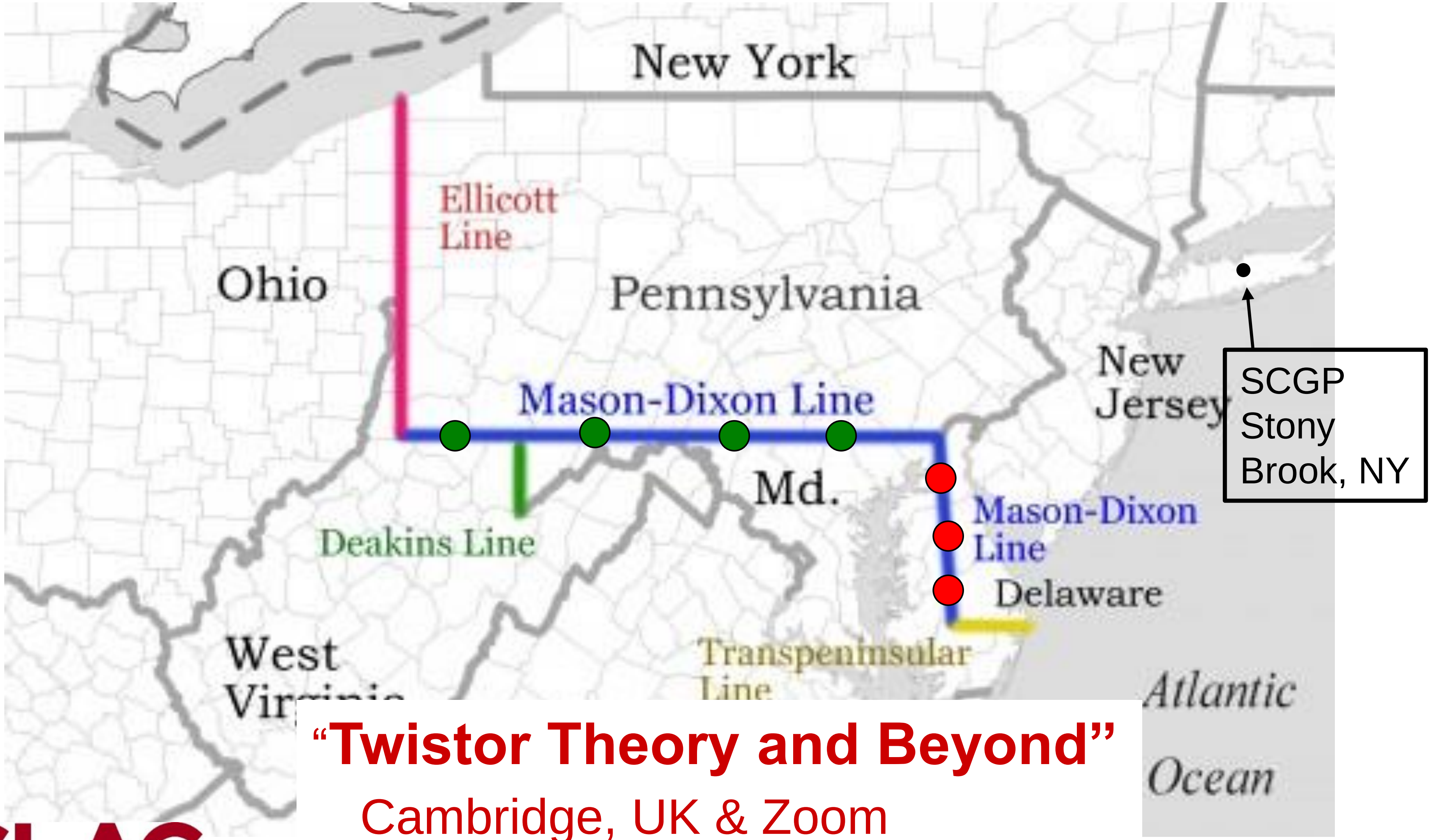
# Surveying the Mason-Dixon Line



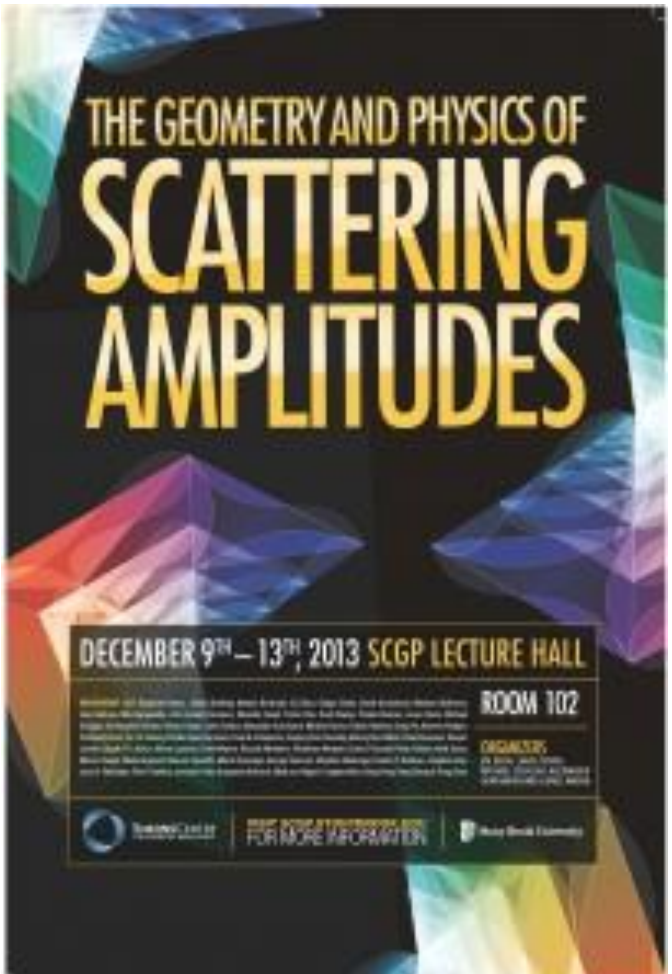

**"Twistor Theory and Beyond"**

Cambridge, UK & Zoom

**Lance Dixon**

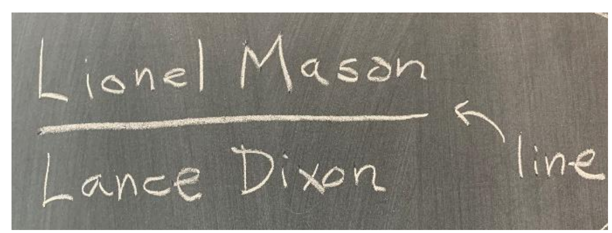
SLAC NATIONAL ACCELERATOR LABORATORY

# In Fall 2013, I co-organized a program and workshop with Lionel (and Zvi)




Simons Center for Geometry and Physics, Stony Brook, NY

I shared an office with Lionel, with the (customized) plaque:



# Now, a Mason turns 60

But 60 is just some arbitrary base 10 number (especially during a pandemic)


Western survey started
1765
+256
---------
2021
→ $2^8$ years old!

