A conformally invariant Yang-Mills energy and equation on 6-manifolds

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Links: e.g.
G.+ Peterson+ Sleigh, arXiv:2107.08515
Branson + G. Commun. Contemp. Math. 9 (2007)
Branson +G. (2005), Comm. Partial Differential Equations 30 (2005),
Bailey+Eastwood+G. *Rocky Mtn* 1994

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Plan

- The conformal Laplacian aka Yamabe operator
- The GJMS operators and Q-curvature
- "Long" operators on differential forms and Q-operators
- Bach Gravity
- Conformal tractors
- The Yang-Mills equations
- A Q-like object for gauge connections and higher Yang-Mills
- Links to the obstruction tensor

Toward a link to Lionel - some background

On pseudo-Riemannian n-manifolds the conformal wave operator/conformal Laplacian

$$Y^g u := (\Delta^g + \frac{n-2}{4(n-1)} \mathrm{Sc}^g) u \qquad \text{here } \mathrm{Sc}^g \text{ is the scalar curvature}$$

is **conformally covariant**: If we replace the g with $\widehat{g} = e^{2\omega} g$, then

$$Y^{\widehat{g}} e^{(1-\frac{n}{2})\omega} u = e^{(-1-\frac{n}{2})\omega} Y^g u \qquad \omega \in C^\infty(M).$$

Hence (put $e^{(1-\frac{n}{2})\omega} u = 1$) the **Yamabe equation**

$$\frac{n-2}{4(n-1)} \mathrm{Sc}^{\widehat{g}}\, u^{\frac{n+2}{n-2}} = (\Delta^g + \frac{n-2}{4(n-1)} \mathrm{Sc}^g) u.$$

But in critical dim $n = 2$: $\Delta^{\widehat{g}} u = e^{-2\omega} \Delta^g u$ and (exercise):

$$\mathrm{Sc}^{\widehat{g}} = e^{-2\omega} (\mathrm{Sc}^{\widehat{g}} + 2\Delta\omega).$$

So Gauss curvature prescription is qualitatively different than Scalar curvature prescription in higher dim. and $\int_M \mathrm{Sc}$ is conformally invariant on closed M – as required by Gauss Bonnet!

There is a generalization of this whole picture to higher order/dim! Paneitz produced a conformally covariant $P_4 = \Delta^2 + lots$ and then Graham-Jenne-Mason-Sparling constructed operators

$$P_{2m} = \Delta^m + lots \qquad m \in \mathbb{Z}_+, \quad 2m - n \notin 2\mathbb{Z}_+$$

s.t.

$$P_{2m}^{\widehat{g}} \circ e^{(m-\frac{n}{2})\omega} = e^{-(m+\frac{n}{2})\omega} P_{2m}^{g}.$$

Branson noticed these have the form (FSA by Graham-Zworksi)

$$P_{2m} = \delta S_{2m} d + \frac{n-2m}{2} Q_{2m},$$

thus a generalisation of the Yamabe equation governing $Q_{2m}^{\widehat{g}}$ and, in the critical dim $n = 2m$:

$$\boxed{Q^{\widehat{g}} = e^{-n\omega}(Q^g + P_n\omega).}$$

So Q-curvature prescription is similar to Gauss, and $\int_M Q$ is conformally invariant on closed M of even dimension.

Long and Q-operators on forms

In fact there is a much bigger picture still!! On k-forms $\mathcal{E}^k$ $(0 \le k \le n/2)$ there are similar operators

$$L_{2m} : \mathcal{E}^k[w] \to \mathcal{E}^k[u] \qquad L_{2m} = -2u(\delta d)^m - 2w(d\delta)^m + lots$$

where $w = m + k - \frac{n}{2}$, and δ is formal adjoint of d (really ∇).
Again $w = 0$ (true forms) is the **critical case** and
Thm : [Branson+G. 2007]

$$L_{2m+2} = \delta Q_{2m} d$$

is **conformally invariant and FSA** with

$$\boxed{Q^{\widehat{g}}_{2m} = e^{u\omega}(Q^{g}_{2m} + L_{2m} \circ \omega) \qquad \text{on closed forms.}} \quad \therefore \int_M \langle u_k, Qw_k \rangle \; ci$$

