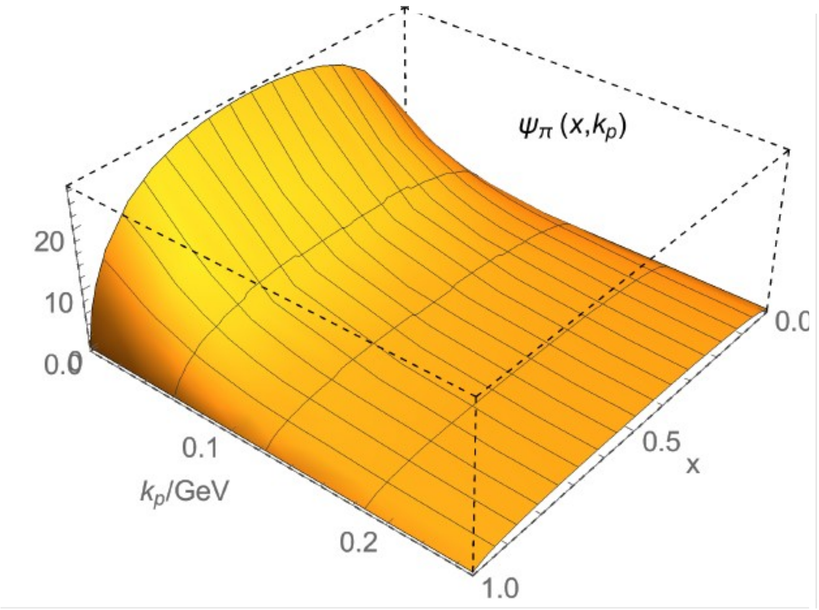
Realistic case: $b_0^{\pi}=0.275, z_0^{\pi}=1.23;$

Asymptotic case: $\rho(z)=(1-z^2)$



See Khépani's talk: PDF as a benchmark!



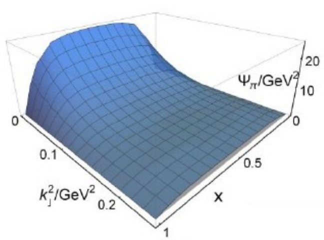


S-S Xu et al., PRD97(2018)094014

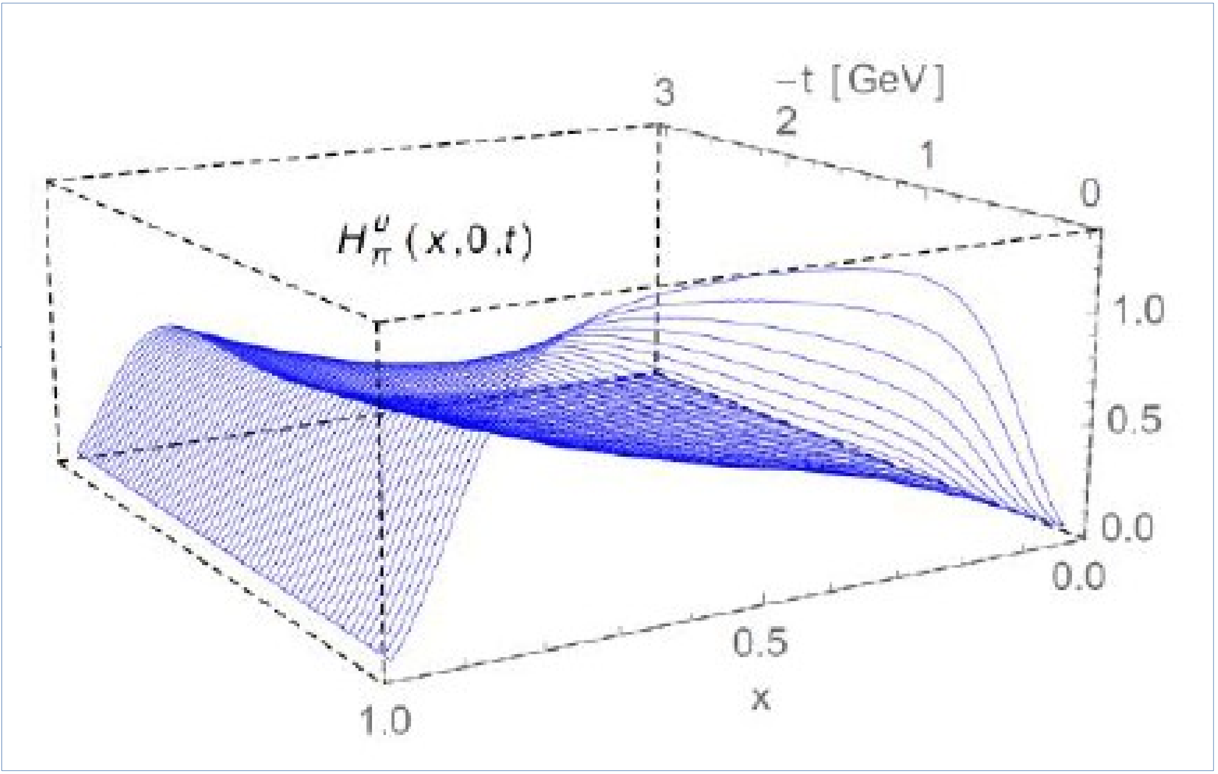
GPD overlap approach: pion case

The overlap valence-quark GPD for a meson in the DGLAP kinematic region reads

$$\boxed{H^q(x,\xi,t)} = \int \frac{d^2\mathbf{k}_\perp}{16\,\pi^3} \boxed{\Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right)} \boxed{\Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)}$$



Phenomenological model



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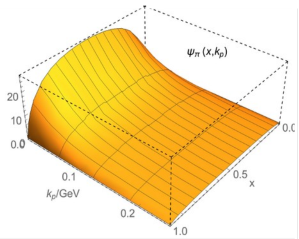

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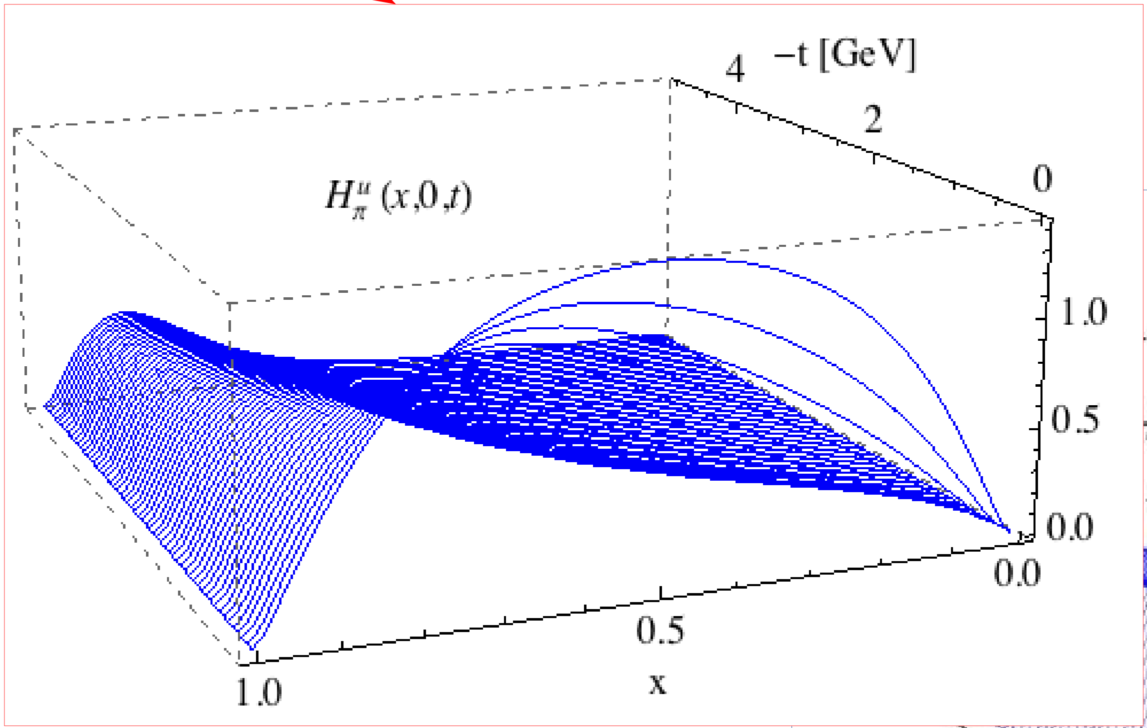
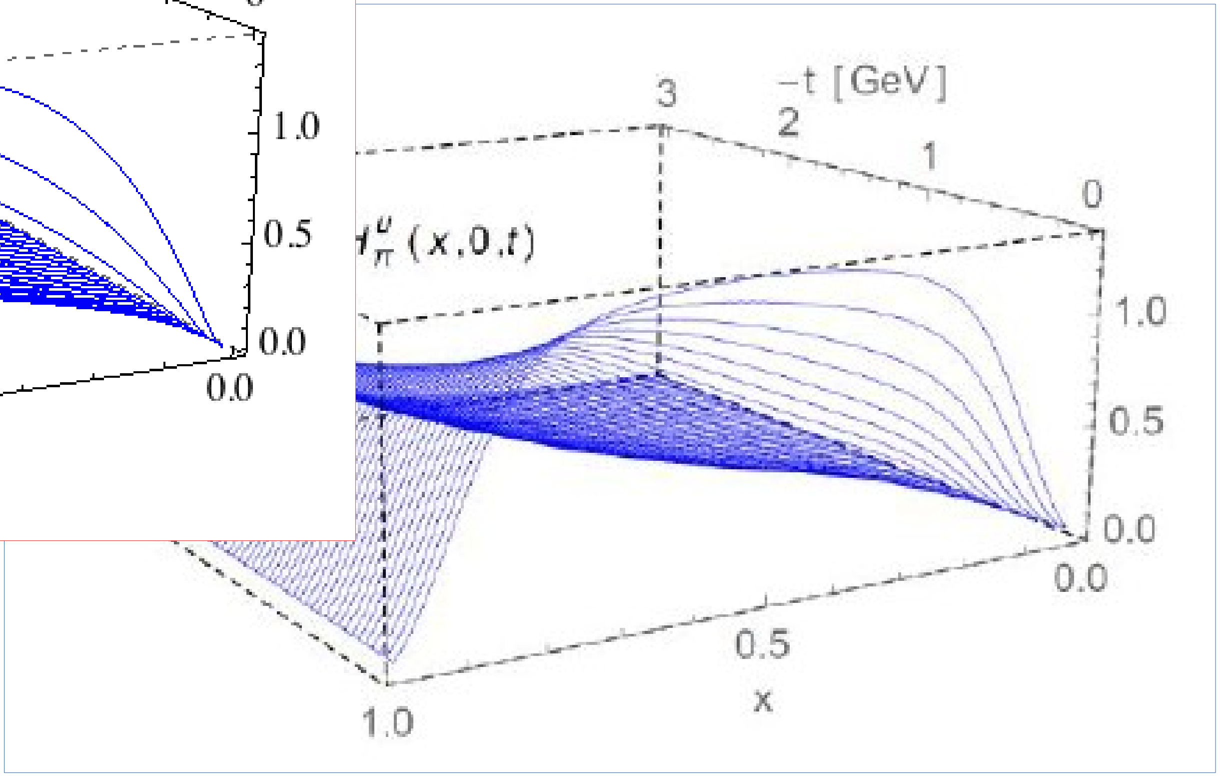


Realistic case



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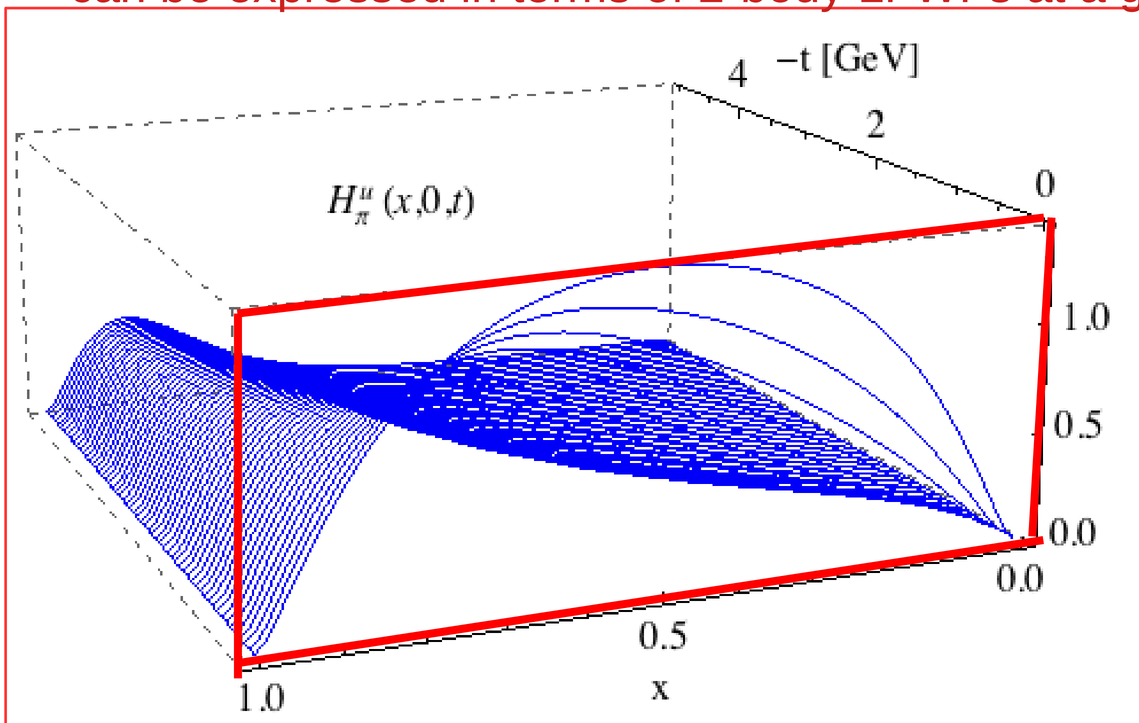
Focus on the forward limit: the PDF that, in the overlap representation at low Fock space, can be expressed in terms of 2-body LFWFs at a given hadronic scale



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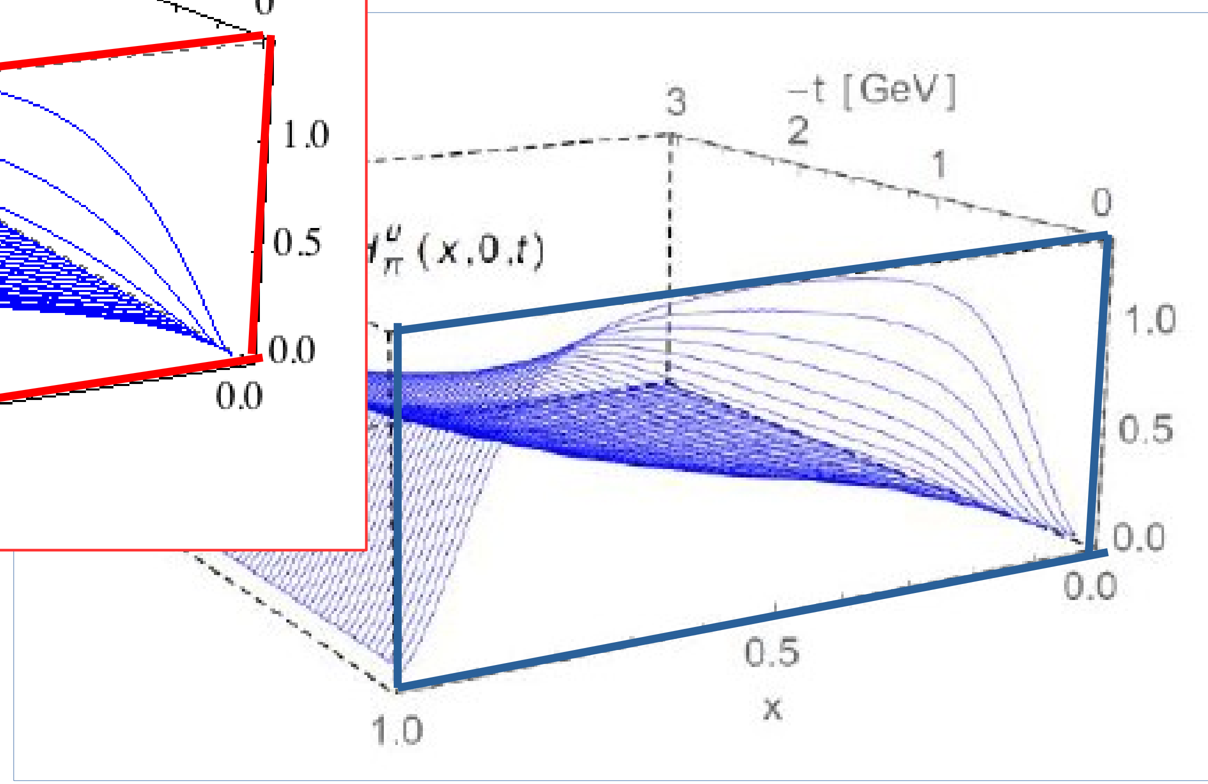