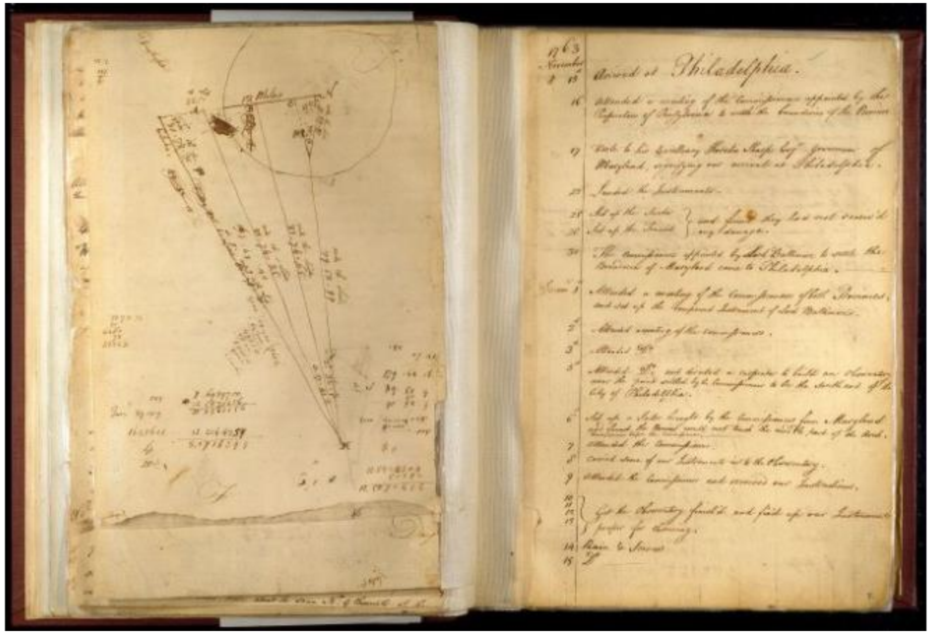
# Mason-Dixon Line: Historical border between North and South leading up to Civil War

- English Crown intervened in Maryland-Pennsylvania border conflict called Cresap's War, ordering Frederick Calvert, 6th Baron of Baltimore to accept 1732 agreement between the 5$^{th}$ Baron and William Penn's sons.
- As part of settlement, Penns and Calverts commissioned the English team of **Charles Mason** and **Jeremiah Dixon** to survey the newly established boundaries between the Province of Pennsylvania, the Province of Maryland, Delaware Colony and parts of Colony and Old Dominion of Virginia.
- Mason was an assistant at Greenwich Observatory, an Anglican widower with two sons. Dixon was a skilled surveyor from Durham, a Quaker bachelor whose Meeting had ousted him for his unwillingness to abstain from liquor.

https://www.southernpartisan.com/the-history-of-the-mason-dixon-line/

https://www.risingsunmd.org/department/division.php?structureid=51

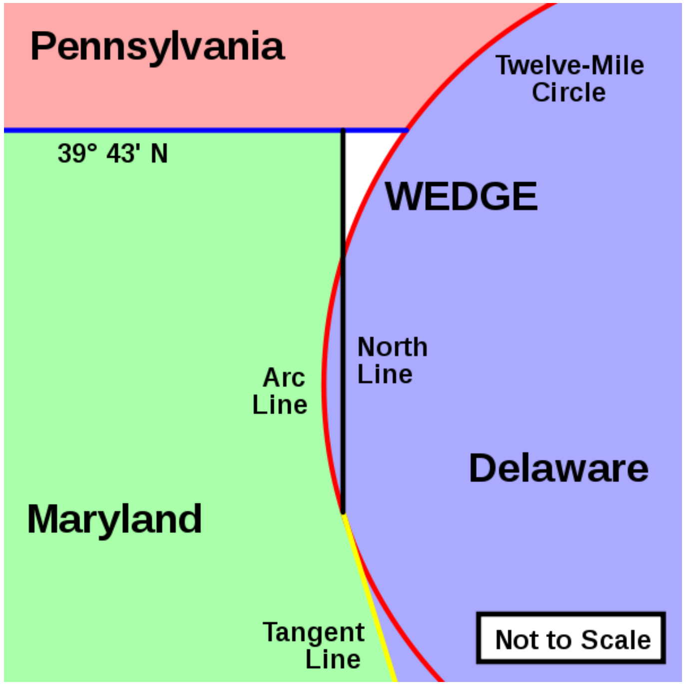
# A precision survey



Mason and Dixon produced “the straightest and most regular” lines ever run because the surveyors took astronomical sightings with a new kind of zenith sector, an instrument “so exact, that they found they could trace out a parallel of latitude by it, without erring above 15 or 20 yards.” – Nevil Maskeleyn, British Astronomer Royal

All without GPS or even twistor theory!

# Not just a line: 2 “lines” + an arc




The surveyors also fixed the boundary between Delaware and Pennsylvania and the approximately north-south portion of the boundary between Delaware and Maryland. Most of the Delaware-Pennsylvania boundary is a circular arc, and the Delaware-Maryland boundary does not run truly north-south because it was intended to bisect the Delmarva Peninsula rather than follow a meridian.

# First time I met Lionel (I think) was at workshop he organized

*Twistor String Theory*

**Conference Poster**

**Link to program and transparencies**

**Link to `From Twistors to Amplitudes,' a QMUL workshop.**

**THE MATHEMATICAL INSTITUTE**
**University of Oxford**

**London Mathematical Society Workshop**

**10-14 January 2005**


**Abstract**

This meeting was organised to take stock of the rapid progress being made on twistor-string theory and to encourage further cross-fertilization between string-theory, twistor theory and perturbative gauge theory. Twistor string theory was introduced by Witten in hep-th/0312171 as a string theory in twistor space that makes contact with N=4 super Yang-Mills theory on space-time via a generalization of the Penrose-Ward transform augmented by certain D-instanton corrections. It promises to combine many of the most attractive features of string theory and twistor theory and has implications not only for Yang-Mills but also for (conformal) gravity. It has in particular led to major advances in the calculations of Yang-Mills scattering amplitudes with applications to collider physics.

# I was starting to get acquainted with twistors



**All Non-Maximally-Helicity-Violating One-Loop**

**Seven-Gluon Amplitudes in $\mathcal{N} = 4$ Super-Yang-Mills Theory**

Zvi Bern

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*Los Angeles, CA 90095–1547, USA*

Vittorio Del Duca

*Istituto Nazionale di Fisica Nucleare*

*Sez. di Torino*

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*Department of Physics, University of Durham*

*Durham, DH1 3LE, UK*

David A. Kosower

*Service de Physique Théorique, CEA–Saclay*

*F–91191 Gif-sur-Yvette cedex, France*

(Dated: October 21, 2004)

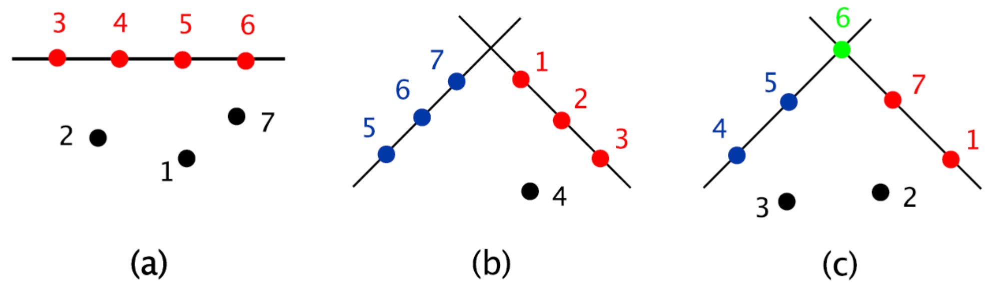
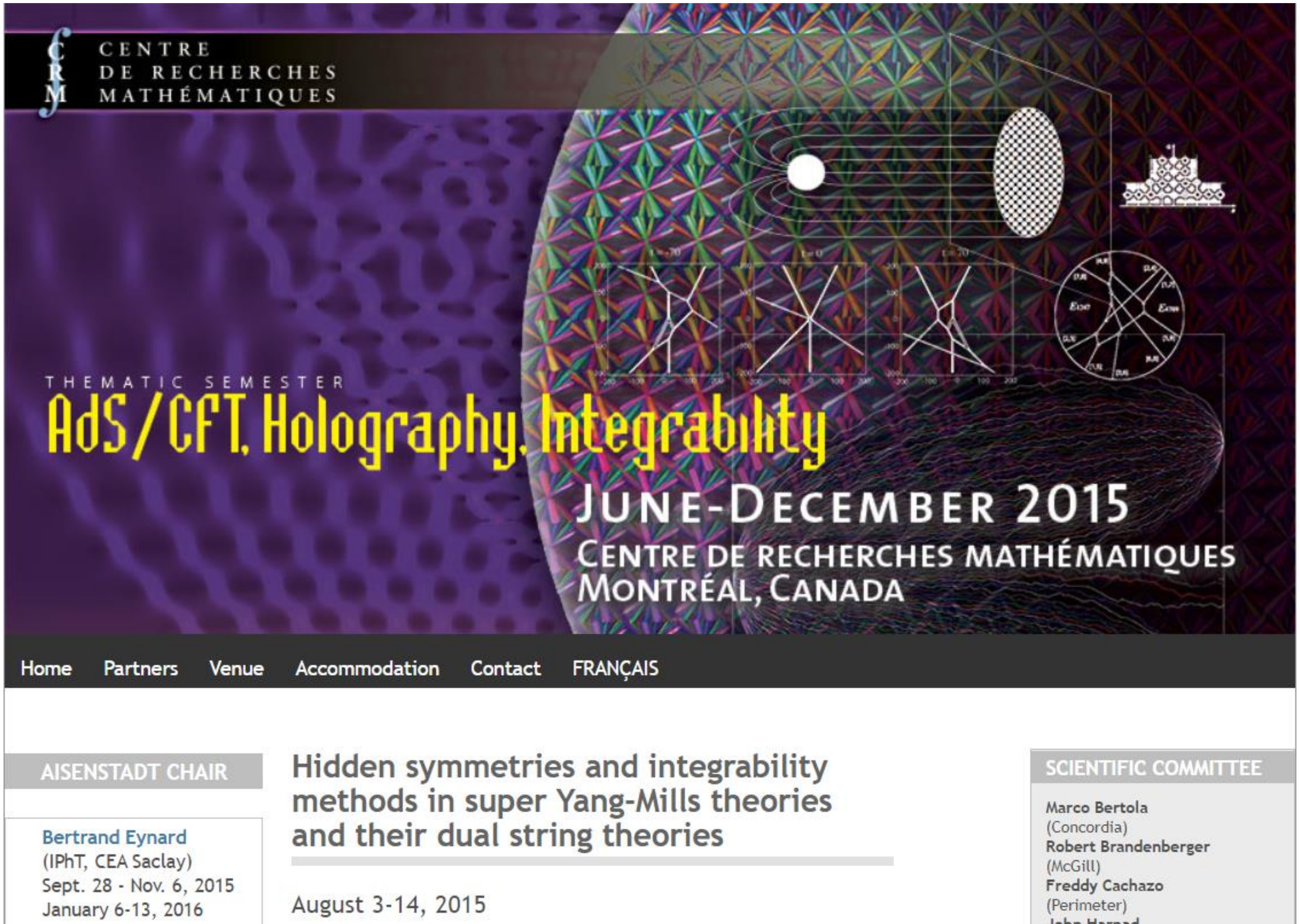



