Twistor Theory and Donaldson-Thomas invariants

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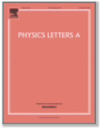
(joint with Tom Bridgeland: Lett.Math.Phys (2021) 111:54)

28^{th} September, 2021

First Meeting

Physics Letters A

Volume 137, Issues 1–2, 1 May 1989, Pages 29-33



Nonlinear Schrödinger and korteweg-de Vries are reductions of self-dual Yang-Mills

L.J. Mason [1, 2, 3], G.A.J. Sparling [3]

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https://doi.org/10.1016/0375-9601(89)90964-X

Background: From $SU(2)$ to $SU(\infty)$: Large N-limits - replacing matrices by vector fields

Lax equations:

$$\mathcal{L}_1 = \frac{\partial}{\partial z_1} + \varepsilon^{-1} A\,, \qquad \mathcal{L}_2 = \frac{\partial}{\partial z_2} + \varepsilon^{-1} B\,,$$

with $A\,, B \in \mathfrak{g}$. Integrability condition/Frobenius integrability:

$$[\mathcal{L}_i, \mathcal{L}_j] = 0$$

gives $A = J^{-1}\partial_{z_1} J\,, B = J^{-1}\partial_{z_2} J$ where $J \in G$ and:

$$\partial_{z_1}\left(J^{-1}\partial_{z_2} J\right) - \partial_{z_2}\left(J^{-1}\partial_{z_1} J\right) = 0$$

What happens for $SU(N)$ as $N \to \infty$?

Use the algebra of volume preserving diffeomorphisms of a surface Σ^2

$$\mathcal{L}_1 = \frac{\partial}{\partial z_1} + \varepsilon^{-1} H_f \,, \qquad \mathcal{L}_2 = \frac{\partial}{\partial z_2} + \varepsilon^{-1} H_g \,,$$

where

$$H_f = \frac{\partial f}{\partial \theta_1}\frac{\partial}{\partial \theta_2} - \frac{\partial f}{\partial \theta_2}\frac{\partial}{\partial \theta_1}$$

and $[H_j, H_g] = H_{\{f,g\}}$.

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and $[H_j, H_g] = H_{\{f,g\}}$.
Then $[\mathcal{L}_1, \mathcal{L}_2] = 0$ implies there exists a function W such that $g = \partial_{z_2} W$, $f = \partial_{z_1} W$ where W satisfies

$$\frac{\partial^2 W}{\partial \theta_1 \partial z_1}\frac{\partial^2 W}{\partial \theta_2 \partial z_2} - \frac{\partial^2 W}{\partial \theta_1 \partial z_2}\frac{\partial^2 W}{\partial \theta_2 \partial z_1} = 1$$

This is Plebanski's first heavenly equation. The function W is the Kähler potential of a metric g .

- The metric is Ricci flat, with (anti)-self-dual Weyl tensor.
- Penrose non-linear graviton construction. There is a twistor space $\mathcal{T}$ - fibred over $\mathbb{P}^1$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

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Generalization:

- Normal bundle $\mathcal{O}(n_1) \oplus \ldots \mathcal{O}(n_N)$.
- HyperKähler , hypercomplex, quaternionically Kähler. All in $4k$-dimensions.

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Aside:
If $\Sigma^2 = \mathbb{T}^2$ then:

$$\left\{ \begin{array}{c} \textit{volume preserving} \\ \textit{diffeomorphism of } \mathbb{T}^2 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{automorphims of an} \\ \textit{algebraic torus} \end{array} \right\}$$

Consider the algebra with basis $x_1^a x_2^b$ and with multiplication $(m \in \mathbb{N})$

$$x_1^a x_2^b \cdot x_1^c x_2^d = (-1)^{m(ad-bc)} x_1^{a+c} x_2^{b+d}$$

Easy to check that the transformation $T_{a,b}$

$$x_1 \mapsto x_1 \cdot (1 - x_1^a x_2^b)^{-mb}\,, \quad x_2 \mapsto x_2 \cdot (1 - x_1^a x_2^b)^{ma}$$

is an automorphism.

