Einstein-Weyl Structures and Dispersionless Equations

Twistor Theory and Beyond

On the occasion of Lionel Mason's 60th birthday

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From the conformal self-duality equations to the Manakov-Santini system (2019) J. Geom. Phys. (PP) arXiv:1902.07844

The quadric ansatz for the $mn-$dispersionless KP equation (2021) Preprint (M. Dunajski, PP)

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Integrability Self-Duality, and Twistor Theory

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OXFORD SCIENCE PUBLICATIONS

Einstein-Weyl Structures and Dispersionless Equations

Twistor theory and Integrability

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Anti-self-dual equations (4 dim)

↙ ↘

Anti-self-dual Yang-Mills eq (ASDYM)

Anti-self-dual (ASD) eq for conformal structures

↓ symmetry reduction ↓

Solitonic equations
e.g. KdV, NLS, sine-Gordon

Dispersionless systems
e.g. dKP, $SU(\infty)$ Toda

Integrability of ASDYM on $\mathbb{C}^4$

- $*F = -F \quad \Leftrightarrow \quad F$ flat on α-planes (SD null 2-planes in $\mathbb{C}^4$)
- **Lax pair** $[L, M] = 0, \quad L = D_w - \lambda D_{\tilde{z}}, \quad M = D_z - \lambda D_{\tilde{w}}$

  $D_a = \partial_a + A_a$
- **Twistor Correspondence** (Ward 1977)

  Solutions of ASDYM on $\mathbb{C}^4$ $\longleftrightarrow$ Holomorphic vector bundles over twistor space $\mathcal{P} = \mathbb{CP}^3 - \mathbb{CP}^1$
- **Symmetry reductions:** Most (low dim) solitonic integrable systems can be realised as symmetry reductions of ASDYM eq (Ward 1985)
  - Classification (Mason and Woodhouse 1996)
- **Example:** $G = SL(2\mathbb{C})$ 2-dim translation group H_{+0}

  Complete classification to either the nonlinear Schrödinger eq (NLS) or Korteweg-de Vries eq (KdV)

  (Mason-Sparling 1989)

Einstein-Weyl Structures and Dispersionless Equations

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ASD conformal structures

$\mathcal{M}$ 4-dim complex manifold, $[g]$ hol. conformal class

$(\mathcal{M}, [g])$ is ASD if the Weyl tensor is ASD.

Theorem (Penrose 1976)

- There exists a three-parameter family of α-surfaces in $\mathcal{M}$ iff $(\mathcal{M}, [g])$ is ASD.

  α-surface: a 2-dim surface in $\mathcal{M}$ s.t. its tangent plane at every point in an α-plane

- Complex ASD conformal structures $\longleftrightarrow$ 3-dim complex manifold with 4-parameter family of holomorphic copies of $\mathbb{CP}^1$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$

Einstein-Weyl Structures

$(X, [h])$ conformal structure, D torsion-free connection

- $(X, [h], D)$ is a Weyl structure if $\exists$ a one-form ν s. t.

$$D_a h_{bc} = \nu_a h_{bc}$$

- $(X, [h], D)$ is a Einstein-Weyl (EW) structure if, in addition,

$$W_{(ab)} = \frac{1}{N} W\, h_{ab} \quad (N = \dim X,\ W_{ab} \text{ Ricci tensor of } D)$$

(e.g. Cartan 1943, Hitchin 1982, Pederden-Tod 1993)

- Twistor correspondence for 3-dim EW (Hitchin 1982)

Solutions to EW eq. in 3 dim $\longleftrightarrow$ 2-dim complex manifold with 3-parameter family of copies of $\mathbb{CP}^1$ with normal bundle $\mathcal{O}(2)$

- **Jones-Tod Correspondence** (Jones-Tod 1985)

ASD conformal structure with a non-null conformal Killing symmetry	$\longrightarrow$	EW structure on 3-dim space of orbits

Conversely,

3-dim EW structure + solution to generalised monopole equation	$\longrightarrow$	ASD metric with a Killing symmetry

Motivation (Dunajski-Mason-Tod 2001)

- Any 3-dim Lorentzian EW structure with a covariantly constant weighted vector field can be locally rep by

$$h = dy^2 - 4dxdt - 4udt^2, \quad \nu = -4u_x dt,$$

where $u_{xt} - (uu_x)_x = u_{yy}$ (dKP).