E.g. in dimension 6

$$\mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \xrightarrow{\to} \mathcal{E}^3 \xrightarrow{\to} \mathcal{E}_2 \xrightarrow{\delta} \mathcal{E}_1 \xrightarrow{\delta} \mathcal{E}_0$$

$L_2 = \delta d : \mathcal{E}^2 \to \mathcal{E}_2$

$L_4 = \delta Q_2 d : \mathcal{E}^1 \to \mathcal{E}_1$

$P_6 = \delta Q_4 d : \mathcal{E}^0 \to \mathcal{E}_0$

Recall Bach's conformal gravity

Riemannian geometry: $(M^n, g) \rightsquigarrow \nabla^g \rightsquigarrow R_{ab}{}^c{}_d$

The **Riemann curvature** $R_{ab}{}^c{}_d$ decomposes:

$$R_{abcd} = \underbrace{W_{abcd}}_{\text{trace-free}} + \underbrace{(P \odot g)_{abcd}}_{\text{trace part}}$$

Under **conformal rescaling** $g \mapsto \widehat{g} = e^{-2\omega} g$, $\omega \in C^\infty(M)$, the **Weyl curvature** $W_{ab}{}^c{}_d$ is conformally invariant:

$$W^{\widehat{g}} = W^g, \quad \text{whence} \quad |W|^2_{\widehat{g}} = e^{-4\omega}|W|^2_g,$$

In dim. $n = 4$, the measure transforms $d\mu_{\widehat{g}} = e^{4\omega} d\mu_g$, and so

$$\int_M |W|^2 d\mu_g \quad \text{is a conformally invariant energy/action}$$

with Euler-Lagrange equation, with respect to **metric variations**

$$B_{bc} := 2\nabla_a \nabla_d W^a{}_{bc}{}^d + Ric_{ad} W^a{}_{bc}{}^d = 0.$$

- The **Bach tensor B_{bc}** and Weyl W are the only linear leading term conformal invariants in dimension 4. Higher dimensions?

Conformal geometry and the tractor bundle/connection

There is in general no preferred conformally invariant connection on TM but there is on a rank 2 extension

$$\mathcal{T} = \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1], \qquad \mathcal{E}[1] := ((\Lambda^n TM)^2)^{\frac{1}{2n}}$$

that we call the **conformal tractor bundle** $\mathcal{T}$. Given $g \in \boldsymbol{c}$ this **connection** $\nabla^{\mathcal{T}}$ is given by

$$\mathcal{T} \stackrel{g}{=} \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1],$$

$$\nabla^{\mathcal{T}}_a(\sigma, \mu_b, \rho) = (\nabla_a \sigma - \mu_a,\ \nabla_a \mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho,\ \nabla_a \rho - P_{ab}\mu^b),$$

and $\nabla^{\mathcal{T}}$ preserves a conformally invariant **tractor metric** h

$$\mathcal{T} \ni V = (\sigma, \mu_b, \rho) \mapsto 2\sigma\rho + \mu_b\mu^b = h(V, V).$$

This gives the conceptual and practical approach to conformal geometry. E.g.

$$|W|^2 = |F^{\nabla^{\mathcal{T}}}|^2 \in \Gamma(\mathcal{E}[-4])$$

where $F^{\nabla^{\mathcal{T}}} = F_{ab}{}^C{}_D$ is the tractor curvature.

G-bundles and connections

We consider:

• $\pi : P \to M$ be a G-bundle, for some Lie group G.
• $\mathbb{V}$ finite-dimensional G-module $\to$ associated vec. bun.
$V = P \times_G \mathbb{V}$, over M.
• A a G-connection on the bundle P – same notation for associated linear connection on V giving covariant deriv. ∇^A on V.
• Write F or F^A for the curvature of A (or equival. of ∇^A).

The curvature satisfies the differential Bianchi identity

$$d^A F = 0,$$

where d^A is the exterior derivative coupled to A.