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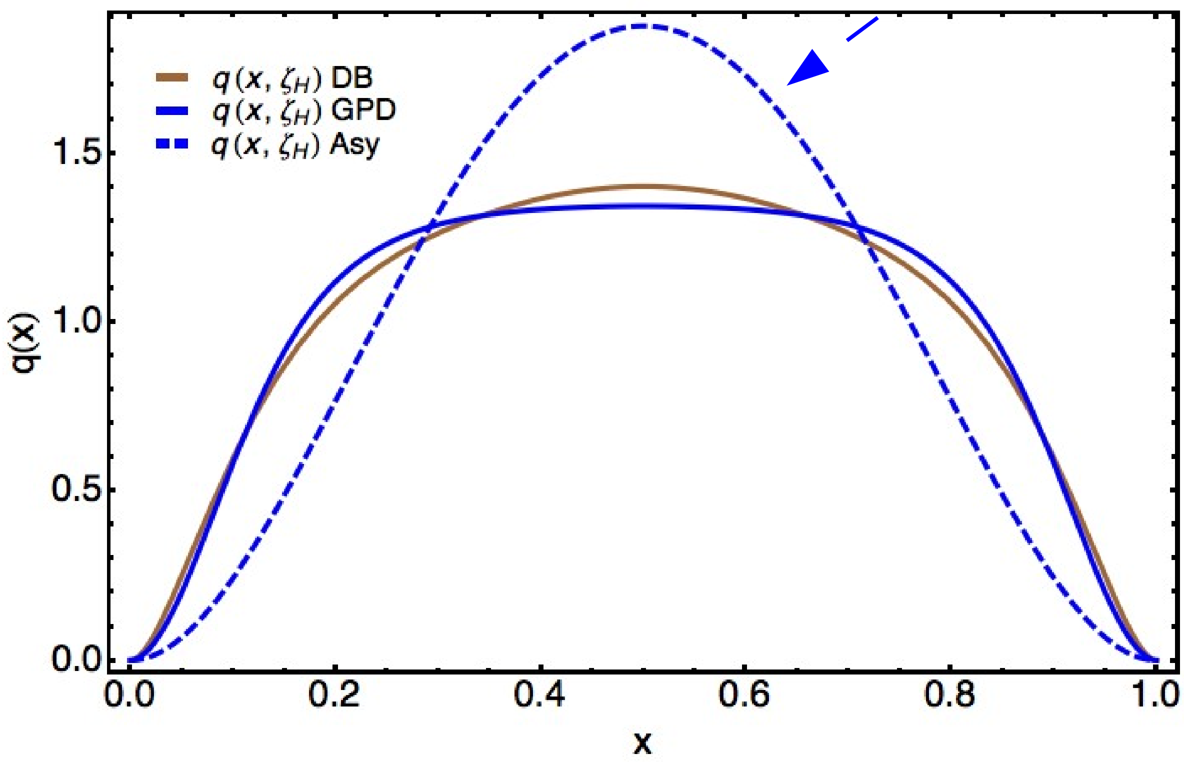
$$q^\pi(x;\zeta_H) = 30x^2(1-x)^2$$

LFWF leading to asymptotic PDA



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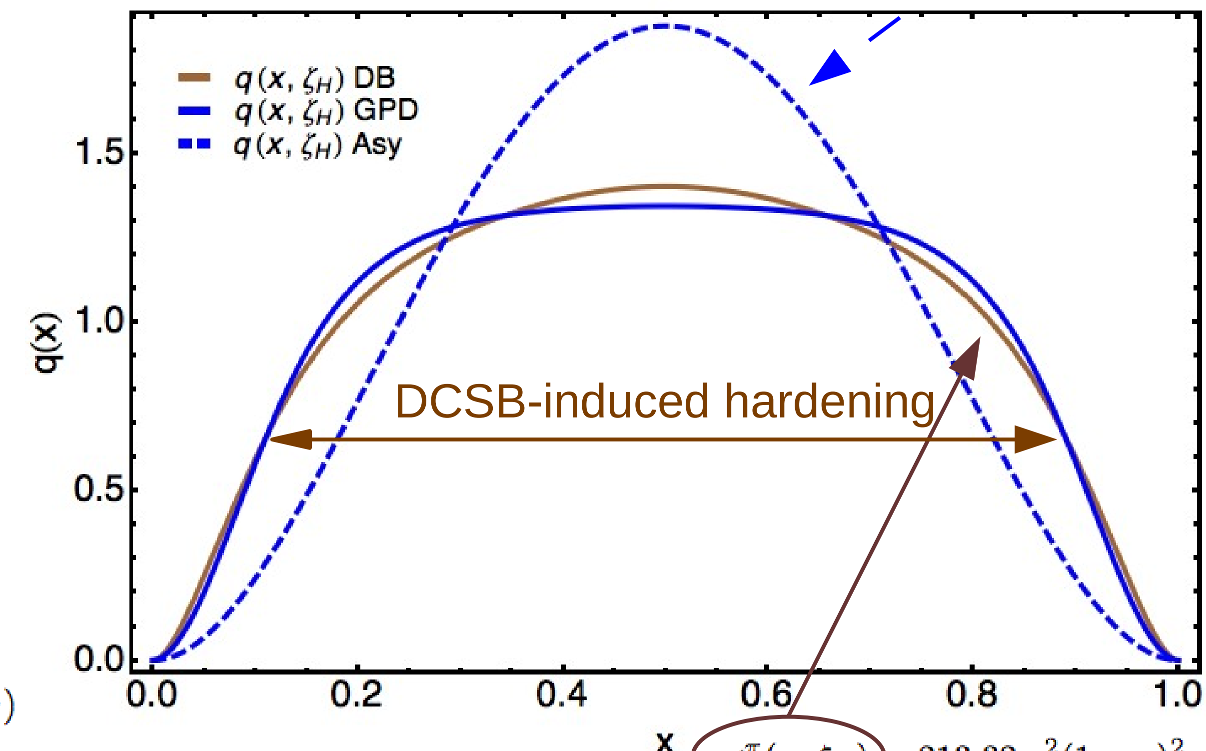

$$q^\pi(x; \zeta_H) = 213.32\, x^2 (1-x)^2 \times \left[1 - 2.9342\sqrt{x(1-x)} + 2.2911\, x(1-x)\right]$$

Direct computation of Mellin moments:

$$\langle x^m \rangle^\pi_{\zeta_H} = \int_0^1 dx\, x^m q^\pi(x; \zeta_H)$$

$$= \frac{N_c}{n \cdot P} \mathrm{tr} \int_{dk} \left[\frac{n \cdot k_\eta}{n \cdot P}\right]^m \Gamma_\pi(k_{\bar{\eta}}, P)\, S(k_{\bar{\eta}})\, n \cdot \partial_{k_\eta} \left[\Gamma_\pi(k_\eta, -P) S(k_\eta)\right]$$

M. Ding et al., PRD01(2020)054014

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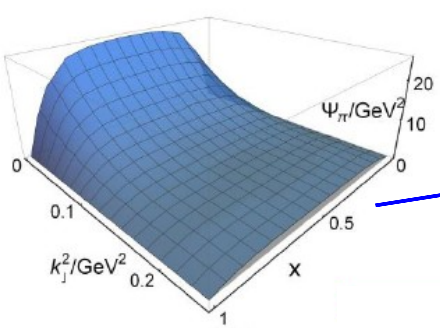

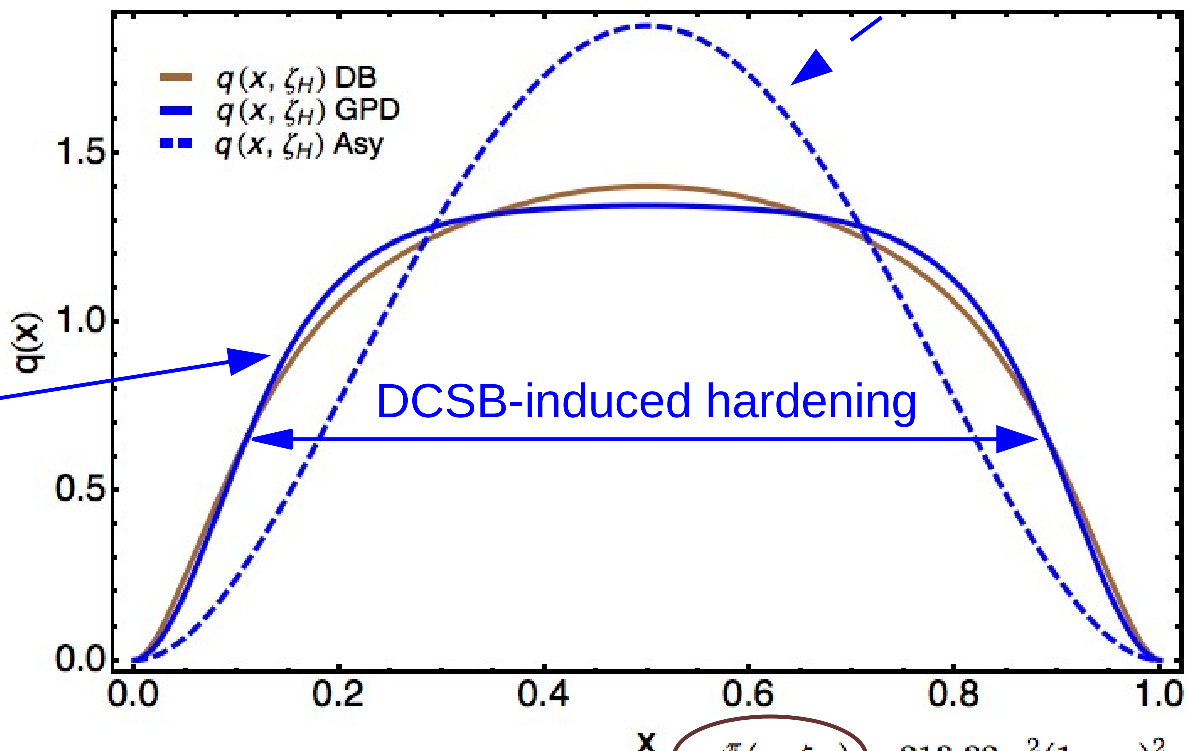

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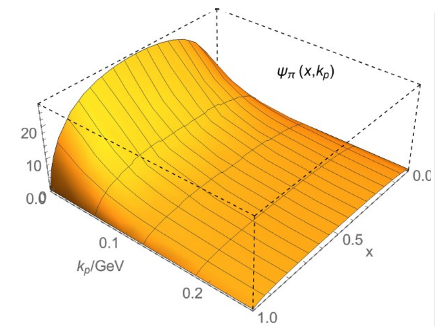

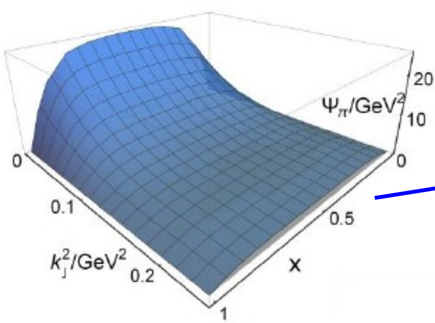

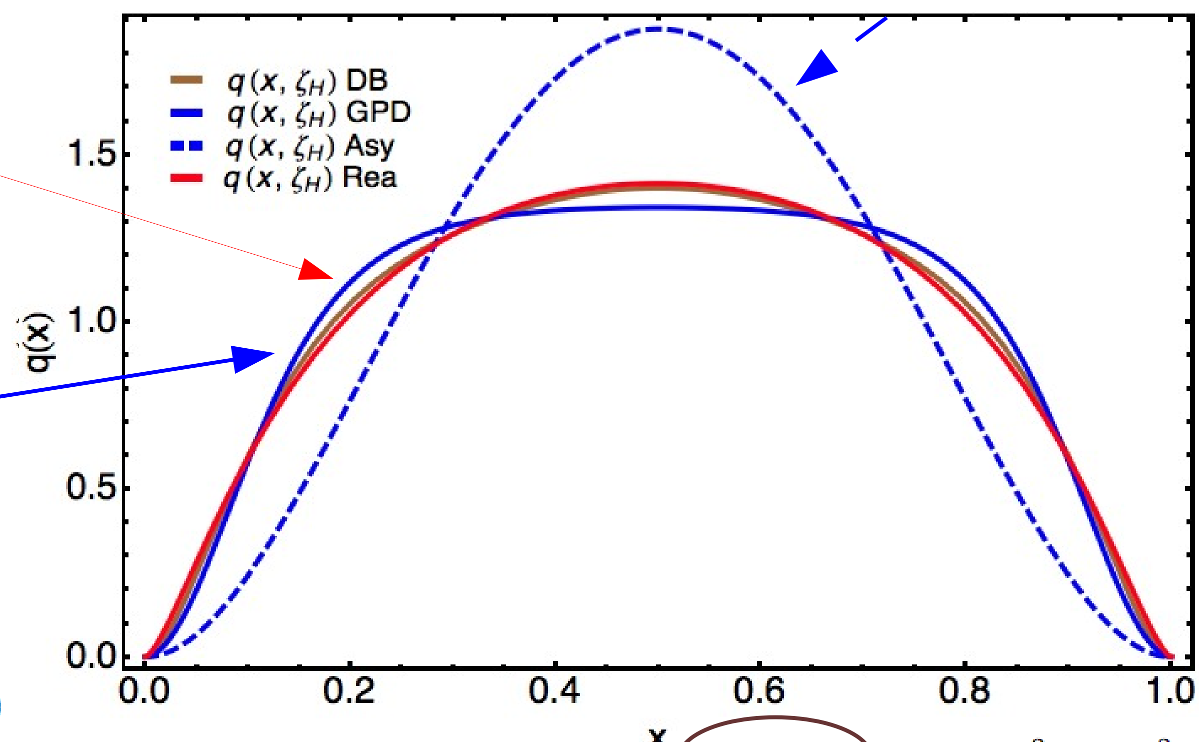

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Comparison with experiment: DGLAP evolution

$$M_n(t) = \int_0^1 dx\, x^n q(x,t)$$

$$t = \ln\left(\frac{\zeta^2}{\zeta_0^2}\right)$$

A master equation for the (1-loop) moments' evolution:

$$\frac{d}{dt} q(x,t) = \frac{\alpha(t)}{4\pi} \int_x^1 \frac{dy}{y} q(y,t) P\left(\frac{x}{y}\right) + \ldots$$

$$-\int_0^1 dx\, x^n P(x) = \gamma_0^n$$

Moments' evolution (1-loop):

$$\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t) + \ldots$$

$$P(x) = \frac{8}{3}\left(\frac{1+z^2}{(1-x)_+} + \frac{3}{2}\delta(x-1)\right)$$

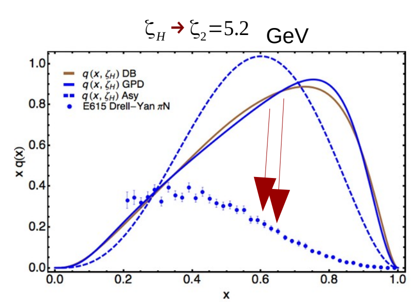
$$\frac{d}{dt}\alpha(t) = -\frac{\alpha^2(t)}{4\pi}\beta_0 + \ldots$$

$$\gamma_0^n = -\frac{4}{3}\left(3 + \frac{2}{(n+2)(n+3)} - 4\sum_{i=1}^{n+1}\frac{1}{i}\right)$$

$$\alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)} + \ldots$$

$$t_\Lambda = \ln\left(\frac{\Lambda^2}{\zeta_0^2}\right)$$

$$M_n(t) = M_n(t_0)\left(\frac{\alpha(t)}{\alpha(t_0)}\right)^{\gamma_0^n/\beta_0}$$



Comparison with experiment: DGLAP evolution

Which value of Lambda?

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The evolution will thus depend on the scheme *via* the perturbative truncation

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The evolution will thus depend on the scheme *via* the perturbative truncation and the usual prejudice is that truncation errors are optimally small in MS scheme.