- Via the Jones-Tod correspondence

Such EW structure	$\longleftrightarrow$	ASD Ricci-flat metric
(with certain solution to generalised monopole eq)		(with certain Killing symmetry)

- Appendix: 2nd heavenly $\longrightarrow$ dKP

- **This talk**:
  1. Anti-self-dual conformal eqs $\longrightarrow$ Manakov-Santini sys
  2. The *mn*-dKP equation: its (non)-integrability (via quadric ansatz) and related EW structures

ASD conformal eqs → Manakov-Santini sys

Theorem Any ASD conformal structure, signature $(2,2)$ can be locally rep. by

$$g = dWdX + dZdY + F_Y dW^2 - (F_X + G_Y)dWdZ + G_X dZ^2,$$

$$\begin{aligned} \partial_X(Q(F)) - \partial_Y(Q(G)) &= 0, \\ (\partial_W - F_Y\partial_X + G_Y\partial_Y)Q(G) + (\partial_Z + F_X\partial_X - G_X\partial_Y)Q(F) &= 0, \end{aligned}$$

$$Q = \partial_W\partial_X + \partial_Z\partial_Y - F_Y\partial_X^2 - G_X\partial_Y^2 + (F_X + G_Y)\partial_X\partial_Y.$$

Theorem Any 3-dim Lorentzian EW structure can be locally rep. by

$$\begin{aligned} h &= -(dy - v_x dt)^2 + 4(dx - (u - v_y)dt)dt, \\ \nu &= -v_{xx}dy + (4u_x - 2v_{xy} + v_x v_{xx})dt, \end{aligned}$$

Manakov-Santini system

$$P(u) + u_x^2 = 0, \qquad P(v) = 0,$$

$$P = \partial_x\partial_t - \partial_y^2 + (u - v_y)\partial_x^2 + v_x\partial_x\partial_y.$$

(Dunajski-Ferapontov-Kruglikov 2015)

$$g = dWdX + dZdY + F_Y dW^2 - (F_X + G_Y)dWdZ + G_X dZ^2$$

- ASD conformal eq (DFK 2015) can be reduced to the Manakov-Santini sytem by a non-null translation ∂_W

$$\begin{aligned} \partial_X(Q(F)) - \partial_Y(Q(G)) &= 0 \\ (\partial_W - F_Y\partial_X + G_Y\partial_Y)Q(G) + (\partial_Z + F_X\partial_X - G_X\partial_Y)Q(F) &= 0 \end{aligned}$$

$$Q = \partial_W\partial_X + \partial_Z\partial_Y - F_Y\partial_X^2 - G_X\partial_Y^2 + (F_X + G_Y)\partial_X\partial_Y$$

- $\partial_W : \quad F_W = G_W = 0$
- $u_X = Q(G), \quad u_Y = Q(F)$
- $x = -F, \quad y = X, \quad t = -Z,$ Y=Y(x,y,t)
- $v_x = \dfrac{Y_y - G_x}{Y_x} \quad \Rightarrow \quad u, v$ satisfy Manakov-Santini sytem

$$P(u) + u_x^2 = 0, \qquad P(v) = 0,$$

$$P = \partial_x\partial_t - \partial_y^2 + (u - v_y)\partial_x^2 + v_x\partial_x\partial_y.$$

Cf. 2nd heavely $\to$ Laplace's eq (Finley-Plebański 1979)

Quadric ansatz for the $mn-$dKP equation (Dunajski-PP)

The *mn*-dispersionless Kadomtsev-Petviashvili (*mn*-dKP) eq.

$$u_{xt} - (u^m u_x)_x = \sum_{i=1}^{n} \partial_{y_i}^2 u,$$

$u(x, y_1, \ldots, y_n, t)$. (Manakov-Santini 2011, Santucci-Santini 2016)

- Integrable: $n = 0$ (Riemann eq), $m = 0$ (linear wave eq), $m = n = 1$ (dKP).
- Results: (Dunajski-PP (Preprint))
  - Non-integrability for $n > 2$ (quadric ansatz reduction)
  - Related EW structures in $n + 2$ dimensions

Quadric ansatz

(e.g. Tod 1995, Dunajski-Tod 2002, Ferapontov *et al.* 2012)

$$\frac{\partial}{\partial x^c}\left(b^{cd}(u)\,\frac{\partial u}{\partial x^d}\right) = 0.$$