Donaldson-Thomas Invariants

Consider the algebra with basis $x_1^a x_2^b$ and with multiplication $(m \in \mathbb{N})$

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Theorem (Integrality conjecture/Reineke, Kontsevich & Soibelman)

There exists integers $N(a,b)$ - called numerical Donaldson-Thomas (DT) invariants such that

$$T_{1,0} \cdot T_{0,1} = \prod_{\frac{a}{b}\uparrow} T_{a,b}^{N(a,b)}$$

The product is over increasing values of a/b.

Example

$(m=1)$ $\mathsf{T}_{1,0} \cdot T_{0,1} = T_{0,1} \cdot T_{1,1} \cdot T_{1,0}$

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Easily generalised:
Let $\langle -, - \rangle$ be a skew form on $\Gamma \cong \mathbb{Z}^r$ Define multiplication

$$x^\alpha \cdot x^\beta = (-1)^{\langle \alpha, \beta \rangle} x^{\alpha + \beta}$$

and automorphims

$$T_\alpha : x^\beta \mapsto x^\beta \cdot (1 - x^\alpha)^{\langle \alpha, \beta \rangle}$$

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$$\{x^\alpha, x^\beta\} = \langle \alpha, \beta \rangle x^\alpha \cdot x^\beta \,.$$

This is best seem by first proving

$$T_\alpha = Ad \exp\left\{-Li_2(x^\alpha)\right\} \,.$$

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N.B. Connections to quivers: the original example is the m-Kronecker quiver, and spaces of stability conditions.

Key idea I (Joyce): combine DT-invariants into generating functions, and see what differential equations they satisfy

$$dF_\gamma = \sum_{\gamma=\alpha+\beta} [F_\alpha, F_\beta] d \log Z(\beta)$$

Note:

"$d\Gamma + \frac{1}{2}\Gamma \wedge \Gamma = 0$ but we do not expect that $d\Gamma = 0$ and $\Gamma \wedge \Gamma = 0$ as happens in the Gromov-Witten case, so we do not have a 1-parameter family of flat connections and a Frobenius manifold"

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Key idea II (Bridgeland & Toledano Laredo): Interpret equations as isomonodromy equations, with Stokes data coming from the automorphism T_α

Key idea III (Bridgeland) Derive further set of conditions, together with some algebraic conditions and homogeneity conditions.

Bridgeland, Strachan

These equations define a (complex) hyperKähler metric on the total space $X = \mathcal{T}_M$, with a conformal (homothetic) Killing vector field

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Integrable complex structure I , J , K satisfying the quaterion relations, $dI = 0$ etc., and

$$\mathcal{L}_E(g) = g, \qquad \mathcal{L}_E(I) = 0, \qquad \mathcal{L}_E(J \pm iK) = \mp (J \pm iK);$$

HyperKähler geometry on $X = \mathcal{T}_M$ and Plebanski's 2^{nd} Heavenly equation

Let z^i be local coordinates on M and let θ^i be natural linear coordinates on $\mathcal{T}_{M,p}$ $(i = 1, \ldots, n)$
Let η^{ij} be a constant, non-degenerate, skew matrix with inverse ω_{ij}.

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Let z^i be local coordinates on M and let θ^i be natural linear coordinates on $\mathcal{T}_{M,p}$ $(i = 1, \ldots, n)$
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Plebanski's second heavenly equation

Let $W : X \to \mathbb{C}$ satisfy the equation

$$\frac{\partial^2 W}{\partial\theta_i \partial z_j} - \frac{\partial^2 W}{\partial\theta_j \partial z_i} = \sum_{p,q} \eta^{pq} \frac{\partial^2 W}{\partial\theta_i \partial\theta_p} \frac{\partial^2 W}{\partial\theta_j \partial\theta_q} .$$

Then the metric on $X = \mathcal{T}_M$

$$g = \sum_{i,j} \omega_{ij}(d\theta_i \otimes dz_j + dz_j \otimes d\theta_i) - \frac{\partial^2 W}{\partial\theta_i \partial\theta_j}(dz_i \otimes dz_j + dz_j \otimes dz_i)$$

is hyperKähler.