How can we construct (conformal) invariants that couple A to the underlying (conformal) geometry?

- Yang-Mills Lagrangian density: $\mathcal{L}(A) := \mathrm{Trace}(g^{ab}g^{cd}F_{ac} \circ F_{bd})$.

Under a conformal rescaling $g \mapsto \widehat{g} = e^{2\omega}g$, the Lagrangian transforms to $e^{-4\omega}\mathcal{L}(A)$. Again, in dimension 4, $d\mu^g \mapsto d\mu^{\widehat{g}} = e^{4\omega}d\mu^g$, so the YM action $\mathcal{S}(A)$ given by

$$\mathcal{S}(A) := \int_{M^4} \mathcal{L}(A)\ d\mu^g, \tag{1}$$

is conformally invariant. It follows at once that the functional gradient, with respect to variations of connection, is also conformally invariant. (This "uses" that the obvious bilinear form on $\mathrm{ad}(V)$ is non-degenerate.) The corresponding Euler-Lagrange equations are the Yang-Mills equations,

$$\delta^A F = 0, \tag{2}$$

and are thus conformally invariant as well. Here δ^A is the formal adjoint of d_A. **NB:** $A \mapsto \mathcal{S}(A)$ in (1) and $A \mapsto \delta^A F^A$ in (2) are conformal invariants of connections.

The conformal Yang-Mills system

We can now ask what are these invariants $\mathcal{S}(A)$ and $\delta_A F$ if we specialise to the case that A is the tractor connection – i.e. $\nabla^A := \nabla^{\mathcal{T}}$.

In fact we recover some old favourites.

Proposition:

$$\text{Tractor YM action} = \mathcal{S}(\nabla^{\mathcal{T}}) = \int_M |F^{\nabla^{\mathcal{T}}}|^2 \, d\mu_g = \int_M |W|^2 d\mu_g = \text{Bach action}$$

and

Theorem:[Merkulov '84, G+Somberg+Soucek '08] The dim. 4 tractor Yang-Mills equations are exactly the Bach flat condition:

$$\delta^{\nabla^{\mathcal{T}}} F^{\nabla^{\mathcal{T}}} = (0, 0, 0, B_{ac}).$$

NB: The Theorem is **not a trivial Corollary** of the Proposition. As the YM equations come from variation of the connection.

Higher dimensions ??

Q: Is it a dim 4 story only? Is there a picture like the above in higher dimensions?

There are candidates on the conformal side: The **Fefferman-Graham obstruction** tensor

$$\mathcal{B}_{ac} = \Delta^{n/2-2}\nabla^b\nabla^d W_{abcd} + \text{lower order}$$

generalises Bach to higher even dimensions n. It is the functional gradient **wrt metric variations** of

$$\int_M Q^g \, d\mu^g, \qquad \leftarrow \text{conf. invariant on closed } M$$

where Q^g is **Branson's Q-curvature** [Graham-Hirachi].

NB: Under conformal transformations $Q^{\widehat{g}} = e^{-n\omega}(Q^g + P\omega)$, where

$$P = \Delta^{n/2} + \text{lower order} \quad \text{is the order n GJMS conformal Lapacian opertor.}$$

Question: $\exists$ Higher Yang-Mills? Higher **conformal invariants** of connections?

The Q_2 operator

Now as mentioned above there are operators on closed differential forms that are analogues of Q-curvature:

Theorem [Branson+G CPDE] On $(M^n \text{ even}, g)$ n there are differential operators Q^g_ℓ of order $\ell = n - k$, for integers $0 \leq k \leq n/2 - 1$ s.t. **on closed k-forms**

$$e^{\ell\omega} Q^{\widehat{g}}_\ell = Q^g_\ell + \delta Q^g_{\ell-2} d \circ \omega.$$