- Ansatz $\quad M_{ab}(u)x^a x^b = C \quad \Rightarrow \quad$ an ODE for $\mathbf{M}(u)$.
- Introduce $\quad \dfrac{dg}{du} = \dfrac{1}{2}\mathrm{Tr}(\mathbf{bM}), \quad \mathbf{N} = -\mathbf{M}^{-1}$

$$g\frac{d\mathbf{N}}{du} = \mathbf{b}, \qquad g^2 \det \mathbf{N} = \zeta$$

- dKP: $\quad u_{xt} - (uu_x)_x = u_{yy} \quad$ (Dunajski-Tod 2002)

$$\mathbf{b} = \begin{pmatrix} -u & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \quad \mathbf{N} = \begin{pmatrix} Y & \beta & Z \\ \beta & X & \epsilon \\ Z & \epsilon & \phi \end{pmatrix}$$

- dKP: $u_{xt} - (uu_x)_x = u_{yy}$ (Dunajski-Tod 2002)

$$\mathbf{b} = \begin{pmatrix} -u & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \quad \mathbf{N} = \begin{pmatrix} Y & \beta & Z \\ \beta & X & \epsilon \\ Z & \epsilon & \phi \end{pmatrix}$$

- Second order ODE for $Z(u)$
  - $\phi = \epsilon = 0$ Solvable (1st order ODE for $Z(u)$)
  - $\phi = 0,\ \epsilon \neq 0$ Painlevé I
  - $\phi \neq 0$ Painlevé II

Quadric ansatz for the mn-dKP

$$u_{xt} - (u^m u_x)_x = \sum_{i=1}^{n} \partial_{y_i}^2 u$$

$$\mathbf{b}(u) = \left(\begin{array}{c|ccc|c} -u^m & 0 & \cdots & 0 & \frac{1}{2} \\ \hline 0 & & & & 0 \\ \vdots & & -\mathbf{1}_n & & \vdots \\ 0 & & & & 0 \\ \hline \frac{1}{2} & 0 & \cdots & 0 & 0 \end{array}\right), \quad \mathbf{N} = \left(\begin{array}{c|ccc|c} Y & \beta_1 & \cdots & \beta_n & Z \\ \hline \beta_1 & & & & \epsilon_1 \\ \vdots & & \mathbf{X} & & \vdots \\ \beta_n & & & & \epsilon_n \\ \hline Z & \epsilon_1 & \cdots & \epsilon_n & \phi \end{array}\right)$$

- Second order ODE for $Z(u)$ $\qquad \dfrac{\partial}{\partial x^c}\left(b^{cd}(u)\, \dfrac{\partial u}{\partial x^d} \right) = 0$

- $\phi = \epsilon_1 = \epsilon_2 = \cdots = \epsilon_n = 0 \quad \Rightarrow \quad 4\,\zeta \left(\dfrac{dZ}{du}\right)^2 = -Z^2 \det \mathbf{X}$

$$u = 2 \int \left(\frac{-\zeta}{Z^2 \det \mathbf{X}} \right)^{\frac{1}{2}} dZ, \qquad Y = -2 \int u^m \frac{dZ}{du}\, du$$

E.g. (m=1) $\quad u\left(4u(\ln u)\, t^2 - 4xt + y_1^2 + y_2^2\right) \; = \; C, \quad n = 2$

$$u^{2/n}\left(\frac{8}{n-2} u\, t^2 - 4xt + \sum_{i=1}^{n} y_i^2 \right) \; = \; C, \quad n \neq 2$$

Other cases of quadric reduction $\quad M_{ab}(u)x^a x^b = C$

$$\mathbf{N} = -\mathbf{M}^{-1}, \quad \mathbf{N} = \left(\begin{array}{c|ccc|c} Y & \beta_1 & \cdots & \beta_n & Z \\ \hline \beta_1 & & & & \epsilon_1 \\ \vdots & & \mathbf{X} & & \vdots \\ \beta_n & & & & \epsilon_n \\ \hline Z & \epsilon_1 & \cdots & \epsilon_n & \phi \end{array}\right)$$

$$\frac{d^2 Z}{du^2} = A(Z)\left(\frac{dZ}{du}\right)^2 + B(Z) + C(Z)u^m \qquad (*)$$