PDG2018:
[PRD98(2018)030001]

$$\Lambda^{(5)}_{\overline{MS}} = (210 \pm 14)\ \ \mathrm{MeV}, \tag{9.24b}$$

$$\Lambda^{(4)}_{\overline{MS}} = (292 \pm 16)\ \ \mathrm{MeV}, \tag{9.24c}$$

$$\Lambda^{(3)}_{\overline{MS}} = (332 \pm 17)\ \ \mathrm{MeV}, \tag{9.24d}$$

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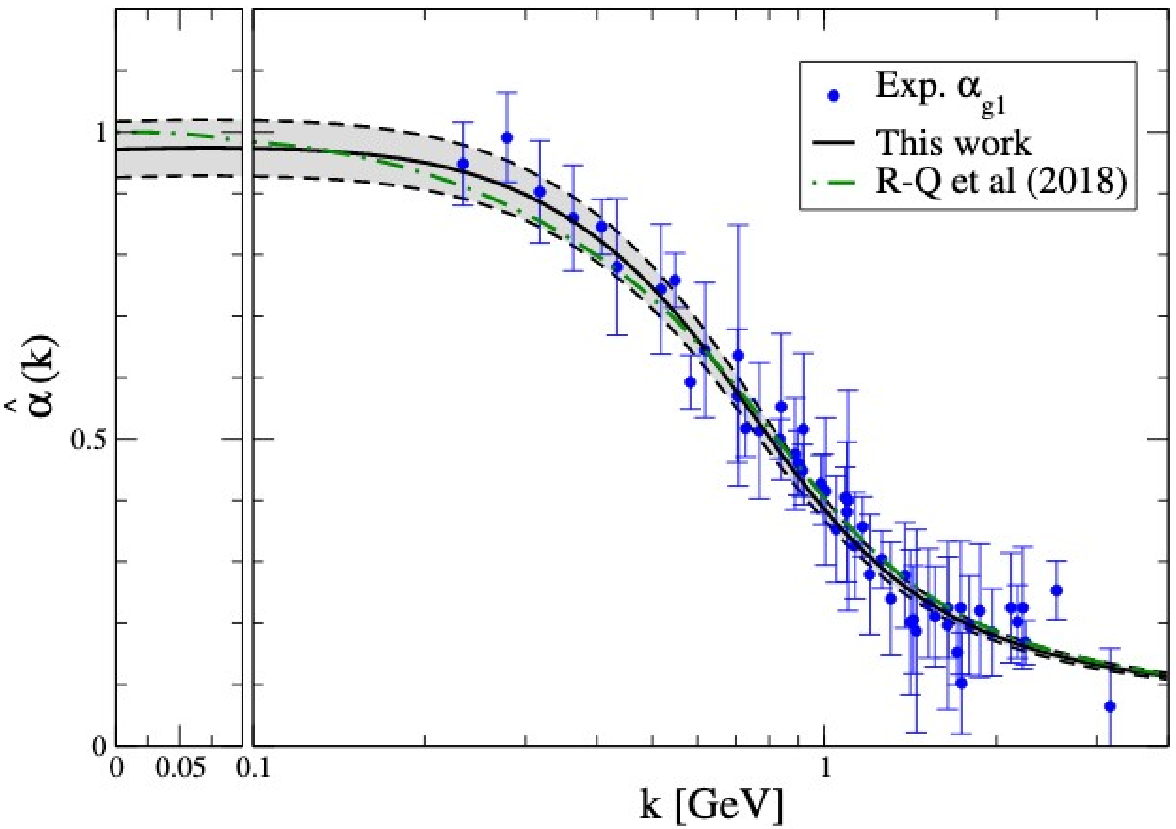
Comparison with experiment: PI effective charge

D.B et al., PRD96(2017)054026
J.R-Q et al., FBS59(2018)121
Z-F Cui et al., arXiv:1912.08232

Process-independent charge, defined as an analogue of the QED Gell-Man-low, on the basis of the PT-BFM truncation of DSEs in the gluon sector

Gauge-independent, no Landau pole, fully determined by the gluon sector, known to unify a wide range of observables, it compares very well with the Bjorken sum rule charge…



It emerges as a strong candidate to represent the interaction strength of QCD at any scale

Assumption: PI effective charge corresponds with the effective charge for the PDF evolution

$m_0 = 0.43(1)$ GeV emerges as natural nonperturbative scale marking the boundary between soft and hard physics, thus ensuring that parton modes are screened from interaction

$$\boxed{\zeta_H = m_0}$$

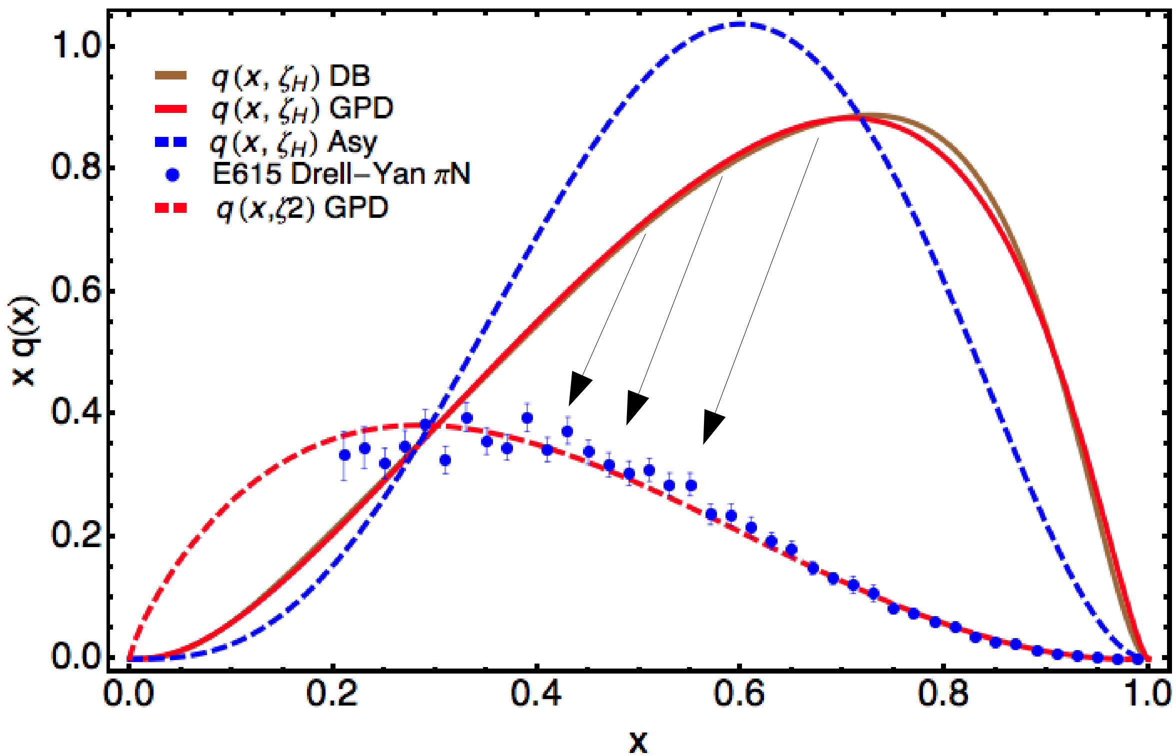
Comparison with experiment:

Then, one can evolve the pion PDF, by using one-loop DGLAP evolution and the effective charge, from the hadronic scale up to the relevant one for the E615 experiment:

$$M_n(t) = M_n(t_0)\exp\left(-\frac{\gamma_0^n}{4\pi}\int_{t_0}^{t} dz\,\alpha(z)\right)$$

[Aicher et al., PRL105(2010)252003]

$$\zeta_H \equiv m_0 \rightarrow \zeta_2 = 5.2 \text{ GeV}$$



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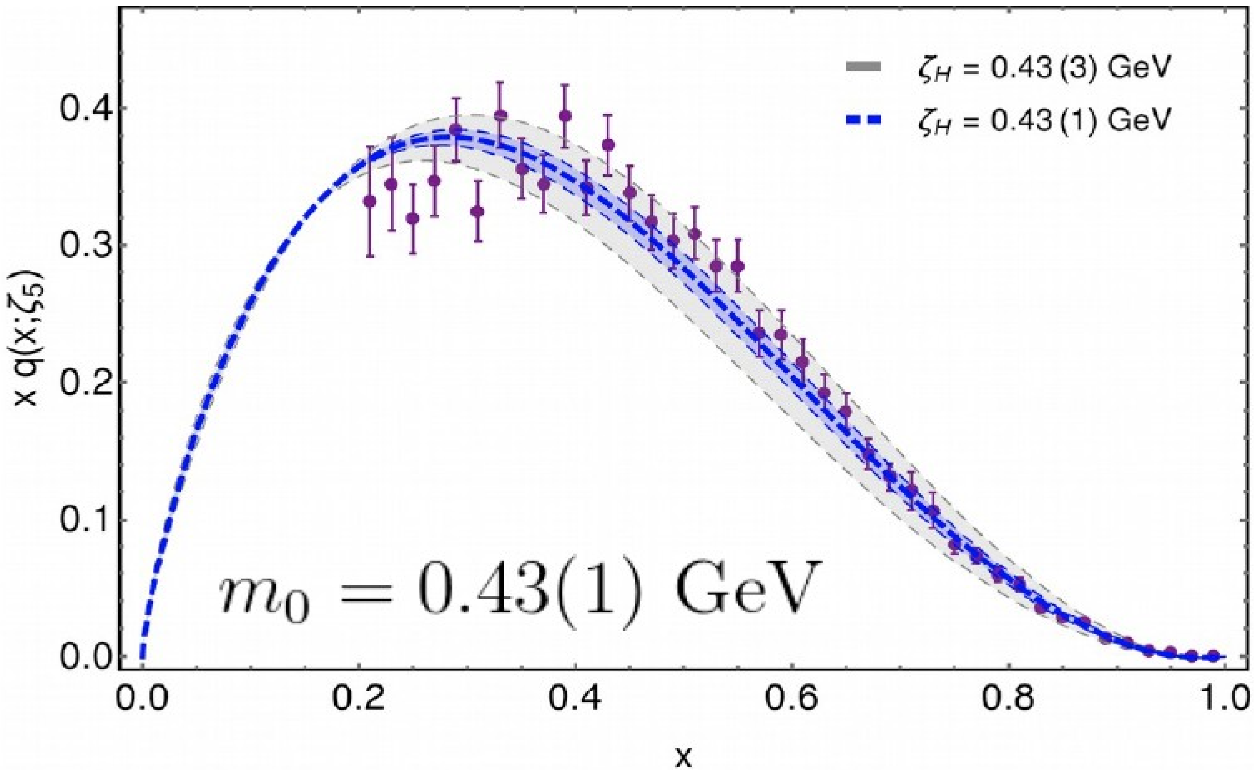

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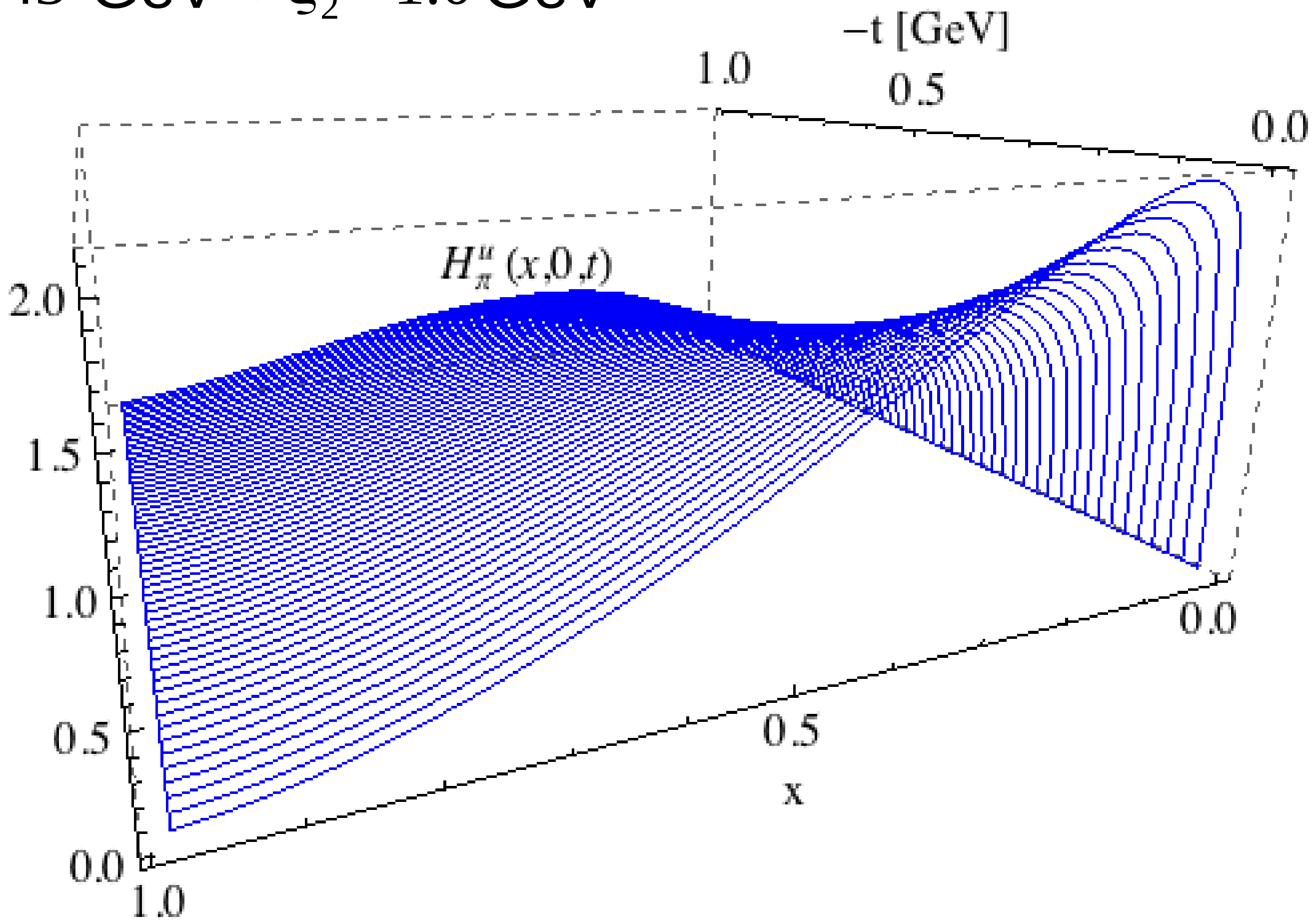
GPD overlap approach: pion case

The overlap valence-quark GPD for a meson in the DGLAP kinematic region reads

$$H^q(x,\xi,t) = \int \frac{d^2\mathbf{k}_\perp}{16\,\pi^3}\Psi^*_{u\bar f}\left(\frac{x-\xi}{1-\xi},\mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right)\Psi_{u\bar f}\left(\frac{x+\xi}{1+\xi},\mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$

$\zeta_0=\zeta_H=0.43$ GeV $\rightarrow \zeta_2=1.0$ GeV



Singlet components evolution

Let us also consider the singlet components:
(an almost textbook exercise)

$$\gamma_{0,AB}^{S,(m)} = -\int_0^1 dx\, x^m P_{0,AB}^S(x)$$

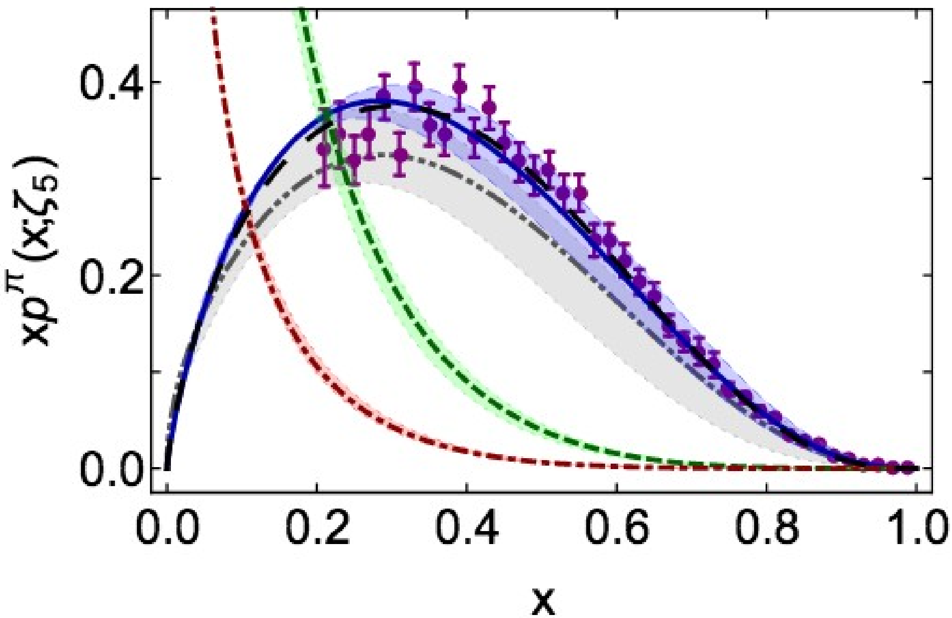
$$\zeta^2 \frac{d}{d\zeta^2} P^{-1} \begin{pmatrix} M_q^{(m)}(\zeta) \\ M_G^{(m)}(\zeta) \end{pmatrix} = -\frac{\alpha(\zeta^2)}{4\pi} \Gamma_D^{(m)} P^{-1} \boxed{\begin{pmatrix} M_q^{(m)}(\zeta_H) \\ 0 \end{pmatrix}}$$

$$P^{-1} \Gamma_0^{S,(m)} P = \begin{pmatrix} \lambda_+^{(m)} & 0 \\ 0 & \lambda_-^{(m)} \end{pmatrix}$$

Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom, can be evolved and shown to produce non-zero gluon and sea-quark components.