FIG. 5: Examples of twistor-space configurations for single-term box coefficients in the helicity amplitude $A_{7;1}^{\mathcal{N}=4}(1^-, 2^-, 3^+, 4^-, 5^+, 6^+, 7^+)$. In every case, all the points lie in a plane. (a) the easy two-mass box coefficient $c_{125}$, (b) the hard two-mass box coefficient $c_{237}$, (c) the three-mass box coefficient $c_{135}$.

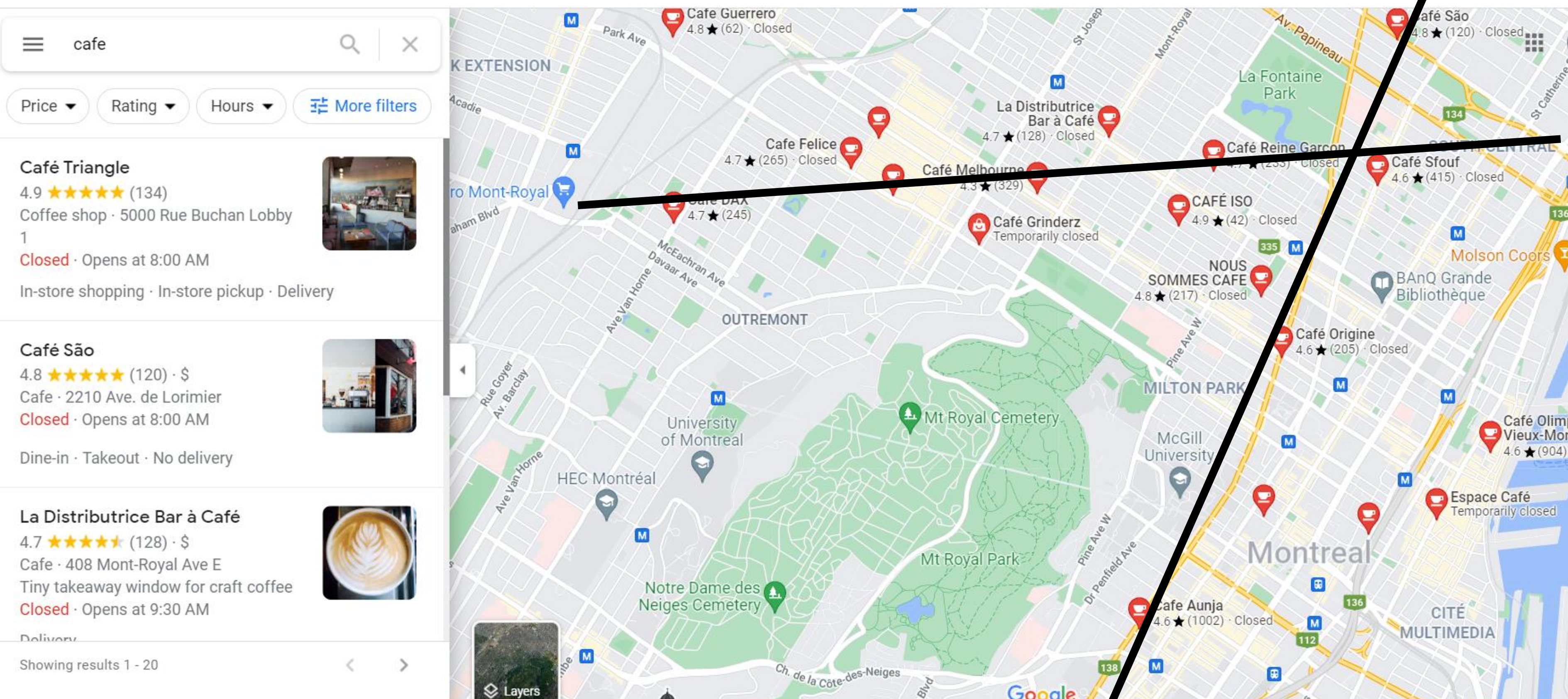
It was fantastic to meet Lionel, a real twistor expert, with such obvious enthusiasm, for twistors, for scattering amplitudes, and for many other things in life!