$A(Z), B(Z)$ rational functions, $C(Z)$ a polynomial

Case 1: $\quad \phi = 0, \;\; \epsilon_i$'s are not all zero

- special case $\quad (\epsilon^2 = \epsilon_1^2 + \cdots + \epsilon_n^2)$

$$2\zeta\frac{d^2 Z}{du^2} = \zeta\frac{(n-1)}{Z}\left(\frac{dZ}{du}\right)^2 - 3(-2)^{n-2}Z^{n+1} - \epsilon^2(-2)^{n-2}Z^{n-1}u^m$$

- **Painlevé analysis:** If $m = n = 1$, $(*)$ is equivalent to Painlevé I. For $n > 2$, $(*)$ does not posses the Painlevé property. (Dunajski-PP: Work in progress)

Other cases of quadric reduction $\quad M_{ab}(u)x^a x^b = C$

$$\mathbf{N} = -\mathbf{M}^{-1}, \quad \mathbf{N} = \left(\begin{array}{c|ccc|c} Y & \beta_1 & \cdots & \beta_n & Z \\ \hline \beta_1 & & & & \epsilon_1 \\ \vdots & & \mathbf{X} & & \vdots \\ \beta_n & & & & \epsilon_n \\ \hline Z & \epsilon_1 & \cdots & \epsilon_n & \phi \end{array}\right)$$

$$\frac{d^2 Z}{du^2} = A(Z)\left(\frac{dZ}{du}\right)^2 + B(Z) + C(Z)u^m \qquad (*)$$

$A(Z), B(Z)$ rational functions, $C(Z)$ a polynomial

Case 2: $\quad \phi \neq 0$

- special case $\quad (\beta^2 = \beta_1^2 + \cdots + \beta_n^2)$

$$2\zeta \frac{d^2 Z}{du^2} = \frac{n\zeta}{Z}\left(\frac{dZ}{du}\right)^2 + (-2)^{n-1}\left(Z^{n+1} + \frac{\phi\beta^2}{4}Z^{n-2}\right) + (-2)^{n-1}\phi Z^n u^m$$

- **Painlevé analysis:** If $m = n = 1$, $(*)$ is equivalent to Painlevé II. For $n > 2$ $(*)$ does not posses the Painlevé property. (Dunajski-PP: Work in progress)

Related EW structures

The *mn*-dKP equation

$$u_{xt} - (u^m u_x)_x = \sum_{i=1}^{n} \partial_{y_i}^2 u$$

the Hodge star operator $*$ defined w.r.t.

$$h = dy_1^2 + \cdots + dy_n^2 - 4dxdt - 4u^m dt^2$$

- $m = n = 1$: $(h,\ \nu = -4u_x dt)$ $\Rightarrow$ dKP EW

  (Dunajski-Mason-Tod 2001)

- For $n > 1$ (set $m = 1$ by redefining u)

  Possible to a one-form ν s.t. the EW condition reduces to a single PDE on u - not the mn-dKP

  $\Rightarrow$ a family of EW structures in $n + 2$ dim

  a class of (Dikarev-Galaev 2021)

Proposition (Dunajski-PP)

The metric h and one-form ν

$$h = dy_1^2 + \cdots + dy_n^2 - 4dxdt - 4udt^2, \ \nu = -\frac{4}{n} u_x dt$$

represent an EW structure iff

$$u_{xt} - (uu_x)_x + \frac{2(n-1)}{n} u_x^2 = \sum_{i=1}^{n} \partial_{y_i}^2 u$$

∂_x is parallel with weight $-\frac{n}{2}$ w.r.t the Weyl connection

Conversely, given an $(n+2)$-dim Lorentzian EW structure, which is non-closed, admits a parallel weighted null vector field, and that the $\mathfrak{so}(\mathfrak{n})$-projection of the holonomy algebra of the Weyl connection is zero, then the EW can be represented locally by (h, ν) as above.

Conclusion

- ASD conformal eqs $\rightarrow$ Manakov-Santini system

  generic ASD conformal (2,2) generic Lorentzian EW

  by a non-null translational Killing symmetry assumption and a simple transformation

- Quadric ansatz for the mn-dKP eq $u_{xt} - (u^m u_x)_x = \sum_{i=1}^{n} \partial_{y_i}^2 u$
  - Some implicit solutions
  - Not posses the Painlevé property for $n > 2$

    (work in progress)

- A class of EW structures in $n+2$ dim

  (belong to a class of (Dikarev-Galaev 2021))

$$u_{xt} - (uu_x)_x + \frac{2(n-1)}{n} u_x^2 = \sum_{i=1}^{n} \partial_{y_i}^2 u$$

Thank you.