ζ_5	$\langle x \rangle_u^\pi$	$\langle x^2 \rangle_u^\pi$	$\langle x^3 \rangle_u^\pi$
Ref. [31]	0.17(1)	0.060(9)	0.028(7)
Herein	0.21(2)	0.076(9)	0.036(5)

$$\langle x \rangle_g^\pi = 0.45(1)\,, \quad \langle x \rangle_{\rm sea}^\pi = 0.14(2)\,.$$

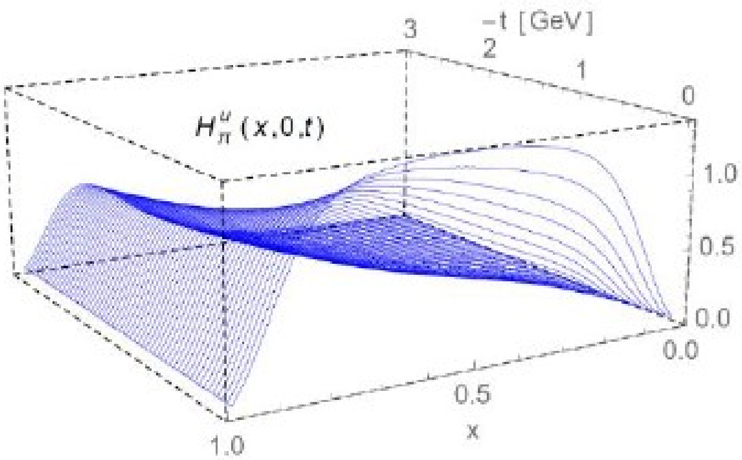


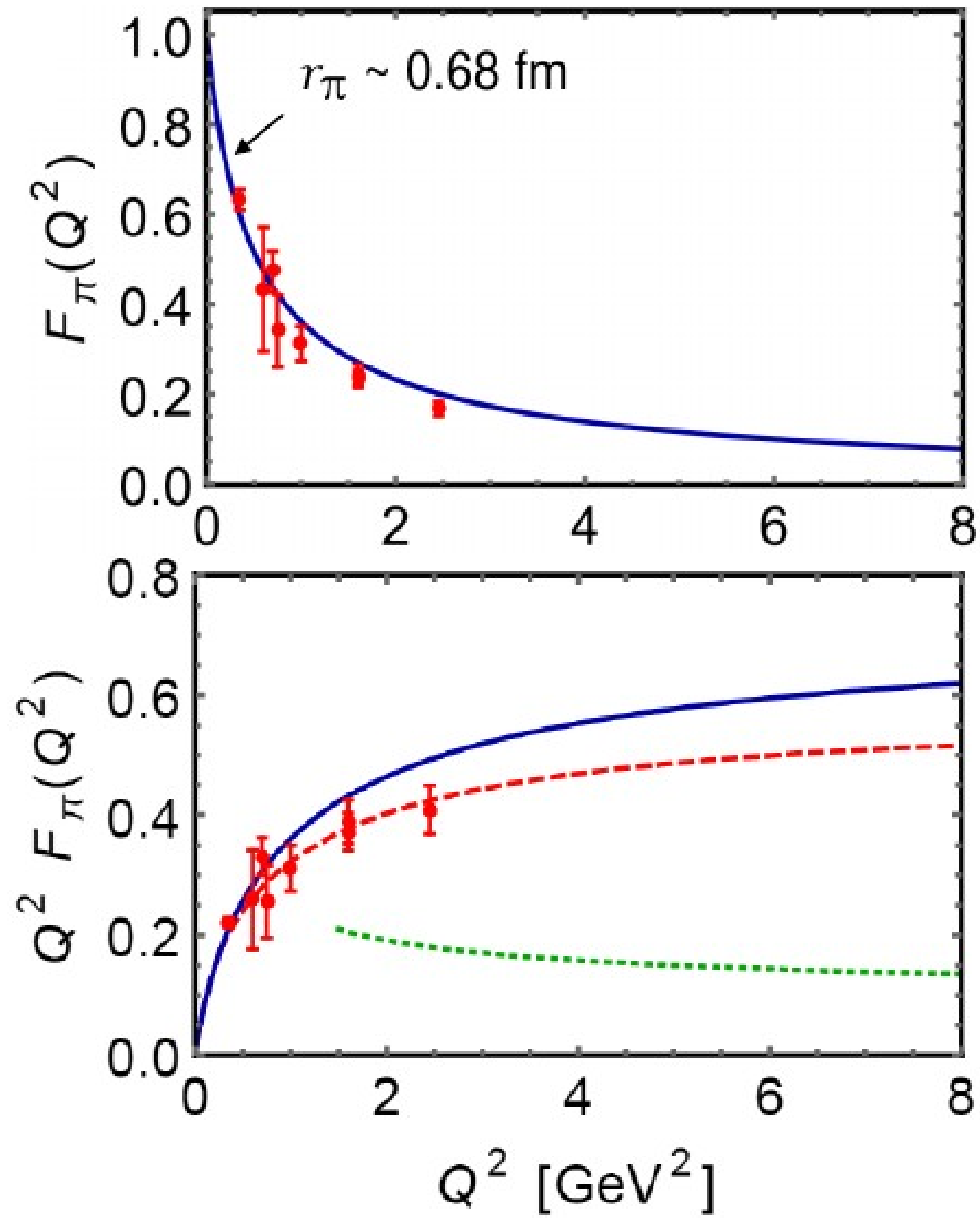
M. Ding et al., ArXiv:1905.05208 [nucl-th]

Pion realistic picture: Electromagnetic Form Factor

$$F_M(\Delta^2) = e_u F_M^u(\Delta^2) + e_f F_M^f(\Delta^2)\ ,\ F_M^q(-t=\Delta^2) = \int_{-1}^{1} dx\ H_M^q(x,\xi,t)$$

Electric charges





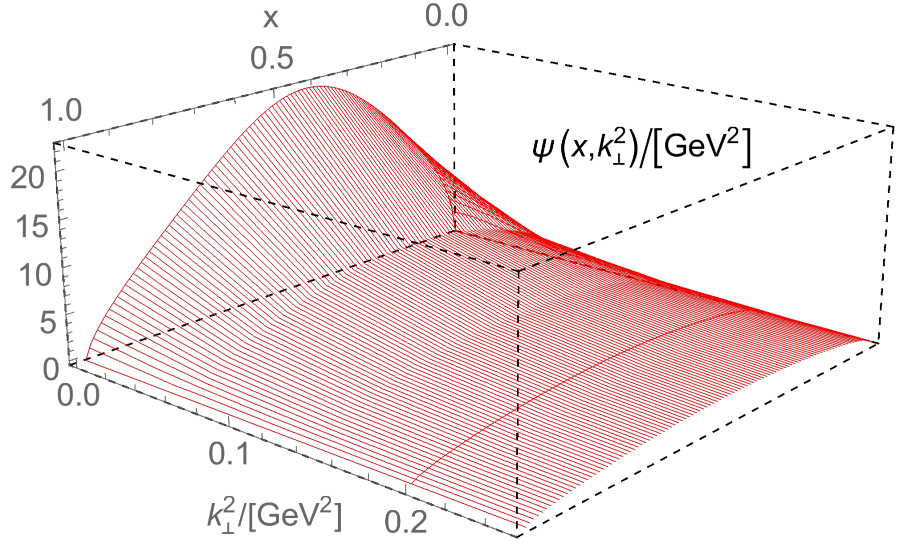
Blue: Computed from GPD

Green: Computed from HS formula

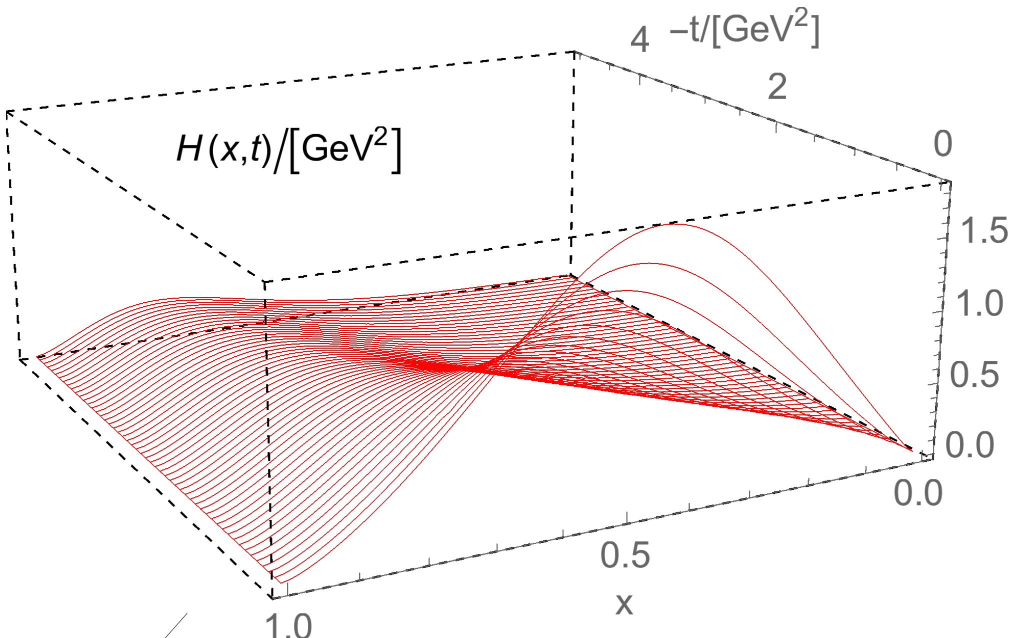
Red: '*Evolved*' form factor

Kaon preliminary results:

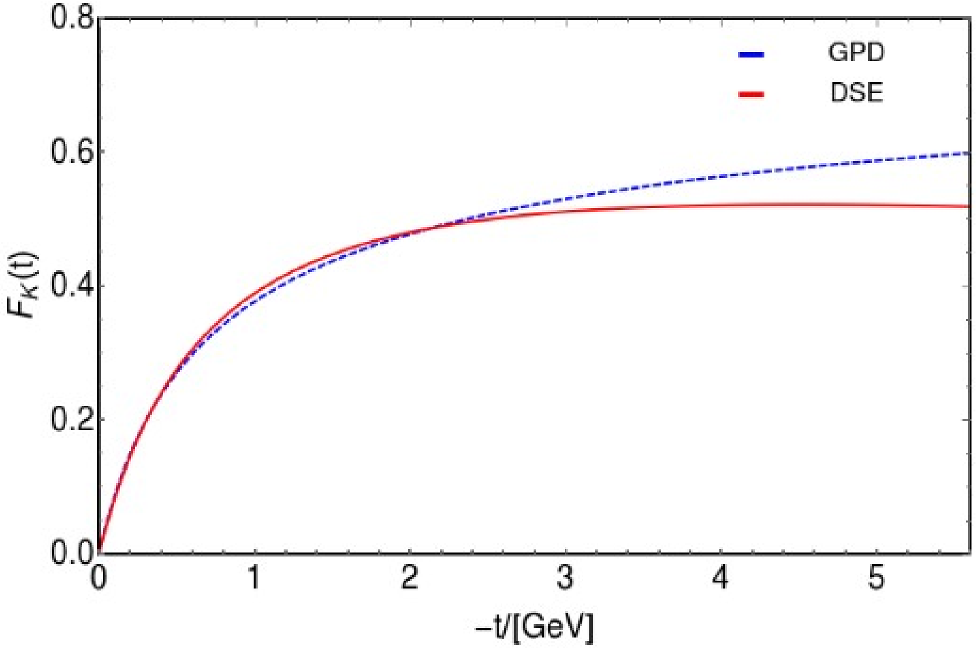
Identifying first the LFWF (see Khépani's talk)



one obtains the DGLAP GPD:



and the electromagnetic form factor.



A word about gravitational form factors:

First, polynomiality:

$$\int_{-1}^{1} dx\, x^m\ H(x,\xi,t) \ = \ \sum_{k=0}^{m+1} C_k^{(m)}(t)\ \xi^k$$

(Time reversal symmetry implies k even)

If one defines a function D such that:

$$\int_{-1}^{1} dx\ x^m D(x,t) = C_{m+1}^{m}(t)$$

where $D(x,t)=0 \quad \forall x \in [-\infty,-1)U(1,\infty]$

$$\int_{-1}^{1} dx\ x^m \left(H(x,\xi,t) - \mathrm{sign}(\xi) D\left(\frac{x}{\xi}\right)\right) \ = \ \sum_{k=0}^{m} C_k^{(m)}(t))\ \xi^k$$

$$\boxed{\int_{\Omega} d\beta d\alpha\ h_{PW}(\beta,\alpha;t)\delta(x-\beta-\alpha\xi) \ = \ \mathcal{R}\left[h_{\mathrm{PW}}\right]}$$

$$\frac{1}{|\xi|} D\left(\frac{x}{\xi},t\right) = \int_{\Omega} d\beta d\alpha\ \delta(\beta)D(\alpha,t)\delta(x-\beta-\alpha\xi) \ = \ \mathcal{R}\left[\delta D)\right]$$

PW D-term
(Pure ERBL contribution)

Specializing for the case m=1

$$\int_{-1}^{1} dx\ x\ H(x,\xi,t) \ = \ c_0^{(1)}(t) \ + \ \xi^2 \int_{-1}^{1} dz\ z\ D(z,t)$$

A word about gravitational form factors:

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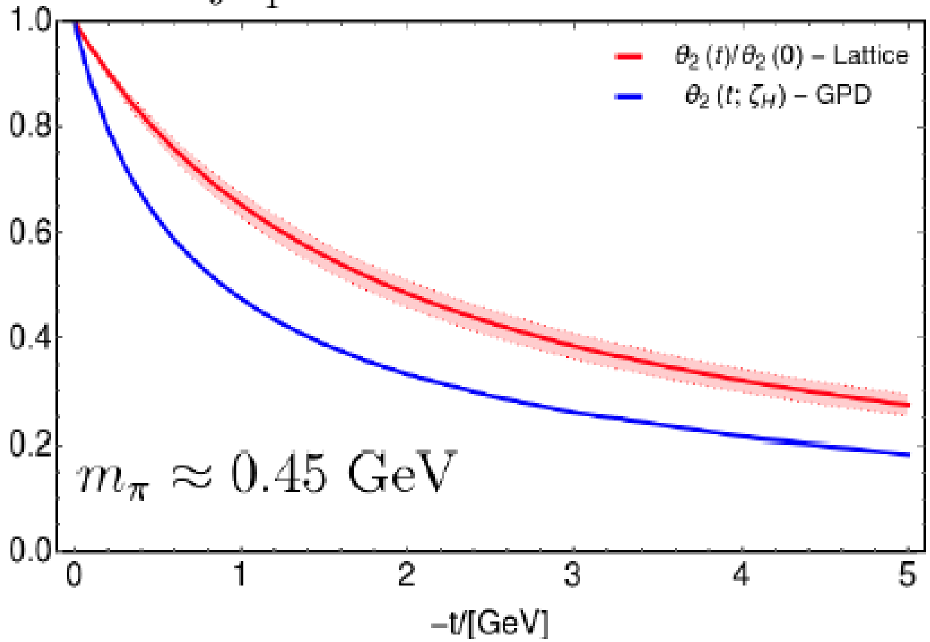
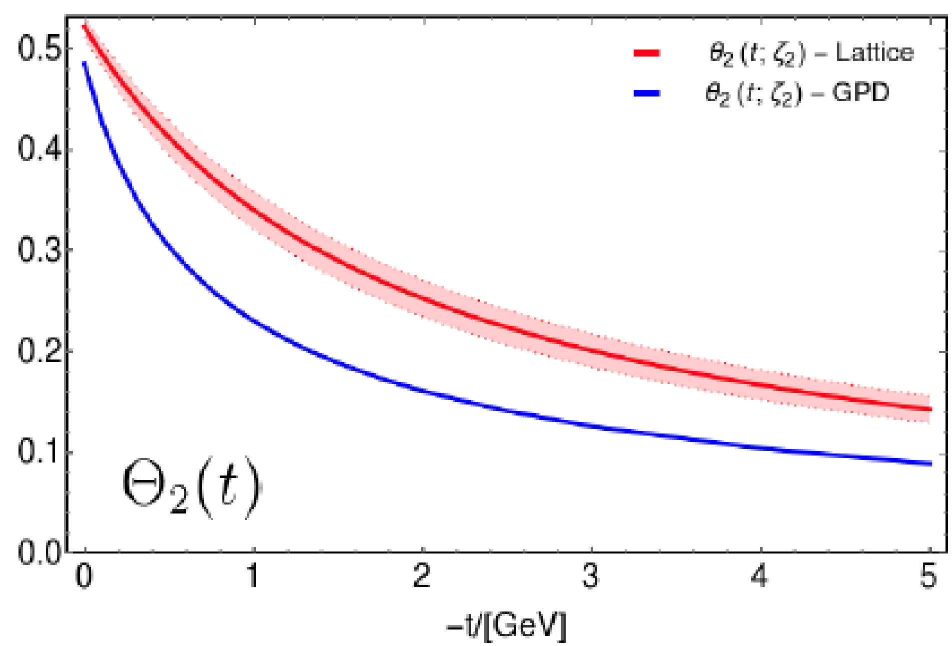
$$\boxed{\frac{1}{2}\theta_2(t)}$$

Isospin-symmetric limit

$$\theta_2(t)\ =\int_{-1}^{1} dx\ x\left(H^u_{\pi^+}(x,0,t)+H^d_{\pi^+}(-x,0,t)\right)$$





Lattice: (2007) Brömmel's dissertation. $\Theta_2(0)/2 =< x >= 0.261(5)$

GPD + Ding et al. $\Theta_2(0)/2 =< x >= 0.242(20)$

Latt.ice data: D. Brommel, Ph.D. thesis, U. Regensburg, Germany (2007), DESY-THESIS-2007-023