# Far from my only next-to-linear interaction with Lionel

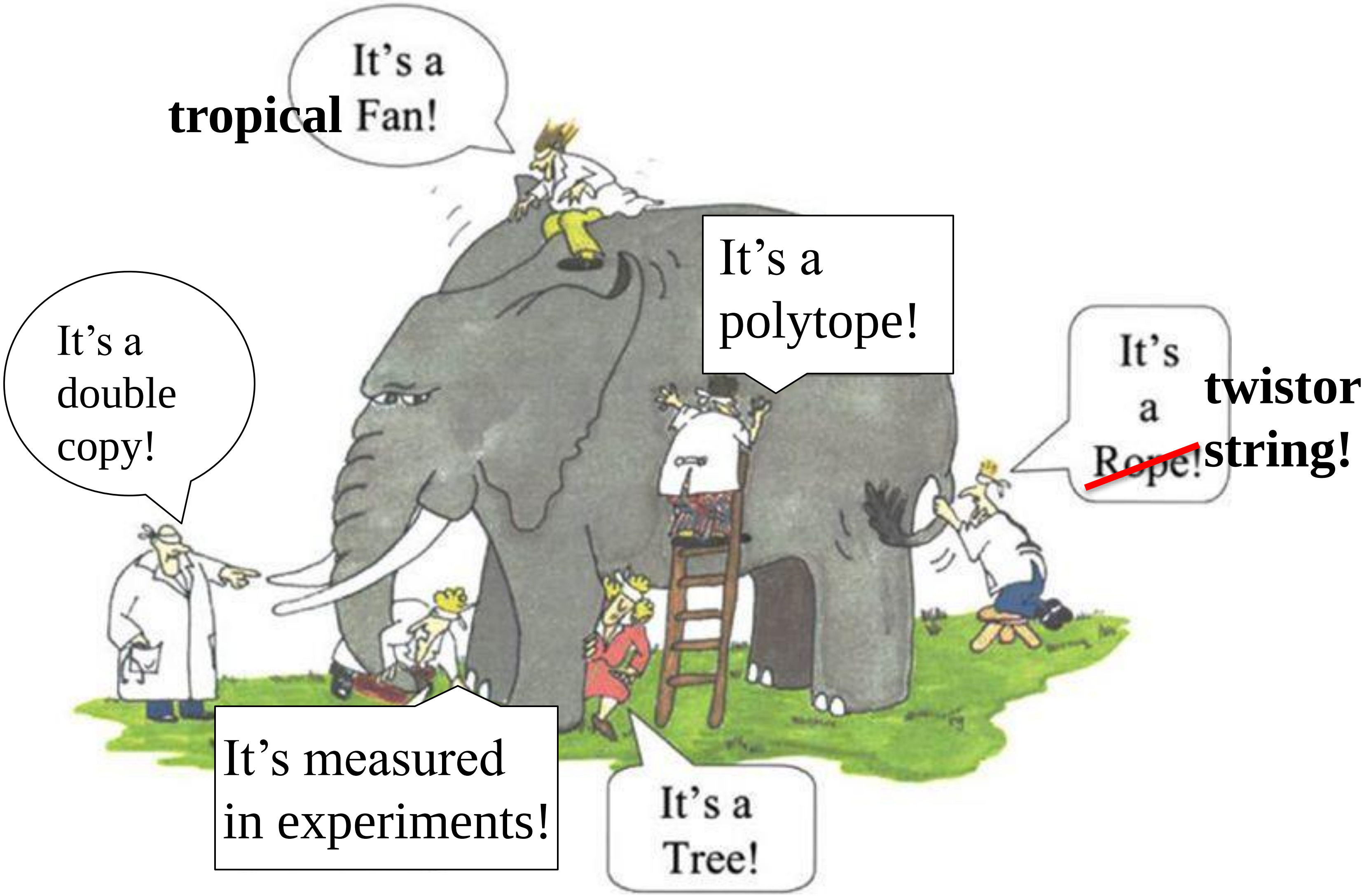


# Lionel the intrepid explorer

tripadvisor®
REVIEWS



# What are amplitudes?



# My current take on what [perturbative] amplitudes "are"

- Functions of external kinematics alone (no Feynman diagrams, no loop integrands!) and the loop order $L$
- Should be **bootstrapped** if possible, by determining the right function space and imposing enough constraints to fix all the unknowns in a linear combination
- Works in "the simplest gauge theory", planar N=4 super-Yang-Mills theory, to a remarkable number of loops, for both pure scattering amplitudes and closely related operator form factors
- The function space can often be refined and restricted with the help of the answers at smaller $L$
- Geometry lurks in the function space

# Planar N=4 SYM
## "toy model for QCD amplitudes"

- QCD's maximally supersymmetric cousin, N=4 super-Yang-Mills theory (SYM), gauge group $\mathrm{SU}(N_c)$, in the large $N_c$ (planar) limit
- Structure very rigid:

  $n$ gluon amplitudes = $\sum_i rational_i \times transcendental_i$
- For planar N=4 SYM, now understand rational structure quite well, basically just those twistor-space localized, dual-conformal invariants seen earlier.
- So focus on the transcendental functions.
- Space of functions so restrictive, physical constraints so powerful, one can **bootstrap**: write $L$ loop answer as linear combination of known weight $2L$ polylogarithms.
- Unknown coefficients found by solving linear constraints

# Hexagon function bootstrap

Loops

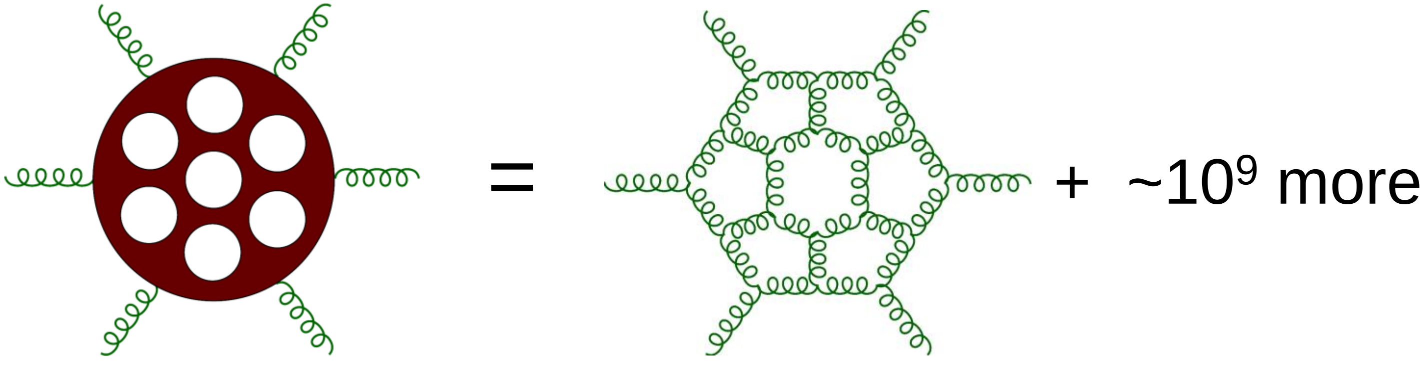
3 LD, Drummond, Henn, 1108.4461, 1111.1704;

4,5 Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington, 1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;

6,7 Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890, 1906.07116; LD, Dulat, 21mm.nnnnn (NMHV 7 loop)

- All based on "letters" from **momentum (super)twistors** [Penrose, Hodges, Mason&Skinner,…]
  → Gr(4,6) = Gr(2,6) cluster algebra (talk by Anastasia Volovich)

- First nontrivial all massless scattering amplitude is for $n = 6$ gluons. Use analytical properties to determine it directly to 7 loops, without ever peeking inside the loops

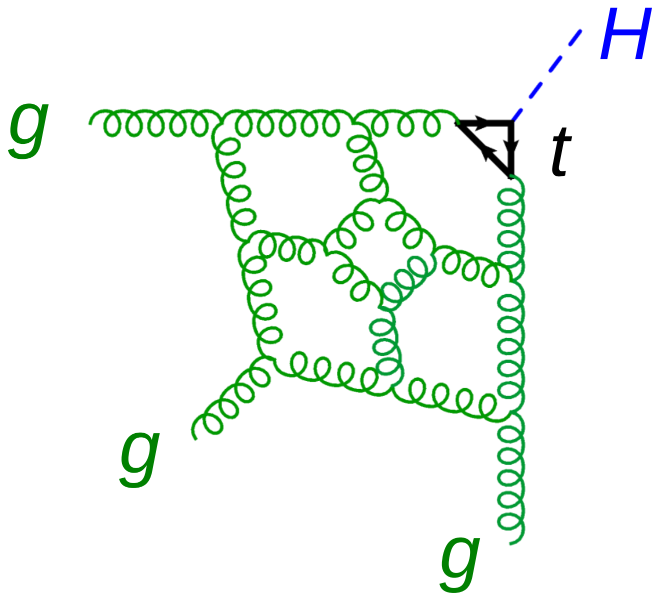


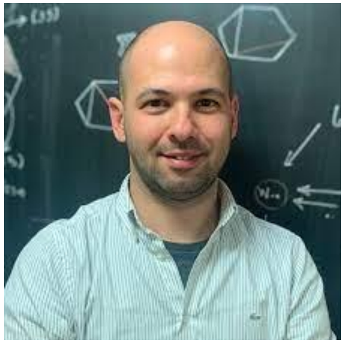
# But…

- Dual conformal symmetry of planar N=4 SYM means that non-trivial part of result depends on 4 fewer variables than in QCD.
- Functions encountered still depend in a rather complicated way on 3 kinematic variables (more for $n > 6$ gluons)
- Can we find a simpler setup where QCD and planar N=4 SYM are even more closely related?

YES!