E.g. $k = \frac{n}{2} - 1$

$$Q_2 u = d\delta u - 4P\#u + 2Ju.$$

where again P is the Schouten tensor and J its metric trace. Because of its low order Q_2 can be coupled to connections A:

$$Q^A_2 U = d_A \delta_A U - 4P\#U + 2JU. \qquad \text{where} \quad d^A U = 0.$$

Then under $g \mapsto \widehat{g} = e^{2\omega} g$: **Thm** [Branson+G]:

$$e^{2\omega} \widehat{Q}^A_2 U = Q^A_2 U + 2\delta_A d_A(\omega U).$$

where U is a $\otimes V$-valued $(n/2 - 1)$-form **that is d^A-closed.**

Dimension 6

Given a connection A then in dimension $n = 6$ its curvature F^A is d^A-closed $n/2 - 1 = 2$-form (by Bianchi id.). Thus :

$$Q^A := \langle F^A, Q_2^A F^A \rangle$$

transforms "like a Q-curvature". Under $g \mapsto \widehat{g} = e^{2\omega} g$:

$$Q^A_{\widehat{g}} = e^{-6\omega}(Q^A_g + \delta F \circ F \wedge d\omega)$$

where, as usual, most constants have been set to 1 and details are impressionistic. The measure transforms $d\mu_{\widehat{g}} = e^{6\omega} d\mu_g$ in dim. 6. Thus on closed manifolds:

Theorem (G.+ Peterson+Sleigh)

$$\mathcal{S}(A) := \int_M Q^A \, d\mu$$

is conformally invariant and gauge invariant in dimension 6.

Higher Yang-Mills for dimension 6

We claim that

$$\mathcal{S}(A) := \int_{M^6} Q^A \, d\mu$$

is a suitable higher YM action energy for dimension 6! Some evidence is its functional gradient wrt **variations of connection**:

$$\mathfrak{D}(A) := \delta_A Q_2^A F_A - [\delta_A F_A, F_A] = \delta_A d_A \delta_A F_A + \text{lower order.}$$

Which has "the right" leading order terms. By construction:

Theorem (GPS)

$\mathfrak{D}(A)$ *is conformally invariant and gauge invariant.*

This is also easily verified directly. As a nice test when we take A to be the tractor connection we have:

Theorem (GPS)

$\mathfrak{D}(\nabla^{\mathcal{T}}) = (0, 0, 0, \mathcal{B}_{ac})$, *where* $\mathcal{B}$ *is the Fefferman-Graham obstruction tensor.*

Application: Einstein equations

The Fefferman-Graham obstruction tensor is the **obstruction** to smoothly formally solving for the so called **Fefferman-Graham ambient metric** associated to an even dim. conformal manifold. E.g. For a conformal 6 manifold this associates, at least formally, a dimension 8 Ricci flat manifold. But if the original even dimensional manifold is conformally Einstein then an "all orders" ambient metric exists, by an easy explicit construction. Thus:

Theorem (FG): The Fefferman-Graham obstruction tensor $\mathcal{B}_{ac}$ vanishes on conformally Einstein manifolds.

So the **FG obstruction-flat** equations,

$$\mathcal{B}_{ac} = 0,$$

are a conformal weakening of the Einstein equations.

Then:

- For dimension 4 this is a Yang-Mills equation on the Tractor connection.
- For dimension 6 this is a higher Yang-Mills equation on the Tractor connection $\mathfrak{D}(\nabla^{\mathcal{T}}) = 0$.

Thank you for Linking in!

Thank you for Listening!

Abstract

A conformally invariant Yang-Mills energy and equation on 6-manifolds. The gauge field equations known as the Yang-Mills equations are extremely important in both mathematics and physics, and their conformal invariance in dimension 4 is a critical feature for many applications. In dimension 4, and when specialised to the Cartan/tractor connection, the Yang-Mills current recovers exactly the Bach tensor. This provides a nice link between the Yang-Mills equations and (for example) the conformally-Einstein condition.

We show that there is a simple and elegant route to higher order equations in dimension 6 that are analogous, and arise as the Euler-Lagrange equations of a conformally invariant action. The functional gradient of this action recovers the conformal Fefferman-Graham obstruction tensor when the gauge connection is taken to be the conformal Cartan (or tractor) connection – so there is a nice analogy to the dimension 4 case. As well as providing evidence that these provide equations provide a good