So we have complex structures I , J , K satisfying the quaternionic relations ($I^2 = -1$, $IJ = K$ etc, and $dI = 0$ etc.). The geometry is best described in terms of horizontal and vertical vector fields:

$$v_i = \frac{\partial}{\partial \theta_i}\,, \qquad h_i = \frac{\partial}{\partial z_i} + \sum_{p,q} \eta^{pq} \frac{\partial^2 W}{\partial \theta_i \partial \theta_p} \frac{\partial}{\partial \theta_q}$$

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and the structure are defined by:

$$\begin{array}{ccc} I(v_j) = i \cdot v_j, & J(v_j) = h_j, & K(v_j) = -ih_j, \\ I(h_j) = -i \cdot h_j, & J(h_j) = -v_j, & K(h_j) = -iv_j. \\ g(v_i, v_j) = 0, & g(v_i, h_j) = \omega_{ij}, & g(h_i, h_j) = 0. \end{array}$$

With these the Lax equations $[\mathcal{L}_i, \mathcal{L}_j] = 0$ where $\mathcal{L}_i = h_i + \varepsilon^{-1} v_i$.

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The main part of the construction is that the holomophic 2-form

$$\Omega_{-}(v, w) = g(v, (J - iK)w)$$

on X is the pull-back via the natural projection $\pi : X \to M$ of a holomorphic sympletic form $\omega = \omega_{ij} dx_i \wedge dx_j$.

Consider the ansatz:

$$W(\mathbf{z}, \theta) = \sum_{\alpha} \frac{F_{\alpha}(\mathbf{z}) e^{\theta(\alpha)}}{\mathbf{z}(\alpha)} \,.$$

In terms of an Euler operator

$$E = \sum z_i \frac{\partial}{\partial z_i}$$

Joyce showed $E(F_{\alpha}) = 0$. And easy to show that $E(W) = -W$ is equivalent to the geometric condition $\mathcal{L}_E g = g$.

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Joyce showed $E(F_{\alpha}) = 0$. And easy to show that $E(W) = -W$ is equivalent to the geometric condition $\mathcal{L}_E g = g$. The second heavenly equation then implies

$$dF_{\gamma} = \sum_{\alpha+\beta=\gamma} [F_{\alpha}, F_{\beta}] d \log z(\beta)$$

which is the isomonodromy deformation condition for

$$\frac{d}{d\varepsilon} - \left(\frac{z}{\epsilon^2} + \frac{Ham_F}{\varepsilon} \right) .$$

This has an irregular pole at $\varepsilon = 0$ and an regular singularity at $\varepsilon = \infty$. So we have a Riemann-Hilbert problem

Riemann-Hilbert Problems

Lie group G and associated Lie algebra $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{g}^{od}$ and $Ham_F \in \mathfrak{g}^{od}$, $z \in \mathfrak{h}$.

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- Stokes rays/Walls l coming from coincident values in z;

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- For a non-Stokes ray r can solve (Balser, Jurkat, Lutz) - with appropriate boundary conditions as $\varepsilon \to 0$ to get a holomorphic function

$$\Psi_r(\varepsilon) : \mathbb{H}_r \to G ;$$

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$$\Psi_r(\varepsilon) : \mathbb{H}_r \to G \,;$$

- Stokes factors: for $\varepsilon \in \mathbb{H}_{r_1} \cap \mathbb{H}_{r_2}$

$$\Psi_{r_2}(\varepsilon) = \Psi_{r_1}(\varepsilon)\mathbb{S}(\Delta)$$

where Δ is a region bounded by non-Stokes rays, which may contain Stokes rays l and $\mathbb{S}(\Delta) = \prod \mathbb{S}(l)$

Essential feature:

$$\mathbb{S}(l) = \exp\left(\sum DT(\gamma)x^{\gamma}\right)$$

and these are the automorphisms of the torus $\mathbb{T}$.

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Programme

$$\begin{array}{c} quiver/stability\ \ conditions \\ \updownarrow \\ DT\ \ invariants,\ \ automorphims\ \ T_{\alpha} \\ \updownarrow \\ Stokes\ \ data:\ \ solve\ \ Riemann\ \ Hilbert\ \ problem \\ \updownarrow \\ Construct\ \ F_{\alpha} \\ \updownarrow \\ (complex)\ \ hyperK\ddot{a}hler\ \ metric \end{array}$$

Open Problems & Comments

- A lot of work, even for the A_2 quiver!
- What about $\varepsilon \to \infty$ and the Riemann-Hilbert problem across the regions around 0 and ∞?
- Is there a direct construction from the quiver to the twistor space?