# Bootstrapping Form Factors



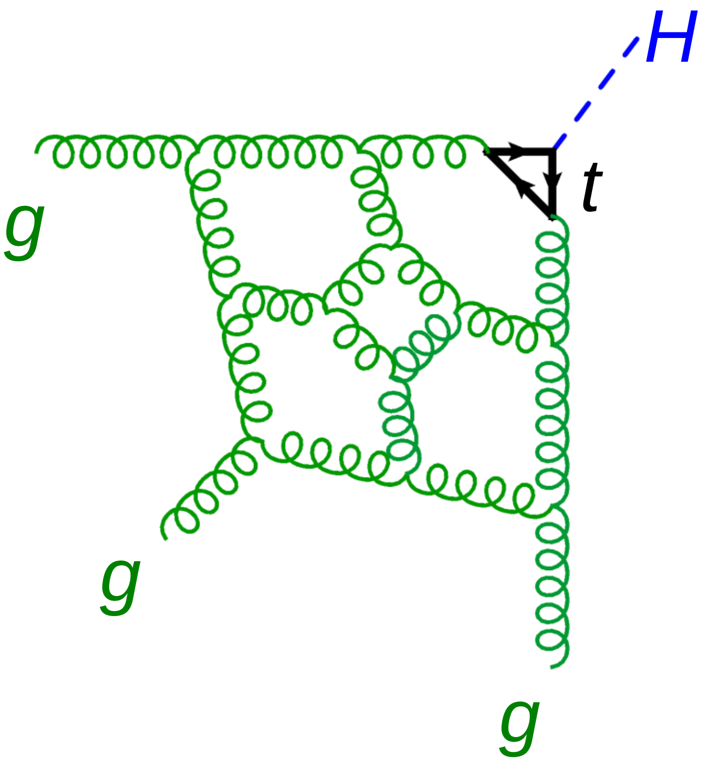
Ömer Gürdoğan

Andrew McLeod

Matthias Wilhelm

# "Higgs" amplitudes and N=4 SYM form factors

LD, A. McLeod, M. Wilhelm, 2012.12286 $L = 3,4,5$ loops
+ in progress also with Ö. Gürdoğan $L = 6,7,8$ loops




- As $\boldsymbol{m_{top}} \to \infty$, integrate out top quark to get operator $H G_{\mu\nu}^{a} G^{\mu\nu\, a}$ (stress tensor supermultiplet in N=4)
- Higgs amplitudes equivalent to matrix elements of this operator with multiple gluons: "form factors"
- *Hgg* Sudakov form factor is "too simple", no kinematic dependence beyond overall $(-s_{12})^{-L\epsilon}$
- *Hggg* is "just right", depends on 2 dimensionless ratios

# *Hggg* kinematics is two-dimensional

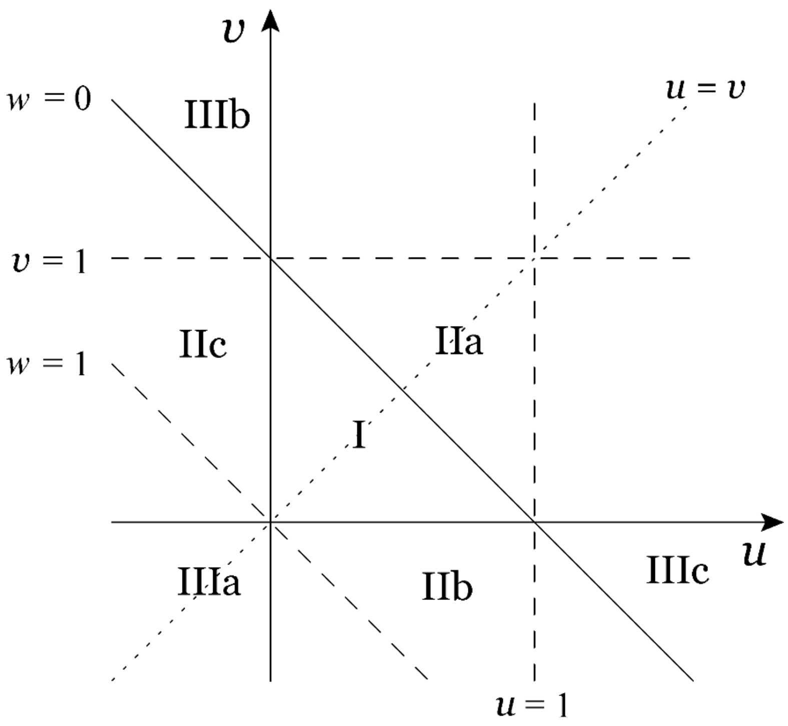
$$p_1 + p_2 + p_3 = -p_H$$

$$s_{123} = s_{12} + s_{23} + s_{31} = m_H^2$$

$$s_{ij} = (p_i + p_j)^2 \qquad p_i^2 = 0$$

$$u = \frac{s_{12}}{s_{123}} \qquad v = \frac{s_{23}}{s_{123}} \qquad w = \frac{s_{31}}{s_{123}}$$

$$u + v + w = 1$$




I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

# A two-loop story

- Gehrmann et al. computed $Hggg$ in QCD at 2 loops
  Gehrmann, Jaquier, Glover, Koukoutsakis, 1112.3554
- Soon after, Brandhuber et al. computed stress tensor 3-point form factor $\mathcal{F}_3$ in N=4 SYM,
  Brandhuber, Travaglini, Yang, 1201.4170
  saw that "maximally transcendental part" of

QCD result was same as N=4 SYM result

# 2d HPLs

Gehrmann, Remiddi, hep-ph/0008287

Space graded by weight $w$. Every function $F$ obeys:

$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{1-u-v} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{u+v}$$

where $F^u, F^v, F^w, F^{1-u}, F^{1-v}, F^{1-w}$ are weight $w$-1 2d HPLs

Special case of iterated integrals $dF = \sum_{s_k \in \mathcal{S}} F^{s_k}\, d\ln s_k$

$F^{s_k} = \{w\text{-}1,1\}$ coproduct of $F$

**Symbol alphabet** $\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w\}$

# Bootstrapping & boundary conditions

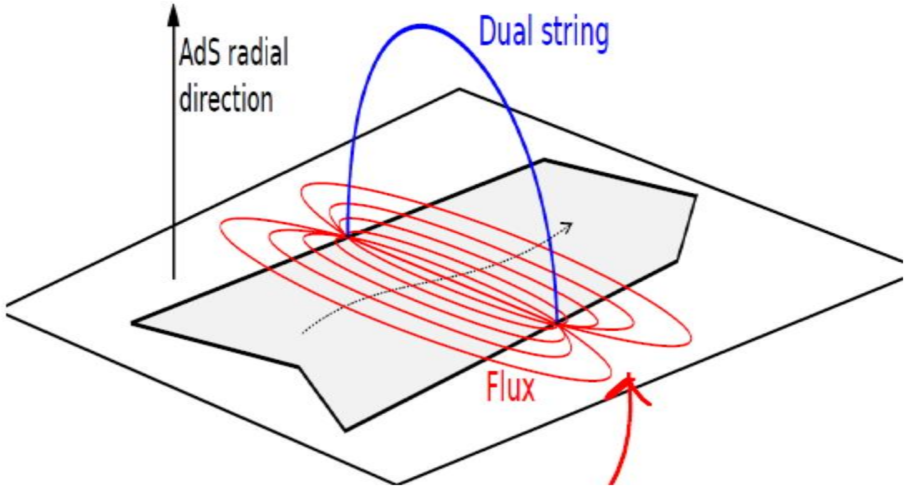
- Symbol alphabet simpler than that for 6- or 7-gluon scattering amplitudes

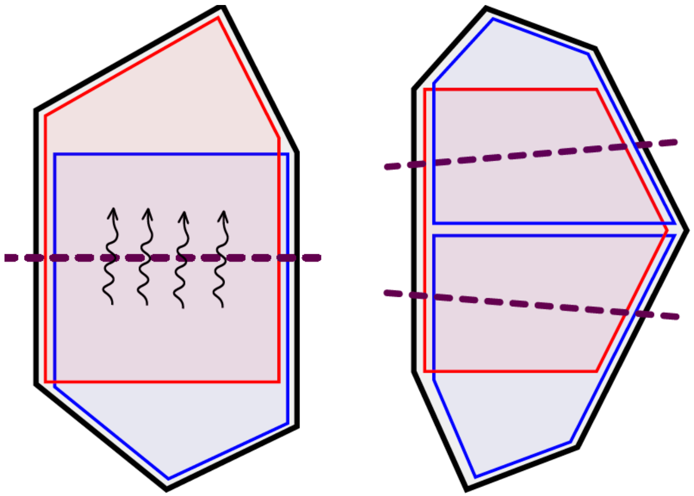
→ "easy" to write down candidate linear combinations of functions.