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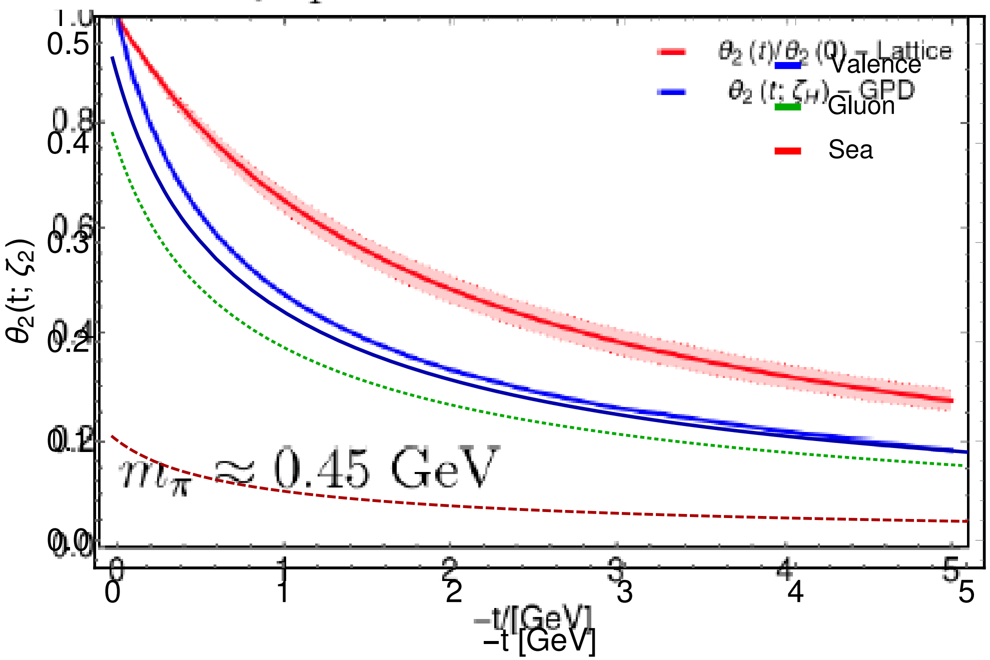

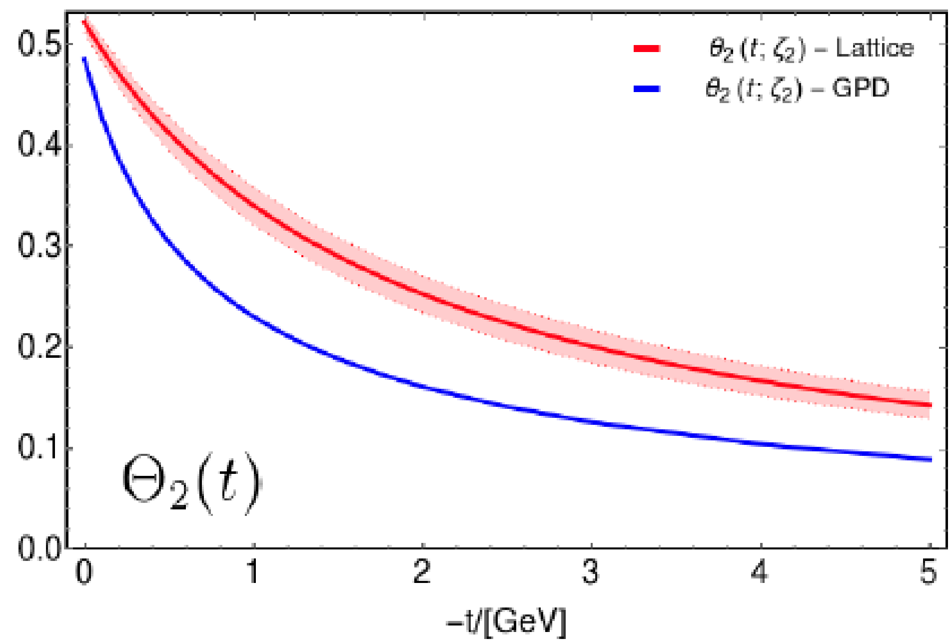

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3/2-Gegenbauer expansion

$$D(z,t) \;=\; (1-z^2) \sum_{k=1,\mathrm{odd}}^{\infty} d_k(t)\; C_k^{(3/2)}(x)$$

$$\theta_2(t) \;=\; \int_{-1}^{1} dx\; x\; \left(H^u_{\pi^+}(x,0,t) + H^d_{\pi^+}(-x,0,t)\right)$$

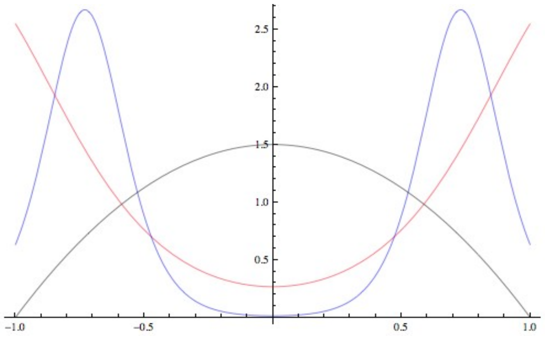
$$\boxed{\frac{4}{5}\; d_1(t)}$$

Only the first coefficient is needed!

LFWF + overlap approach cannot give access to the second gravitational moment. The Radon transform inversion of the DGLAP GPD cannot either (as it is nothing but a D-term contribution).

A possible way-out is considering unsubtracted t-channel dispersion relations to provide with a representation of the D-term form factor
[See Pasquini et al., PLB739(2014)133, precisely determining $d_1(t)$ for a nucleon case]

Conclusions



Owing to a sensible parametrisation of the BSA grounded on the so-called Nakanishi representation, one is left with a flexible algebraic model for the LFWF in terms of a spectral density.

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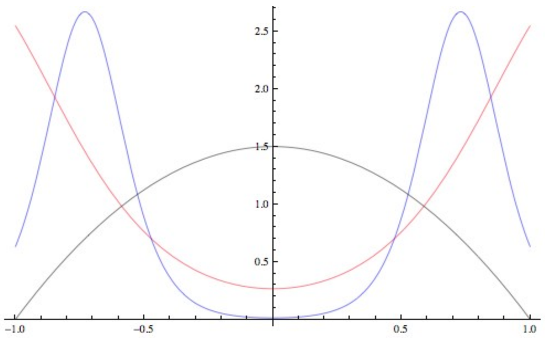

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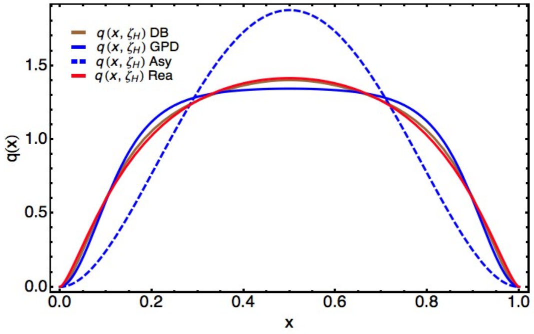

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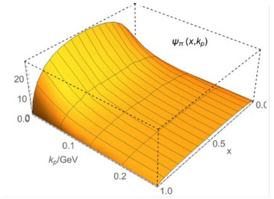

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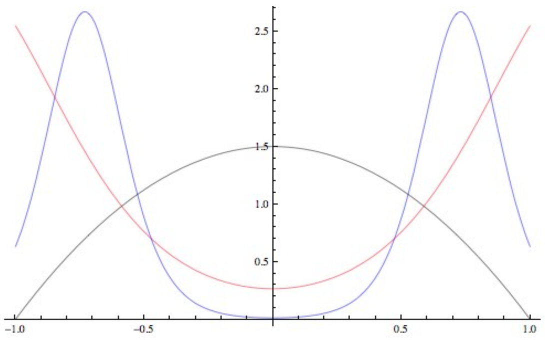

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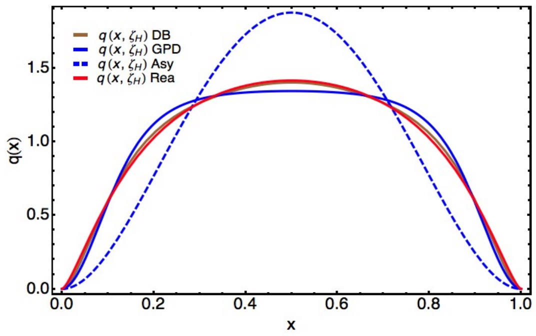

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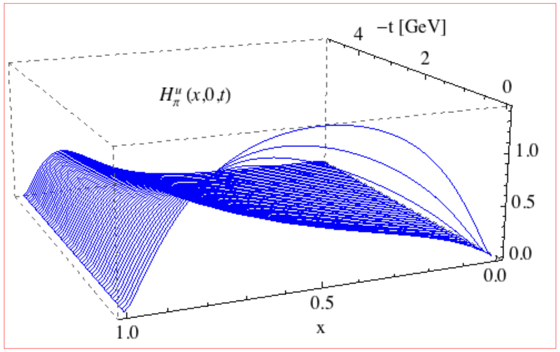

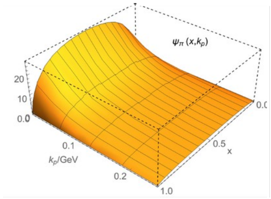

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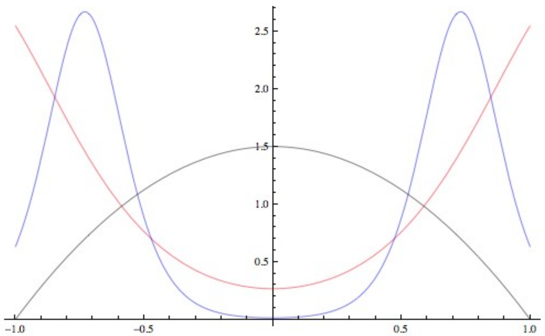

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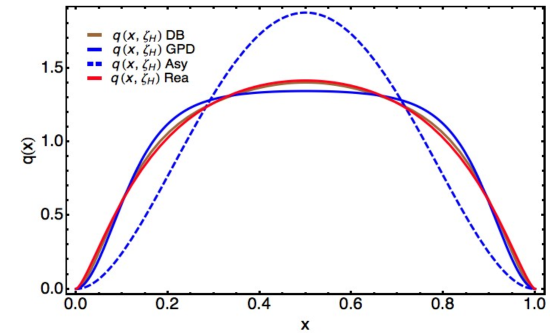

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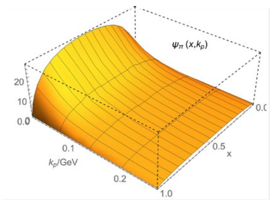

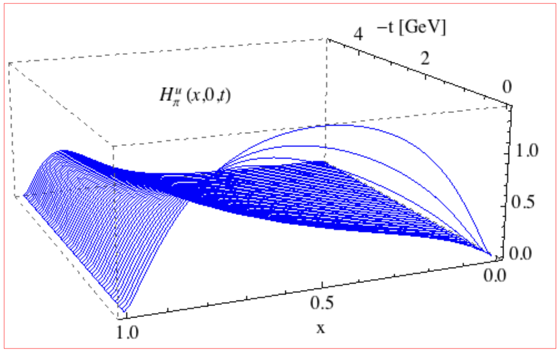

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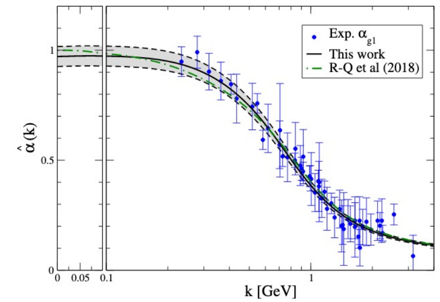
A recently proposed PI effective charge can be used to make the DGLAP GPD evolve from the hadronic scale (where quasi-particle DSE's solutions are the correct degrees-of-freedom) up to any other relevant scale.



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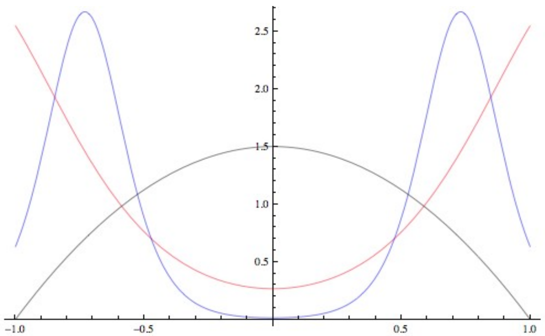

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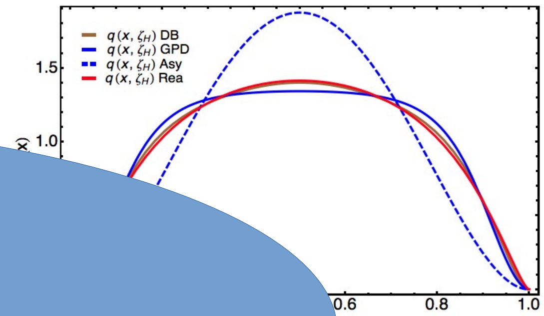
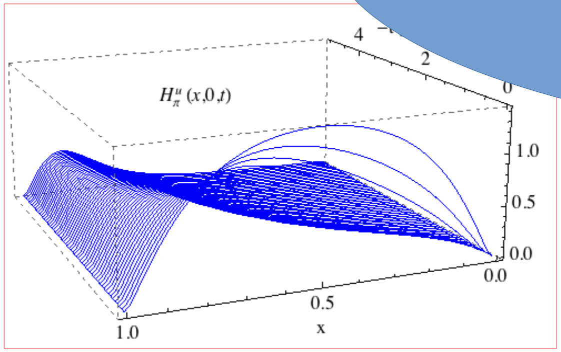
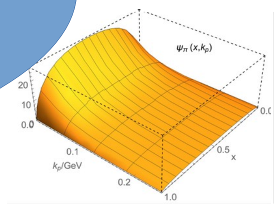


A direct calculation of the PDF f
Bethe-Salpeter equ
kinematical limit)
spectral density

Thank you!!





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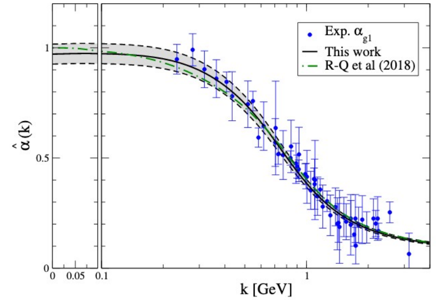


Backslides

PI-effective charge from lattice data with Nf=3 flavors at the physical point

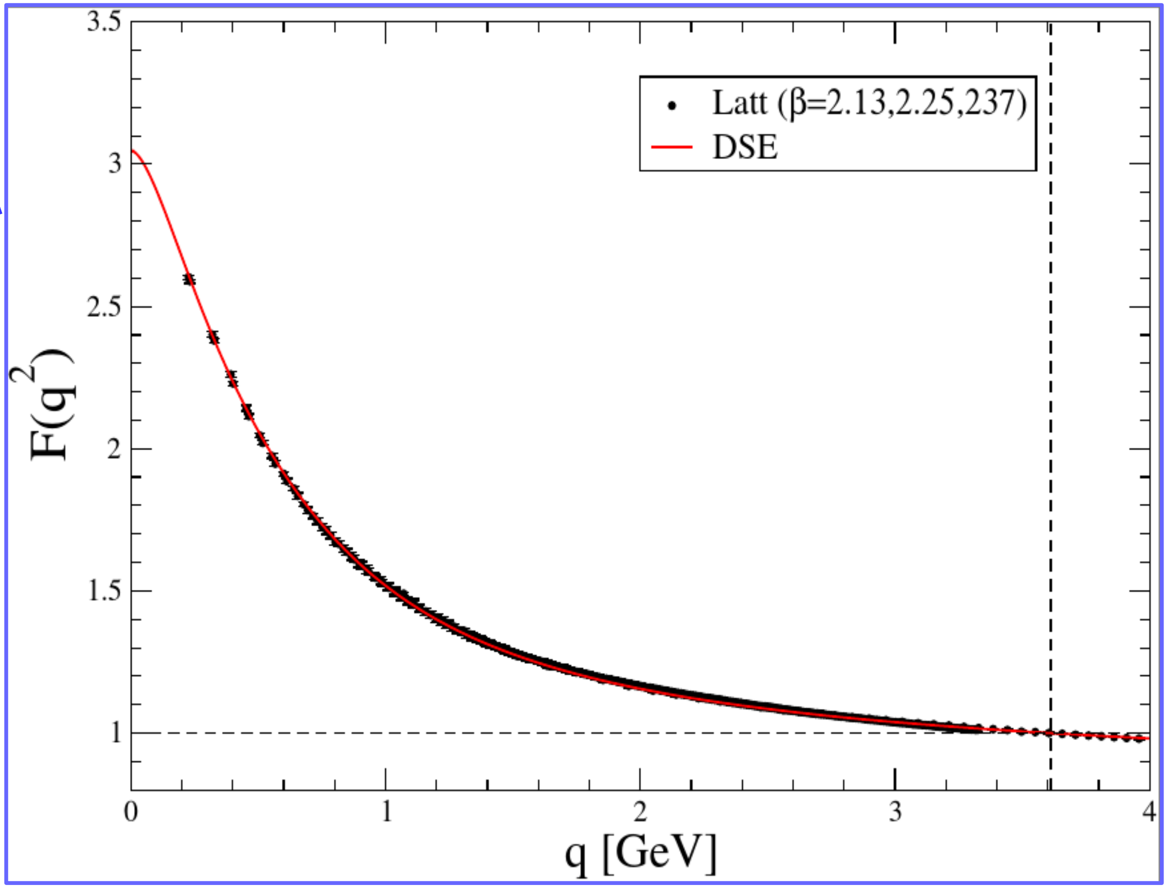
Preliminary results:

$$\widehat{\alpha}_{\mathrm{PI}}(q^2) = \frac{\widehat{d}(q^2)}{\mathcal{D}(q^2)} \simeq \frac{\alpha_T(q^2)}{q^2\left[1 - L(q^2,\zeta^2)F(q^2,\zeta^2)\right]^2} \frac{m_0^2 \Delta_F(0,\zeta^2)}{\Delta_F(q^2,\zeta^2)}$$

$$= \boxed{\alpha_T(\zeta^2)\, \frac{F(q^2,\zeta^2)}{\left[1 - L(q^2,\zeta^2)F(q^2,\zeta^2)\right]^2}\, \Delta_F(0,\zeta^2) m_0^2}$$