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- A lot of work, even for the A_2 quiver!
- What about $\varepsilon \to \infty$ and the Riemann-Hilbert problem across the regions around 0 and ∞?
- Is there a direct construction from the quiver to the twistor space?
- What about quantum DT invariants?

Unsolved problems: What about the KP hierarchy?

Twistor theory (via curved twistor spaces) works beautifully for the dispersionless KP equation:

$$(u_t + uu_x)_x + u_{yy} = 0$$

but not for the full (dispersive) KP equation:

$$(u_t + uu_x + \hbar u_{xxx})_x + u_{yy} = 0$$

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Idea: replace $sdiff\,\Sigma^2$ by the Moyal algebra/pseudo-differential operators.

Return to the start: nonlinear terms come from a Poisson bracket. Such brackets may be deformed.

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Define the operator (where the arrows show which direction the derivatives are to be taken)

$$P = \exp\left[\frac{i\hbar}{2}\eta^{ij}\frac{\overleftarrow{\partial}}{\partial\theta^i}\frac{\overrightarrow{\partial}}{\partial\theta^j}\right].$$

With this, one defines the associative product $f * g = fPg$ and the Moyal bracket

$$\{f,g\}_M = \frac{f*g - g*f}{\hbar} = \{f,g\} + \sum_r \hbar^{2r}\{f,g\}^{(r)},$$

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One can also introduce differential operators $\hat{H}_f$ - with higher derivatives (formally, as these are to all orders) that are deformations of Hamiltonian vector fields, and have the property

$$[\hat{H}_f, \hat{H}_g] = \hat{H}_{\{f,g\}_M}$$

where the l.h.s. is commutator of operators.

Replace Hamiltonian vector fields in Lax equations with the operators $\hat{H}_f$

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Example

Plebanski's first heavenly equation:

$$\left\{ \frac{\partial W}{\partial z^i}, \frac{\partial W}{\partial z^j} \right\}_M = 1\,.$$

Plebanski's second heavenly equation:

$$\frac{\partial^2 W}{\partial z_i \partial \theta_j} - \frac{\partial^2 W}{\partial z_j \partial \theta_i} = \left\{ \frac{\partial W}{\partial \theta_i}, \frac{\partial W}{\partial \theta_i} \right\}_M\,,$$

In terms of a torus basis,

$$\begin{aligned}
\{e^{\theta(\alpha)}, e^{\theta(\beta)}\}_M &= \frac{2}{\hbar}\left\{\frac{e^{\frac{+i\hbar<\alpha,\beta>}{2}} - e^{\frac{-i\hbar<\alpha,\beta>}{2}}}{2i}\right\} e^{\theta(\alpha+\beta)}\,, \\
&= \frac{2}{\hbar}\sin\left[\frac{\hbar<\alpha,\beta>}{2}\right] e^{\theta(\alpha+\beta)}\,, \\
&= \frac{1}{i\hbar}\left[(-q^{\frac{1}{2}})^{<\alpha,\beta>} - (-q^{-\frac{1}{2}})^{<\alpha,\beta>}\right] e^{\theta(\alpha+\beta)}\,.
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\end{aligned}$$

Let $\mathbb{L} = e^{i\hbar}$ and $q = e^{i(\hbar+2\pi)}$. And,

$$\lim_{\hbar\to 0}\{e^{\theta(\alpha)}, e^{\theta(\beta)}\}_M =< \alpha,\beta > e^{\theta(\alpha+\beta)}\,.$$

Two names: sine-algebra or the quantum torus algebra. Or, using the $*$-product, just

$$e^{\theta(\alpha)} * e^{\theta(\beta)} = (-q^{\frac{1}{2}})^{<\alpha,\beta>} e^{\theta(\alpha+\beta)}\,.$$

With the ansatz

$$W = \sum_{\alpha} F_{\alpha}(z) \frac{e^{\theta(\alpha)}}{z(\alpha)}$$

with F_{α} of degree zero, you get the isomonodromy equation

$$dF_{\gamma} = \sum_{\alpha+\beta=\gamma} \frac{1}{i\hbar} \left\{ \mathbb{L}^{\frac{1}{2}<\alpha,\beta>} - \mathbb{L}^{-\frac{1}{2}<\alpha,\beta>} \right\} F_{\alpha} F_{\beta} d \log z(\beta) \, .$$

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But where is the geometry? What are q-deformations of hyperKähler geometry, and q-deformations of twistor theory.