- But we also need boundary data – at least to start with, until we understand the function space well enough
- Amplitude boundary data: (near)collinear limits, related to an OPE for Wilson loops
- Until recently, no OPE for form factors

# Flux tubes at finite coupling

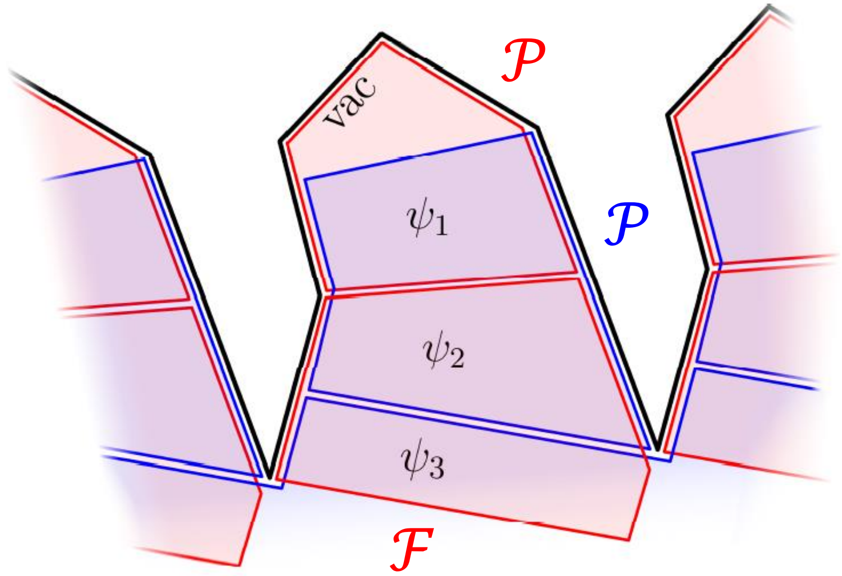
Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987



- Tile $n$-gon with pentagon transitions.
- Quantum integrability → compute pentagons exactly in ’t Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit

# The new FFOPE




- Form factors are Wilson loops in a periodic space, due to injection of operator momentum

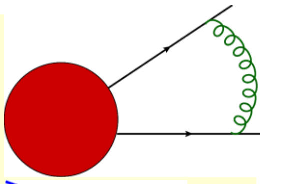
Brandhuber, Spence, Travaglini, Yang, 1011.1899

- Besides pentagon transitions $\mathcal{P}$, this program needs an additional ingredient, the form factor transition $\mathcal{F}$

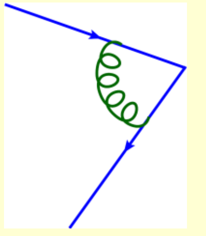
**Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367**

# Removing Amplitude (or Form Factor) Infrared Divergences

- On-shell amplitudes IR divergent due to long-range gluons



- Polygonal Wilson loops UV divergent at cusps, anomalous dimension $\Gamma_{\mathrm{cusp}}$



**– known to all orders in planar N=4 SYM:**
Beisert, Eden, Staudacher, hep-th/0610251

- Both removed by dividing by BDS-like ansatz
  Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708
- Normalized [MHV] amplitude is finite, dual conformal invariant, also uniquely (up to constant) maintains important symbol adjacency relations due to causality (Steinmann relations for 3-particle invariants):

$$\mathcal{E}(u_i) = \lim_{\epsilon\to 0} \frac{\mathcal{A}_6(s_{i,i+1},\epsilon)}{\mathcal{A}_6^{\mathrm{BDS-like}}(s_{i,i+1},\epsilon)} = \exp\left[\frac{\Gamma_{\mathrm{cusp}}}{4}\mathcal{E}^{(1)} + R_6\right]$$

remainder function

# BDS & BDS-like normalization for $\mathcal{F}_3$

$$\frac{\mathcal{F}_3}{\mathcal{F}_3^{\mathrm{MHV,\,tree}}} = \exp\left\{ \sum_{L=1}^{\infty} g^{2L} \left[ \left( \frac{\Gamma_{\mathrm{cusp}}^{(L)}}{4} + \mathcal{O}(\epsilon) \right) M^{1-\mathrm{loop}}(L\epsilon) + C^{(L)} + R^{(L)}(u,v,w) \right] \right\}$$

BDS ansatz

remainder function only a function of $u, v, w$; vanishes in all collinear limits, but no adjacency constraints

split 1-loop amplitude judiciously:

$$\frac{\mathcal{F}_3^{1-\mathrm{loop}}}{\mathcal{F}_3^{\mathrm{MHV,\,tree}}} \equiv M^{1-\mathrm{loop}}(\epsilon) = M(\epsilon) + \mathcal{E}^{(1)}(u,v,w)$$

$$M(\epsilon) = -\frac{1}{\epsilon^2}\sum_{i=1}^{3}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon} - \frac{7}{2}\zeta_2 + \sum^{3} \ldots$$

$$\mathcal{E}^{(1)}(u,v \ldots \left(1 - \frac{1}{v}\right) + \mathrm{Li}_2\left(1 - \frac{1}{w}\right)\right]$$

$$\mathcal{E}^{(1),u} + \mathcal{E}^{(1),1-u} = 0$$

$\mathcal{E}$ obeys "adjacency constraints"

Now div... by:

$$\frac{\mathcal{F}_3^{\mathrm{BDS-like}}}{\mathcal{F}_3^{\mathrm{MHV,\,tree}}} = \exp\left\{ \sum_{L=1}^{\infty} g^{2L} \left[ \left( \frac{\Gamma_{\mathrm{cusp}}}{4} + \mathcal{O}(\epsilon) \right) M(L\epsilon) + C^{(L)} \right] \right\} \Rightarrow$$

$$\mathcal{E} = \exp\left[ \frac{\Gamma_{\mathrm{cusp}}}{4} \mathcal{E}^{(1)} + R \right]$$