The IR running of the PI effective charge with momenta only depends on:

- The ghost dressing function



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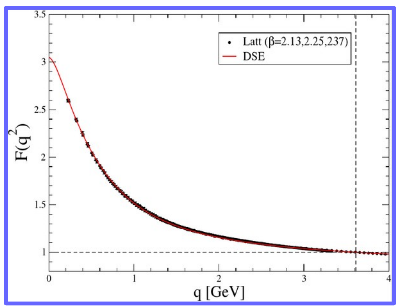

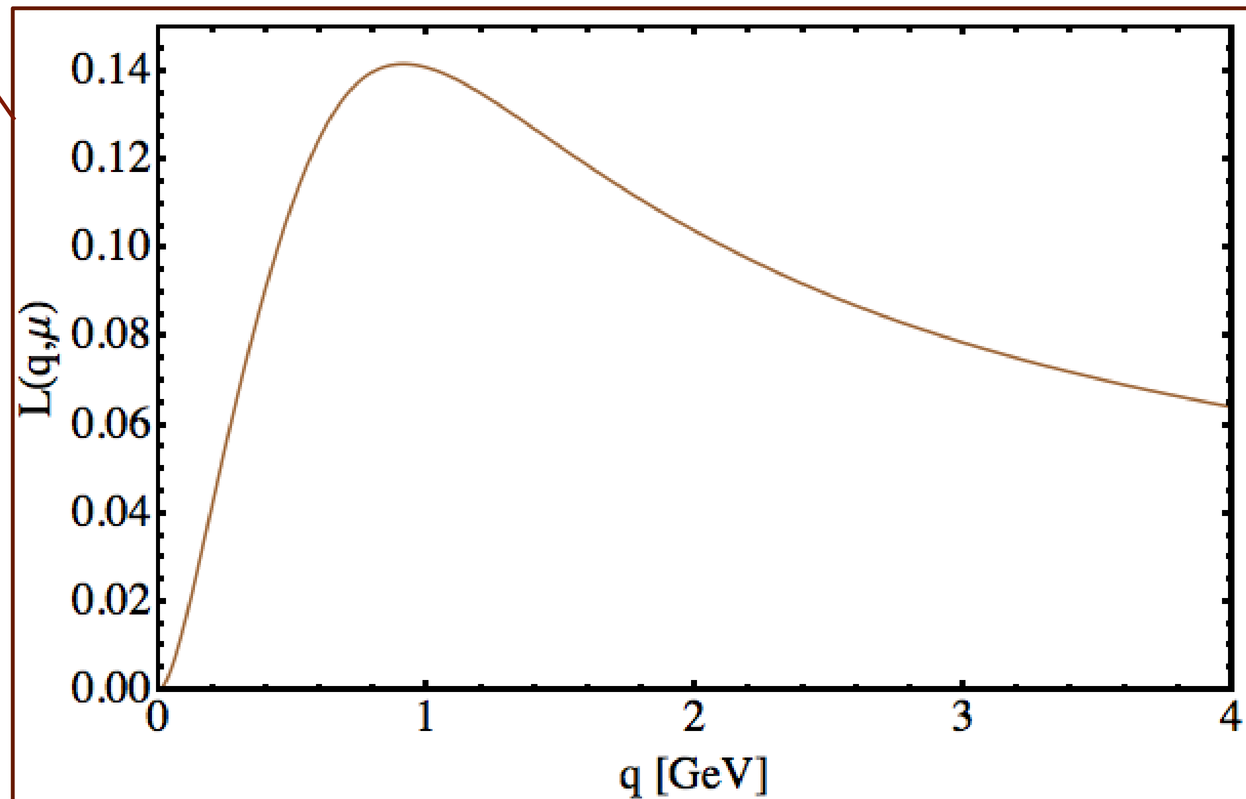

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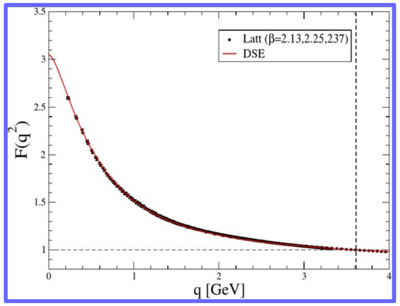
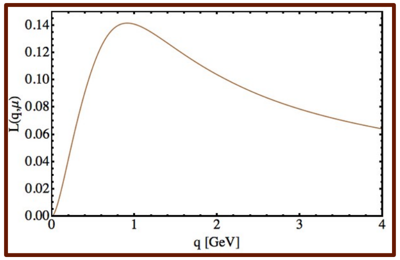
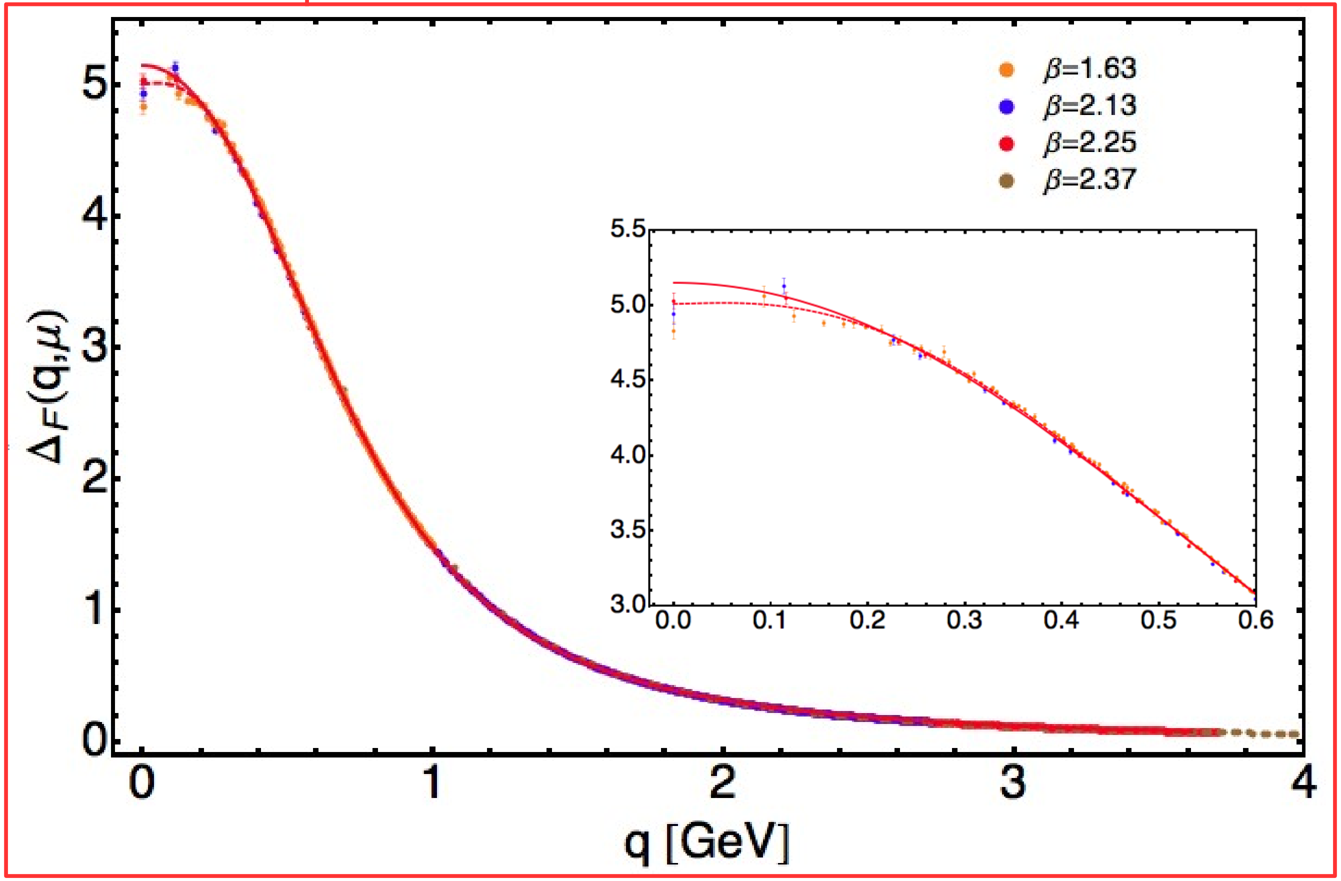
Its strength depends also on the saturation point at zero-momentum of the gluon propagator







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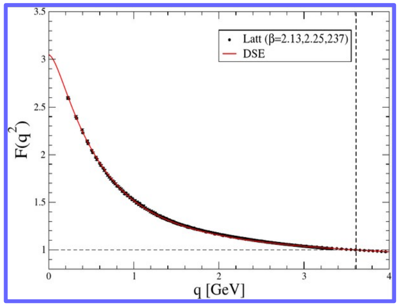

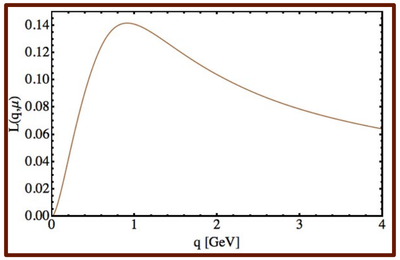

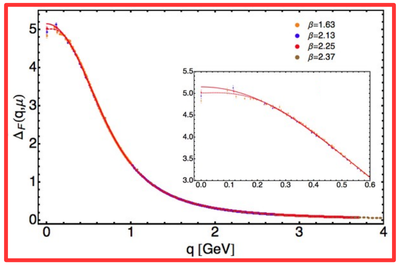

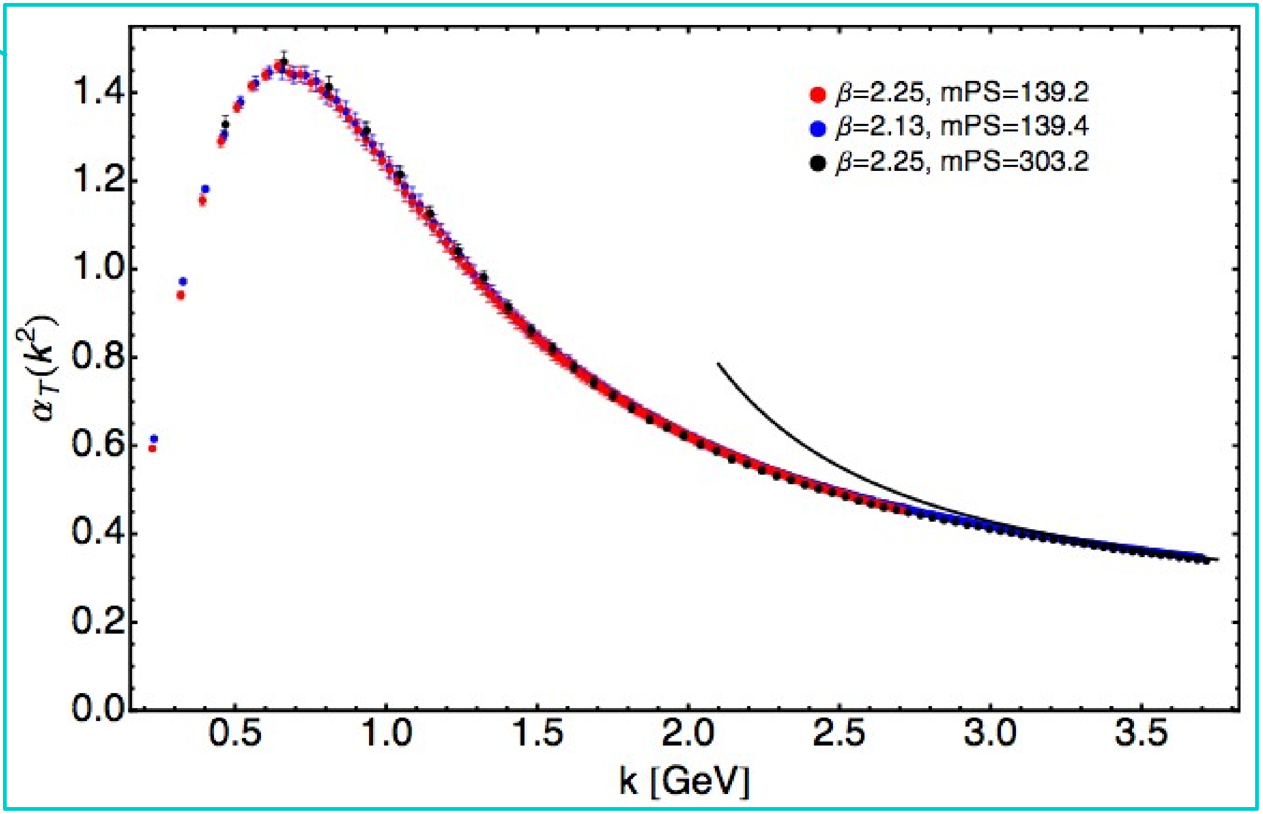

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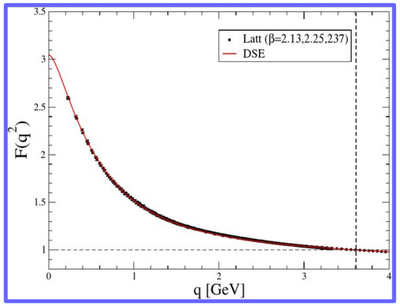
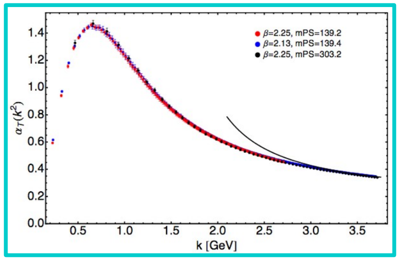
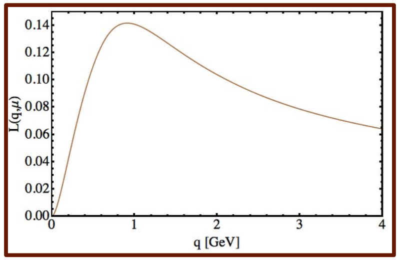
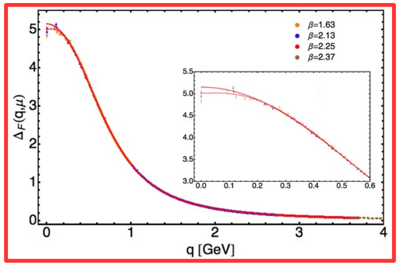


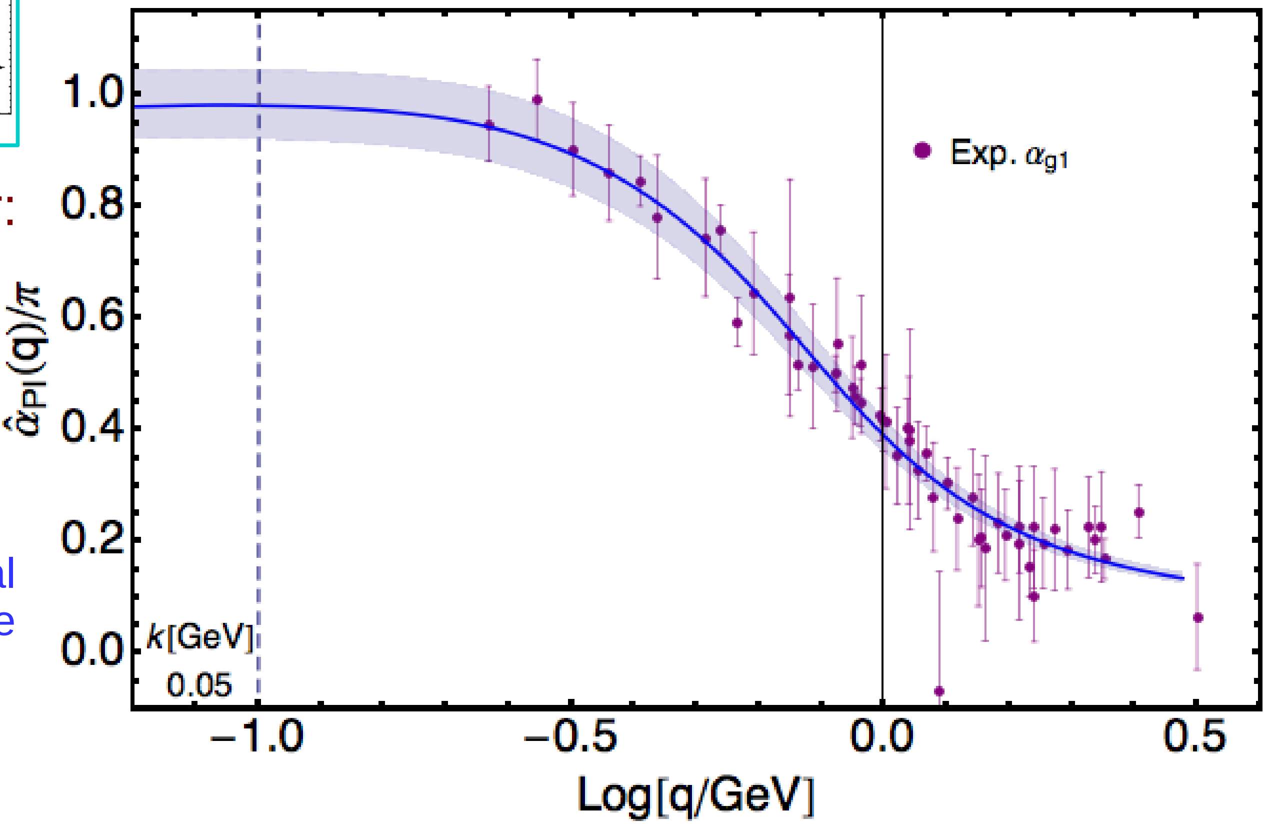






All put together:

Less uncertainties (that of the gluon mass is only left here) and still a better agreement with the world data for the experimental determination of the Bjorken sum-rule effective charge.