# Branch cut conditions

All massless particles

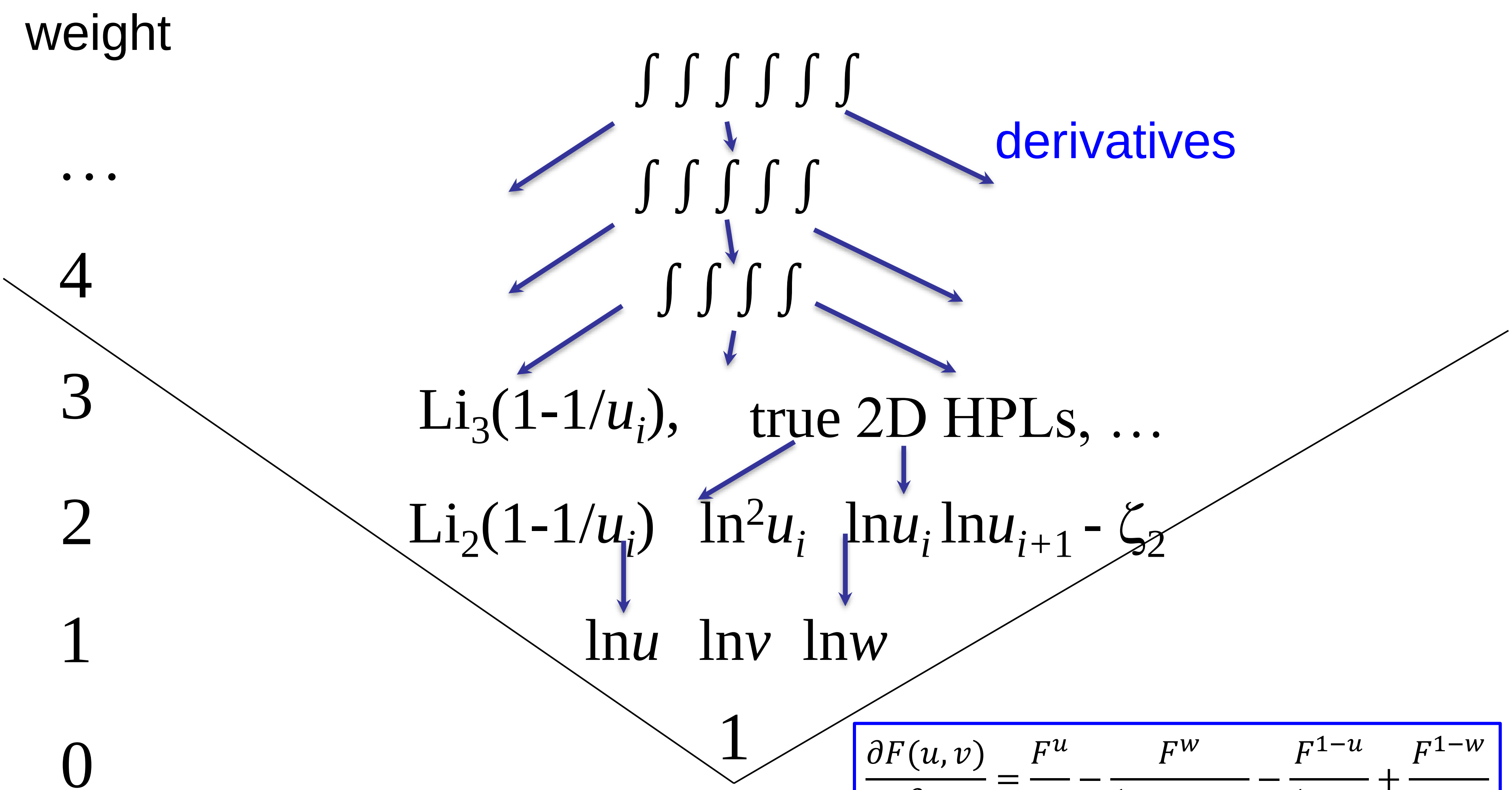
→ all branch cuts start at origin in $s_{i,i+1}$ , $s_{123}$

→ Branch cuts all start from 0 or ∞ in $u = \frac{s_{12}}{s_{123}}$ or $v$ or $w$

→ Only 3 weight 1 functions, not 6: $\{\ln u, \ln v, \ln w\}$ ~~$\ln(1-u)$~~

- Derivatives commute with branch cuts
- Derivatives of higher weight functions must obey branch-cut condition too.
- But also need: $F^{1-u}(1, v, w)|_{v,w\to 0} = 0$
- Powerful constraint, but not powerful enough; number of functions $\propto 4^w$ vs. hexagon functions $\propto 1.8^w$

# Heuristic view of space




$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{1-u-v} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{u+v}$$

# This bootstrap works through 8 loops: even better at 8 than at 7!

| $L$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| symbols in $\mathcal{C}$ | 48 | 249 | 1290 | 6654 | 34219 | ???? | ???? |
| dihedral symmetry | 11 | 51 | 247 | 1219 | ???? | ???? | ???? |
| $(L-1)$ final entries | 5 | 9 | 20 | 44 | 86 | ??? | ??? |
| $L^{\text{th}}$ discontinuity | 2 | 5 | 17 | 38 | 75 | ??? | ?? |
| collinear limit | 0 | 1 | 2 | 8 | 19 | 70 | 6 |
| OPE $T^2 \ln^{L-1} T$ | 0 | 0 | 0 | 4 | 12 | 56 | 0 |
| OPE $T^2 \ln^{L-2} T$ | 0 | 0 | 0 | 0 | 0 | 36 | 0 |
| OPE $T^2 \ln^{L-3} T$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| OPE $T^2 \ln^{L-4} T$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| OPE $T^2 \ln^{L-5} T$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Table 4**: Number of parameters left when bootstrapping the form factor $\mathcal{E}^{(L)}$ at $L$-loop order in the function space $\mathcal{C}$ at symbol level, using all the conditions on the final $(L-1)$ entries, which can be deduced at $(L-1)$ loops.

# Number of (symbol-level) linearly independent $\{n, 1, \ldots, 1\}$ coproducts ($2L - n$ derivatives)

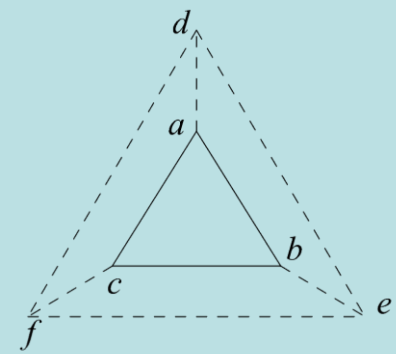
| weight $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $L=1$ | 1 | 3 | 1 | | | | | | | | | | | | | | |
| $L=2$ | 1 | 3 | 6 | 3 | 1 | | | | | | | | | | | | |
| $L=3$ | 1 | 3 | 9 | 12 | 6 | 3 | 1 | | | | | | | | | | |
| $L=4$ | 1 | 3 | 9 | 21 | 24 | 12 | 6 | 3 | 1 | | | | | | | | |
| $L=5$ | 1 | 3 | 9 | 21 | 46 | 45 | 24 | 12 | 6 | 3 | 1 | | | | | | |
| $L=6$ | 1 | 3 | 9 | 21 | 48 | 99 | 85 | 45 | 24 | 12 | 6 | 3 | 1 | | | | |
| $L=7$ | 1 | 3 | 9 | 21 | 48 | 108 | 236 | 155 | 85 | 45 | 24 | 12 | 6 | 3 | 1 | | |
| $L=8$ | 1 | 3 | 9 | 21 | 48 | 108 | 242 | 466 | 279 | 155 | 85 | 45 | 24 | 12 | 6 | 3 | 1 |

- Properly normalized $L$ loop N=4 form factors $\mathcal{E}^{(L)}$ belong to a small space $\mathcal{C}$, dimension saturates on left
- $\mathcal{E}^{(L)}$ also obeys multiple-final-entry relations, saturation on right

# Structure of $C$

- Switch to better alphabet, Caron-Huot, LD, McLeod, von Hippel, 1609.00669
- $\{\, a = \frac{u}{vw}, b = \frac{v}{wu}, c = \frac{w}{uv}, d = \frac{1-u}{u}, e = \frac{1-v}{v}, f = \frac{1-w}{w} \}$
- Inspecting symbols of $\mathcal{E}^{(L)}$,

  we find 12 non-adjacent pairs(!):




$$F^{X,Y} = 0$$

where $\{X, Y\} \in \{a,d\},\{b,e\},\{c,f\},\{d,e\},\{e,f\},\{f,d\}$

(+reverse)

- Plus 3 more double coproduct relations (integrability),

$$F^{a,b} + F^{a,c} - F^{b,a} - F^{c,a} = 0,$$
$$F^{c,a} + F^{c,b} - F^{a,c} - F^{b,c} = 0,$$
$$F^{d,b} - F^{d,c} - F^{b,d} + F^{c,d} + F^{e,c} - F^{e,a} - F^{c,e} + F^{a,e} + F^{f,a} - F^{f,b} - F^{a,f} + F^{b,f}$$
$$+ 4(F^{c,b} - F^{b,c}) = 0,$$

- for a total of 12 + 3 = 15 pair relations. Number of independent pairs: $6^2 - 15 = 36 - 15 = 21$

# Plus new triple relations!

$$F^{a,a,b} + F^{a,b,b} + F^{a,c,b} = 0$$

+ dihedral images (6)

- 4 are independent of the 150 independent triple relations that come from the 15 pair relations, promoted to triples by tacking on any other letter (e.g. $F^{a,d,X} = 0 = F^{X,a,d}$)
- Unlike $n = 6$ gluon amplitudes, where no new triples appear. ($n = 7$ maybe…)

# Empirical multi-final entry relations

1. $\mathcal{E}^{a} = 0$ (plus dihedral images)

2. $\mathcal{E}^{a,e} = \mathcal{E}^{a,f}$ (plus …)