Pion realistic picture: DGLAP evolution

Let us also consider the singlet components:
(an almost textbook exercise)

$$\gamma_{0,AB}^{S,(m)} = -\int_0^1 dx\, x^m P_{0,AB}^S(x)$$

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} M_q^{(m)}(\zeta) \\ M_G^{(m)}(\zeta) \end{pmatrix} = -\frac{\alpha(\zeta^2)}{4\pi} \underbrace{\begin{pmatrix} \gamma_{0,qq}^{S,(m)} & 2n_f \gamma_{0,qG}^{S,(m)} \\ \gamma_{0,Gq}^{S,(m)} & \gamma_{0,GG}^{S,(m)} \end{pmatrix}}_{\Gamma_0^{S,(m)}} \begin{pmatrix} M_q^{(m)}(\zeta) \\ M_G^{(m)}(\zeta) \end{pmatrix}$$

$$P^{-1} \Gamma_0^{S,(m)} P = \underbrace{\begin{pmatrix} \lambda_+^{(m)} & 0 \\ 0 & \lambda_-^{(m)} \end{pmatrix}}_{\Gamma_D^{(m)}}$$

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Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom

Pion realistic picture: DGLAP evolution

Let us also consider the singlet components:
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$$P^{-1} \Gamma_0^{S,(m)} P = \begin{pmatrix} \lambda_+^{(m)} & 0 \\ 0 & \lambda_-^{(m)} \end{pmatrix}$$

Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom, can be evolved and shown to produce non-zero gluon and sea-quark components.

$$\boxed{P^{-1} \begin{pmatrix} M_q^{(m)}(\zeta) \\ M_G^{(m)}(\zeta) \end{pmatrix} = \exp\left(-\Gamma_D^{(m)} \int_{\ln \zeta_H^2}^{\ln \zeta^2} d \ln z^2 \, \frac{\alpha(z^2)}{4\pi} \right) P^{-1} \begin{pmatrix} M_q^{(m)}(\zeta_H) \\ 0 \end{pmatrix}}$$

Pion realistic picture: DGLAP evolution

Let us also consider the singlet components:
(an almost textbook exercise)

$$\gamma_{0,AB}^{S,(m)} = -\int_0^1 dx\, x^m P_{0,AB}^S(x)$$

$$\zeta^2 \frac{d}{d\zeta^2} P^{-1} \begin{pmatrix} M_q^{(m)}(\zeta) \\ M_G^{(m)}(\zeta) \end{pmatrix} = -\frac{\alpha(\zeta^2)}{4\pi} \Gamma_D^{(m)} P^{-1} \boxed{\begin{pmatrix} M_q^{(m)}(\zeta_H) \\ 0 \end{pmatrix}}$$

$$\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

$$P^{-1}\,\Gamma_0^{S,(m)}\,P = \begin{pmatrix} \lambda_+^{(m)} & 0 \\ 0 & \lambda_-^{(m)} \end{pmatrix}$$

Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom, can be evolved and shown to produce non-zero gluon and sea-quark components.

$$\boxed{\begin{aligned} M_q^{(m)}(\zeta) &= M_q^{(m)}(\zeta_H) \\ &\times \left[\frac{w_{11}w_{22}}{\text{Det}\,(P)} \exp\left(-\frac{\lambda_+^{(m)}}{4\pi} \int_{\ln \zeta_H^2}^{\ln \zeta^2} dt\, \alpha(t) \right) - \frac{w_{12}w_{21}}{\text{Det}\,(P)} \exp\left(-\frac{\lambda_-^{(m)}}{4\pi} \int_{\ln \zeta_H^2}^{\ln \zeta^2} dt\, \alpha(t) \right) \right] \\ M_G^{(m)}(\zeta) &= M_q^{(m)}(\zeta_H) \frac{w_{22}w_{21}}{\text{Det}\,(P)} \\ &\times \left[\exp\left(-\frac{\lambda_+^{(m)}}{4\pi} \int_{\ln \zeta_H^2}^{\ln \zeta^2} dt\, \alpha(t) \right) - \exp\left(-\frac{\lambda_-^{(m)}}{4\pi} \int_{\ln \zeta_H^2}^{\ln \zeta^2} dt\, \alpha(t) \right) \right] \end{aligned}}$$

Pion realistic picture: DGLAP evolution

Let us also consider the singlet components:
(an almost textbook exercise)

$$\gamma_{0,AB}^{S,(m)} = -\int_0^1 dx\, x^m P_{0,AB}^S(x)$$

$$\zeta^2 \frac{d}{d\zeta^2} P^{-1} \begin{pmatrix} M_q^{(m)}(\zeta) \\ M_G^{(m)}(\zeta) \end{pmatrix} = -\frac{\alpha(\zeta^2)}{4\pi} \Gamma_D^{(m)} P^{-1} \boxed{\begin{pmatrix} M_q^{(m)}(\zeta_H) \\ 0 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 3/4 \\ -1 & 1 \end{pmatrix}$$

$$P^{-1}\, \Gamma_0^{S,(1)}\, P = \begin{pmatrix} 56/9 & 0 \\ 0 & 0 \end{pmatrix}$$

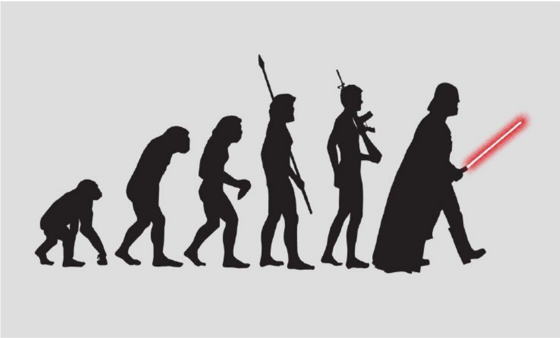
Case m=1

Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom, can be evolved and shown to produce non-zero gluon and sea-quark components.

$$M_q^{(1)}(\zeta) = M_q^{(1)}(\zeta_H) \left[\frac{3}{7} + \frac{4}{7} \exp\left(-\frac{56}{36\pi} \int_{\ln \zeta_H^2}^{\ln \zeta^2} dt\, \alpha(t) \right) \right]$$

$$M_G^{(1)}(\zeta) = \frac{4}{7} M_q^{(1)}(\zeta_H) \left[1 - \exp\left(-\frac{56}{36\pi} \int_{\ln \zeta_H^2}^{\ln \zeta^2} dt\, \alpha(t) \right) \right]$$

PDA and LFWF evolution



Standard PDA evolution:

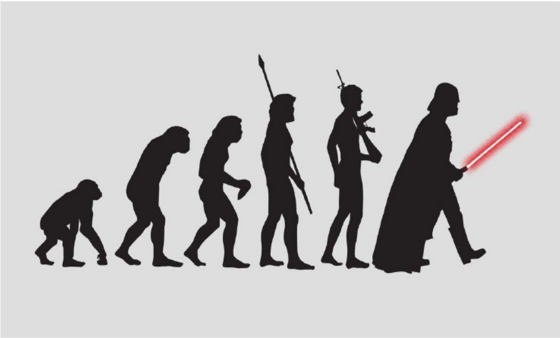
- We project **PDA** onto a 3/2-Gegenbauer polynomial basis. Such that it **evolves**, from an initial scale ζ_0 to a final scale ζ, **according to** the corresponding **ERBL equations**:

$$\phi(x;\zeta) = 6x(1-x)\left[1 + \sum_{n=1} a_n(\zeta) C_n^{3/2}(2x-1)\right] ,$$

$$a_n(\zeta) = a_n(\zeta_0)\left[\frac{\alpha(\zeta^2)}{\alpha(\zeta_0^2)}\right]^{\gamma_0^n/\beta_0} , \; \gamma_0^n = -\frac{4}{3}\left[3 + \frac{2}{(n+1)(n+2)} - 4\sum_{k=1}^{n+1}\frac{1}{k}\right] .$$

- Thus, any PDA at hadronic scale evolves logarithmically towards its conformal distribution, $\phi(x)=6x(1-x)$.
    - ➢ Quark mass and flavor become irrelevant. Broad PDA becomes narrower, skewed PDA becomes symmetric.

PDA and LFWF evolution



LFWF evolution: $\boxed{\phi(x) = \frac{1}{16\pi^3}\int d^2\vec{k}_\perp \psi^{\uparrow\downarrow}(x, k_\perp^2)}$

- We look for a way to evolve the LFWF.
- First, let's assume that the LFWF admits a similar Gegenbauer expansion. That is:

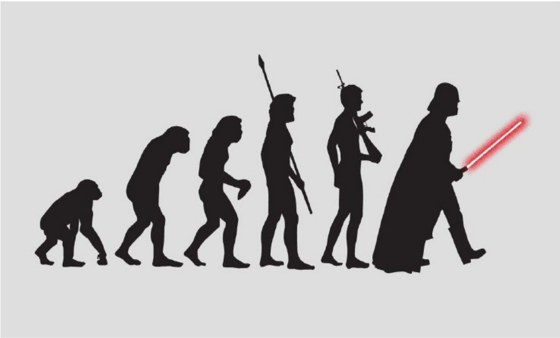
$$\psi(x, k_\perp^2;\zeta) = 6x(1-x)\left[\sum_{n=0} b_n(k_\perp^2;\zeta)\, C_n^{3/2}(2x-1)\right] ,$$

$$a_n(\zeta) \;=\; \frac{1}{16\pi^3}\int d^2\vec{k}_\perp\, b_n(k_\perp^2;\zeta)\ (\text{for } n \geq 1)\ ,\ \frac{1}{16\pi^3}\int d^2\vec{k}_\perp\, b_0(k_\perp^2;\zeta) = 1\ .$$

- 1-loop ERBL evolution of $a_n(\zeta)$ implies:

$$\frac{1}{a_n(\zeta)}\frac{d}{d\ln\zeta^2}a_n(\zeta) \;=\; \frac{\int d^2\vec{k}_\perp\, \frac{d}{d\ln\zeta^2}b_n(k_\perp^2;\zeta)}{\int d^2\vec{k}_\perp\, b_n(k_\perp^2;\zeta)} ,$$

PDA and LFWF evolution



LFWF evolution:

$$\phi(x) = \frac{1}{16\pi^3}\int d^2\vec{k}_\perp \psi^{\uparrow\downarrow}(x,k_\perp^2)$$

- Now, if we take a factorization assumtion, we arrive at:

$$\frac{b_n(k_\perp^2;\zeta)}{b_n(k_\perp^2;\zeta_0)} = \frac{\widehat{b}_n(\zeta)}{\widehat{b}_n(\zeta_0)} = \left[\frac{\alpha(\zeta^2)}{\alpha(\zeta_0^2)}\right]^{\gamma_0^n/\beta_0}, \quad b_n(k_\perp^2;\zeta) \equiv \widehat{b}_n(\zeta)\chi_n(k_\perp^2) .$$

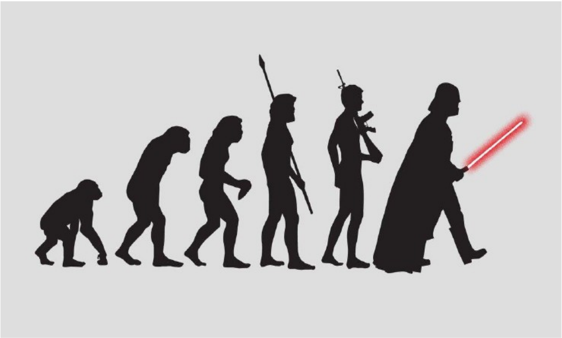
- Suplemented by the condition $\chi_n(k_\perp^2) \equiv \chi(k_\perp^2)$, one gets $\widehat{b}_n(\zeta) \equiv a_n(\zeta)$.
- Such that, the followiong factorised form is obtained:

$$\boxed{\psi(x,k_\perp^2;\zeta) \;\equiv\; \phi(x;\zeta)\,\chi(k_\perp^2)}$$

LFWF Evolves like PDA

- Which is far from being a general result, but an useful approximation instead.

PDA and LFWF evolution

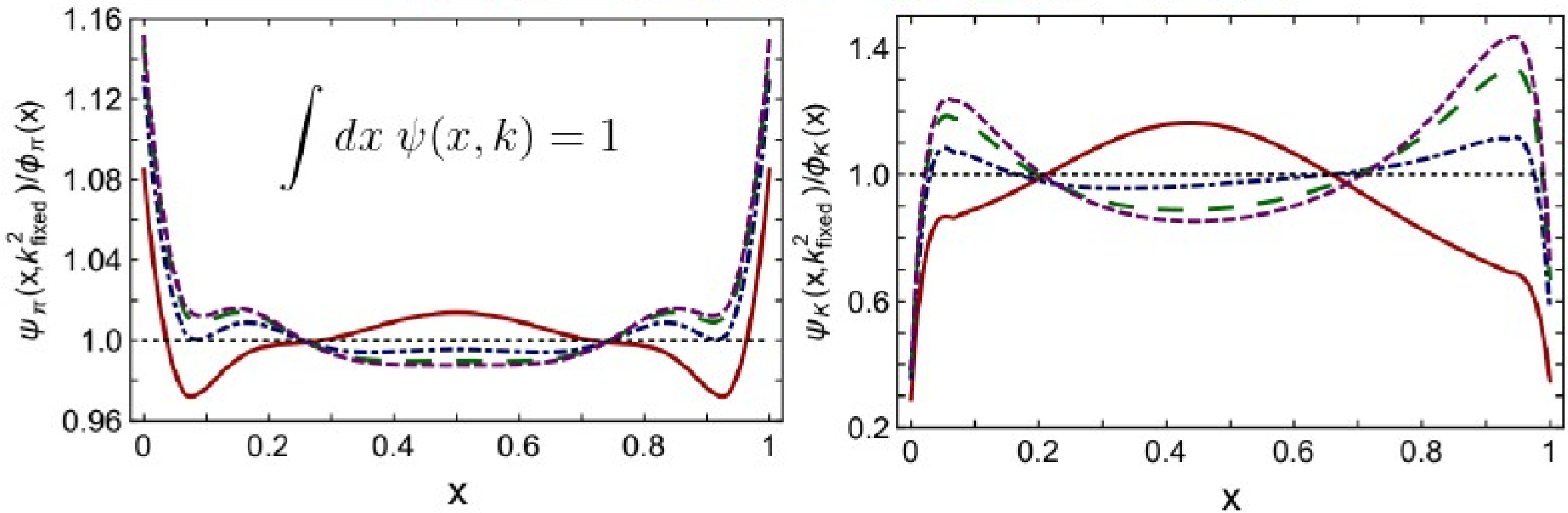


Testing the factorization ansatz:

$$\boxed{\psi(x, k_\perp^2; \zeta) \;\equiv\; \phi(x;\zeta)\,\chi(k_\perp^2)}$$

- A first validation of the factorized ansätz is addressed in **Phys.Rev. D97 (2018) no.9, 094014**:

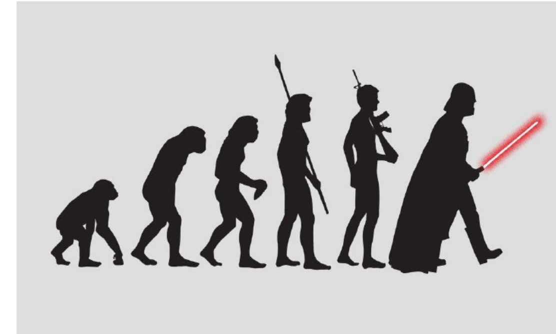
k^2=0, k^2=0.2 GeV, k^2=0.8 GeV, k^2=3.2 GeV

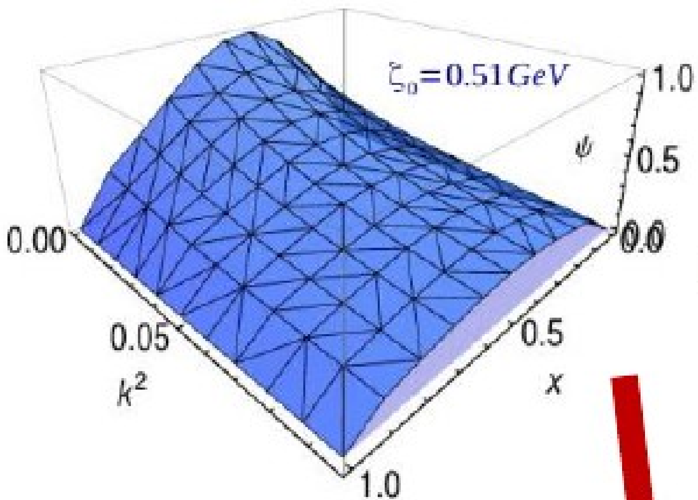


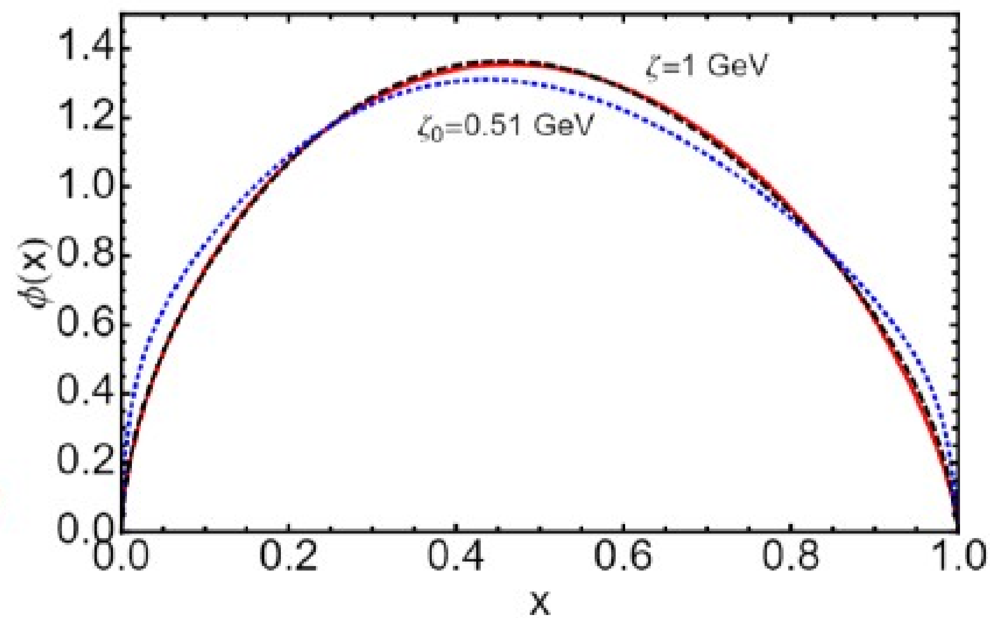
- If the factorized ansatz is a good approximation, then the plotted ratio must be 1. For the pion, it slightly deviates from 1; for the kaon, the deviation is much larger.

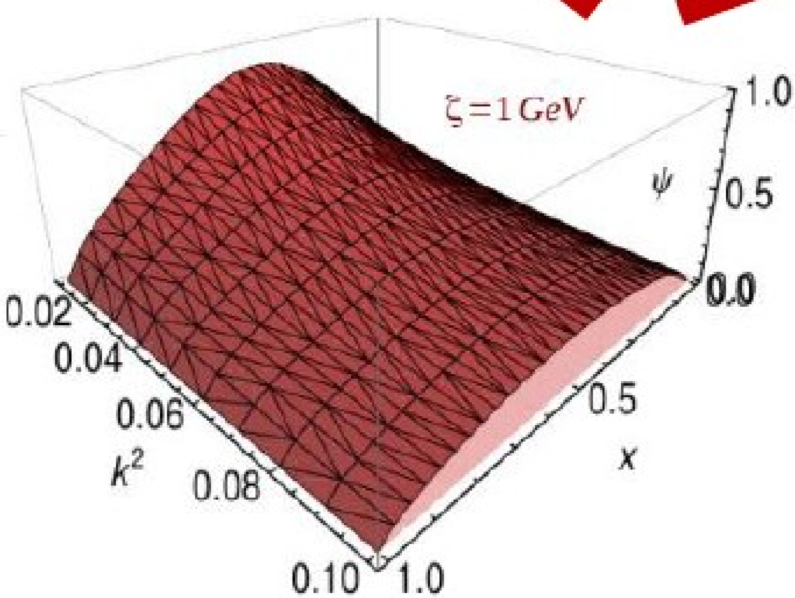
PDA and LFWF evolution

Testing the factorization ansatz:









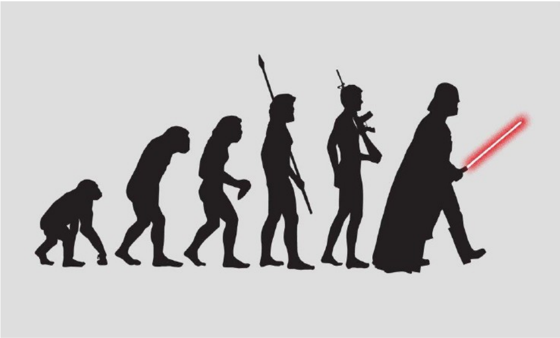
1) Compute LFWF and ERBL running of PDA

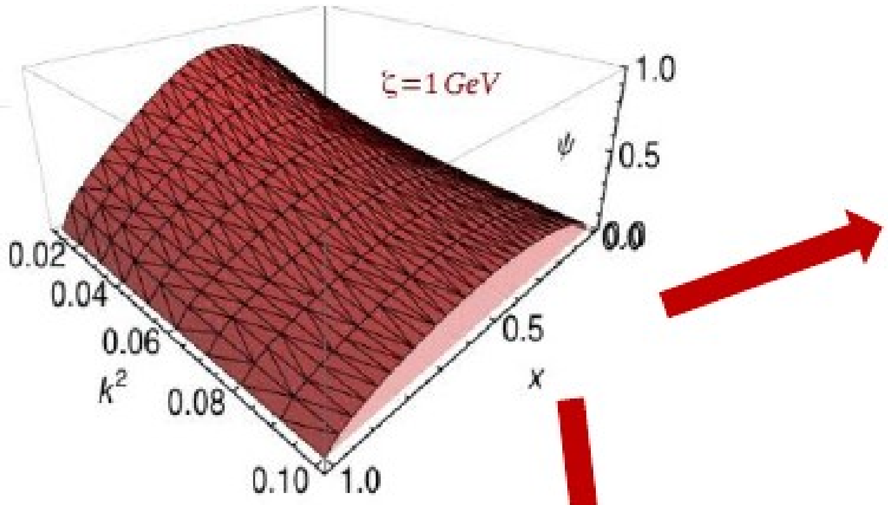
2) ERBL running of LFWF and compute PDA

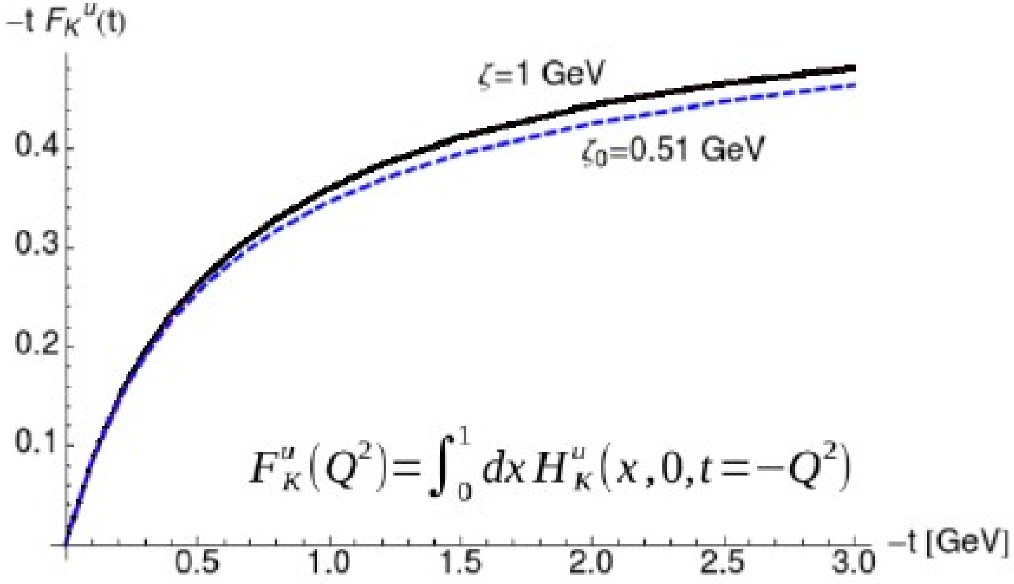
Notably, **1)** and **2)** are **equivalent**. Factorization assumption and evolution seem reasonable.

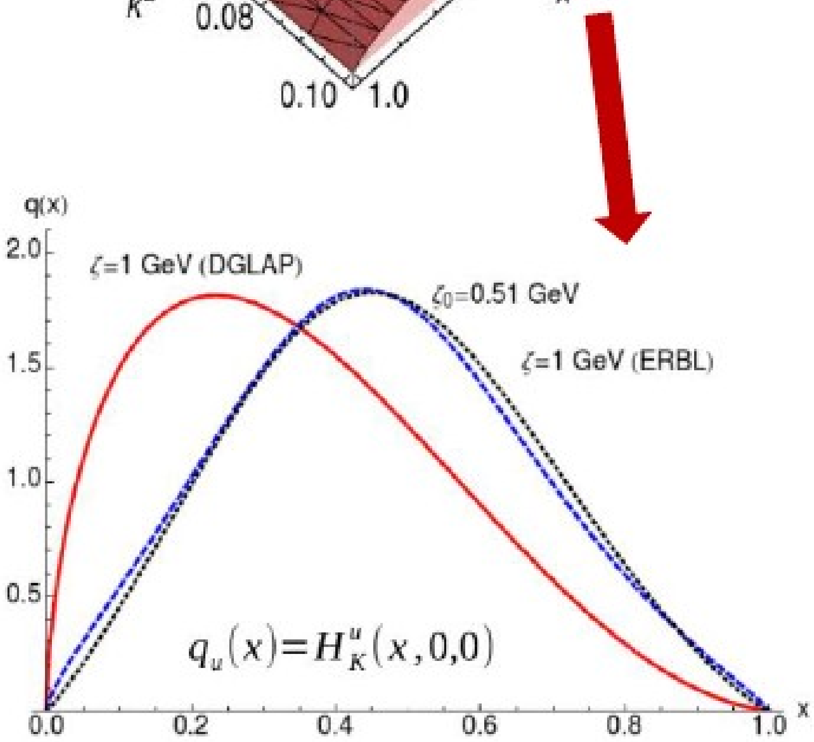
PDA and LFWF evolution

How ERBL and DGLAP evolutions make contact:









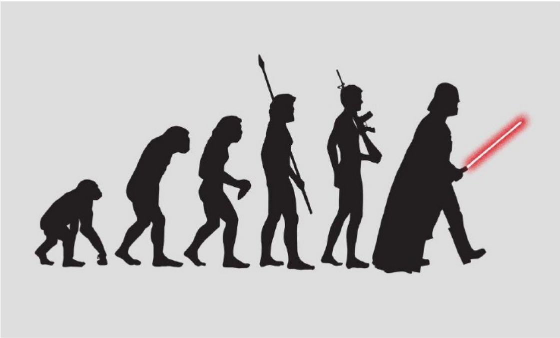
1) Obtained from ERBL evolution of LFWF

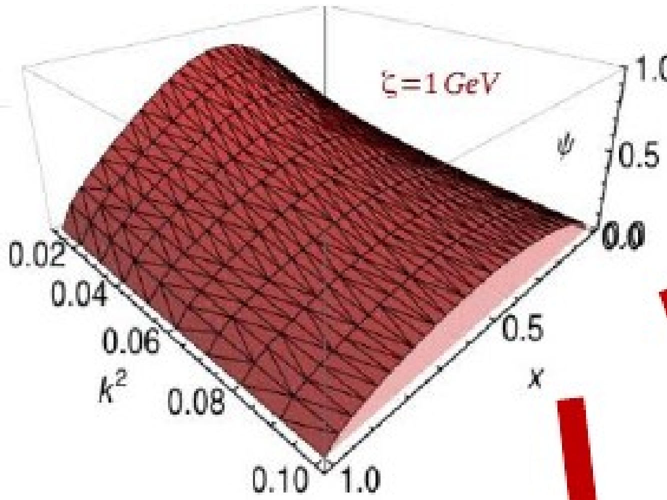
2) Obtained from DGLAP evolution of GPD

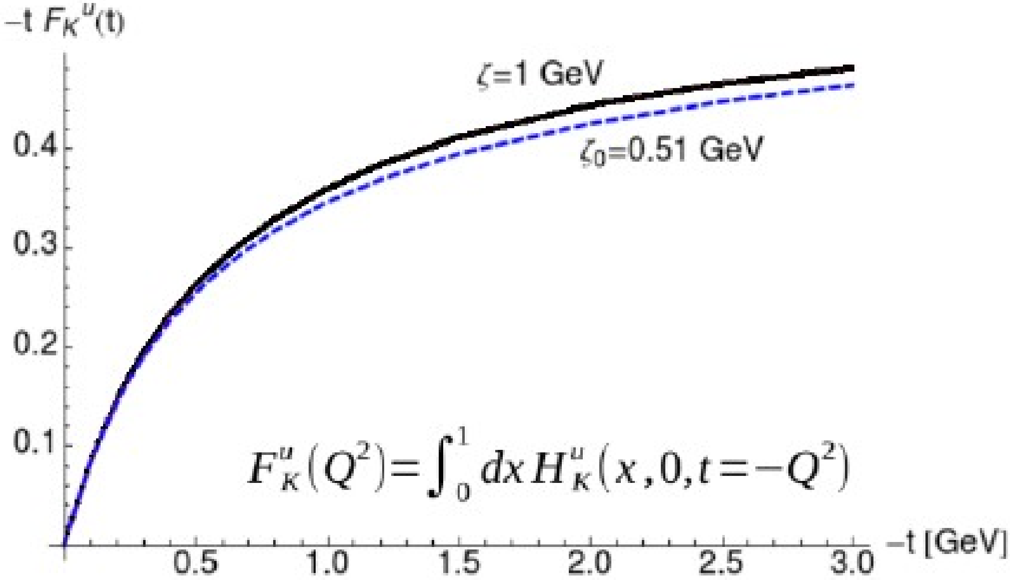
Clearly, **1)** and **2)** are **not equivalent**.

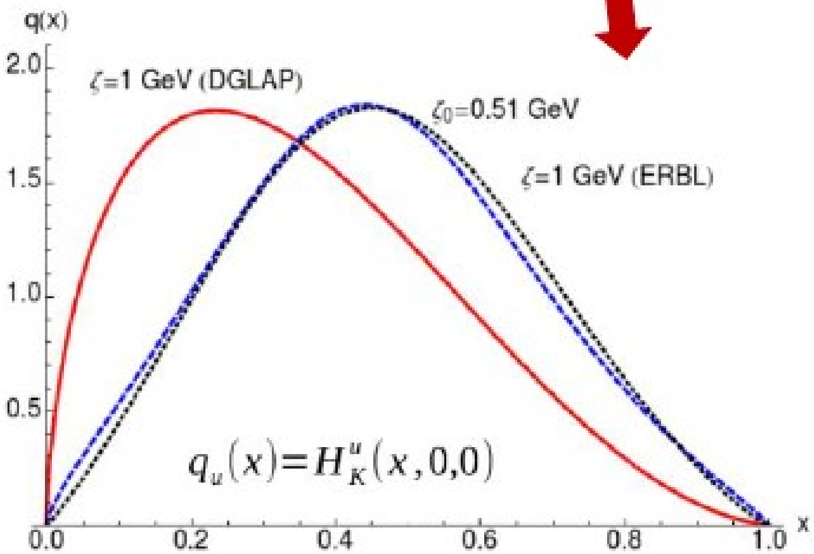
PDA and LFWF evolution

How ERBL and DGLAP evolutions make contact:









1) Obtained from ERBL evolution of LFWF

2) Obtained from DGLAP evolution of GPD

Clearly, **1)** and **2)** are **not equivalent**.

Sea-quark and gluon content incorporated to the parton distribution by DGLAP are obviously not present in the valence-quark PDF from LFWFs!!!