3. $\mathcal{E}^{a,b,d} = 0,\qquad \mathcal{E}^{a,e,e} = -\,\mathcal{E}^{a,f,f},$

$$\mathcal{E}^{e,a,f} = \mathcal{E}^{f,a,f} - \mathcal{E}^{a,f,f}$$

4. …

# Simplicity of low loop symbols

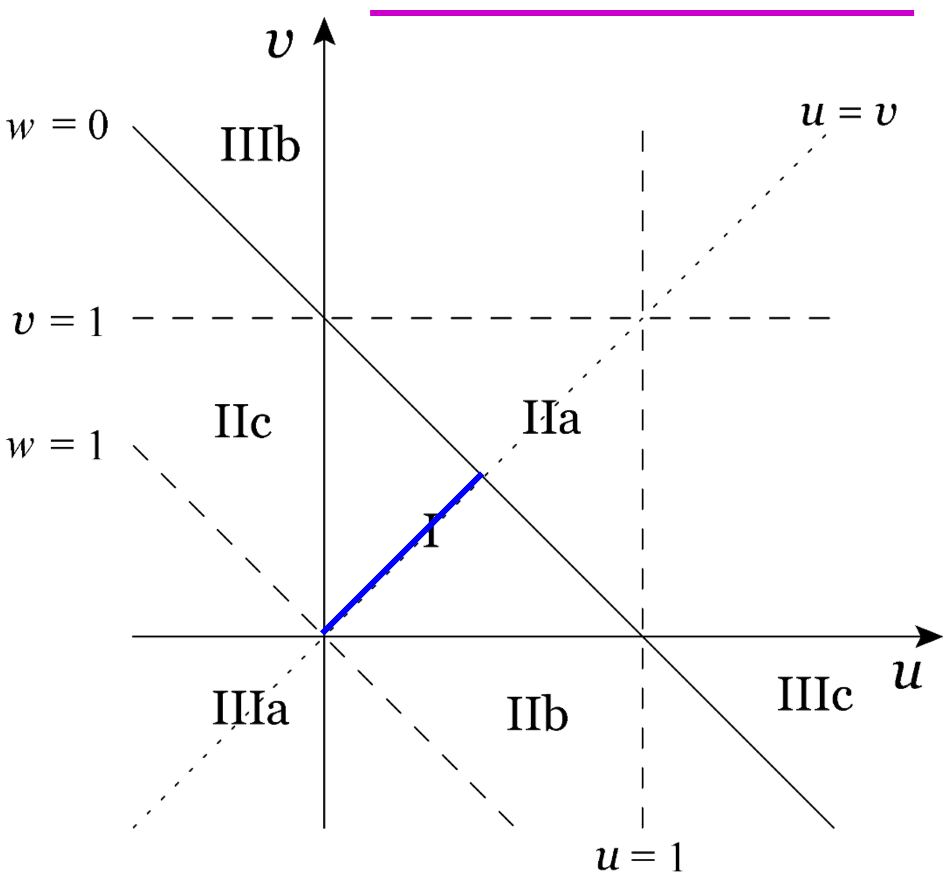
$$S[\mathcal{E}^{(1)}] = (-1)\, b \otimes d + \text{dihedral}$$

$$S[\mathcal{E}^{(2)}] = 4\, b \otimes d \otimes d \otimes d + 2\, b \otimes b \otimes b \otimes d$$
$$+\text{dihedral}$$

$$S[\mathcal{E}^{(3)}] = -48\, b \otimes d \otimes d \otimes d \otimes d \otimes d$$
$$+200\ \mathit{more\ terms} + \text{dihedral}$$

| loop order $L$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| terms in $S[\mathcal{E}^{(L)}]$ | 6 | 12 | 636 | 11,208 | 263,880 | 4,916,466 | 97,594,968 | ??? |

# Some numerics

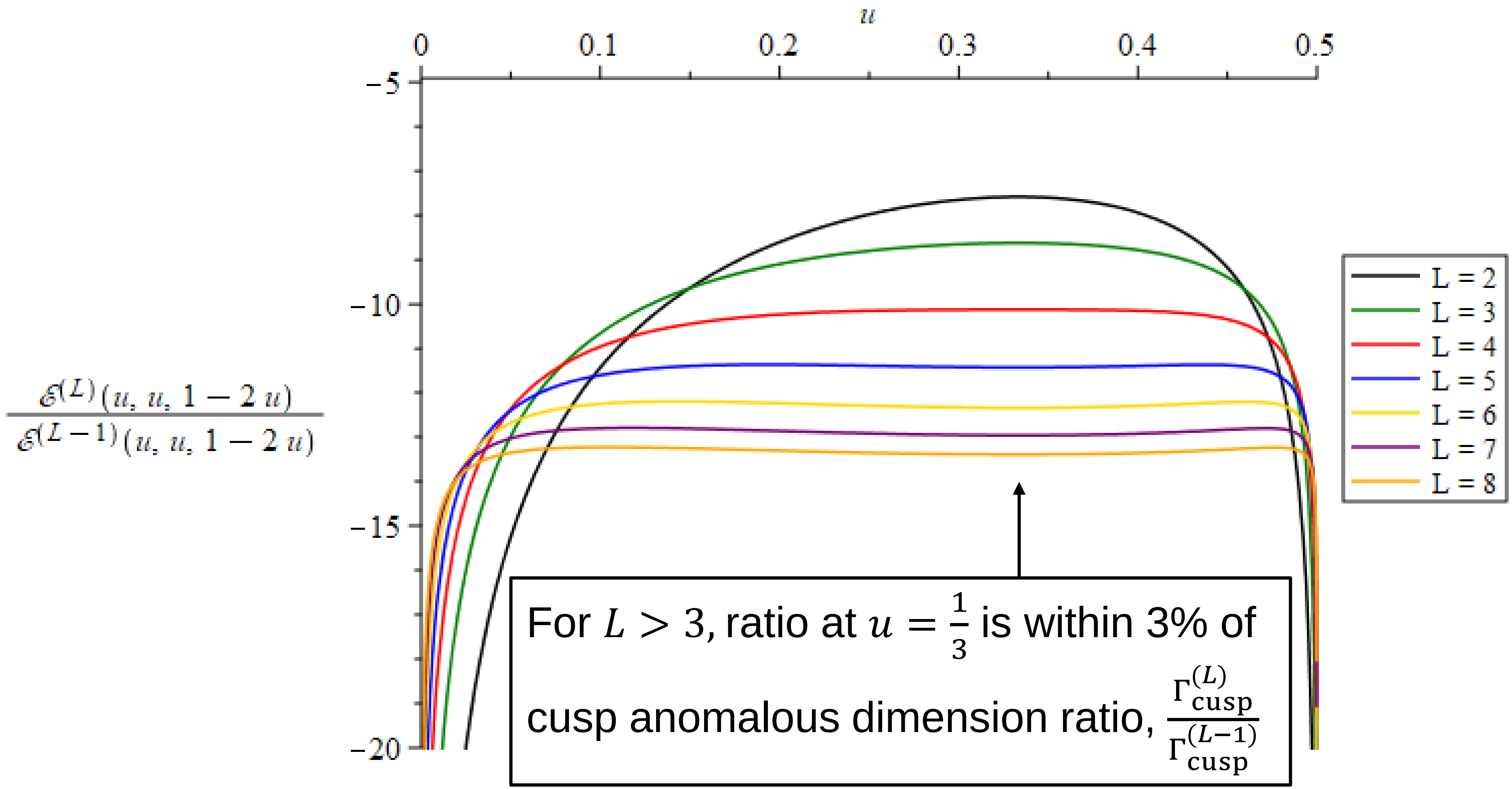



I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

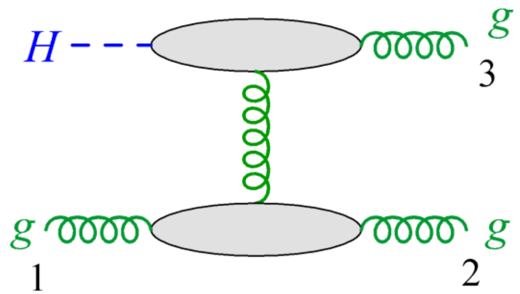
IIIa,b,c = scattering / timelike operator

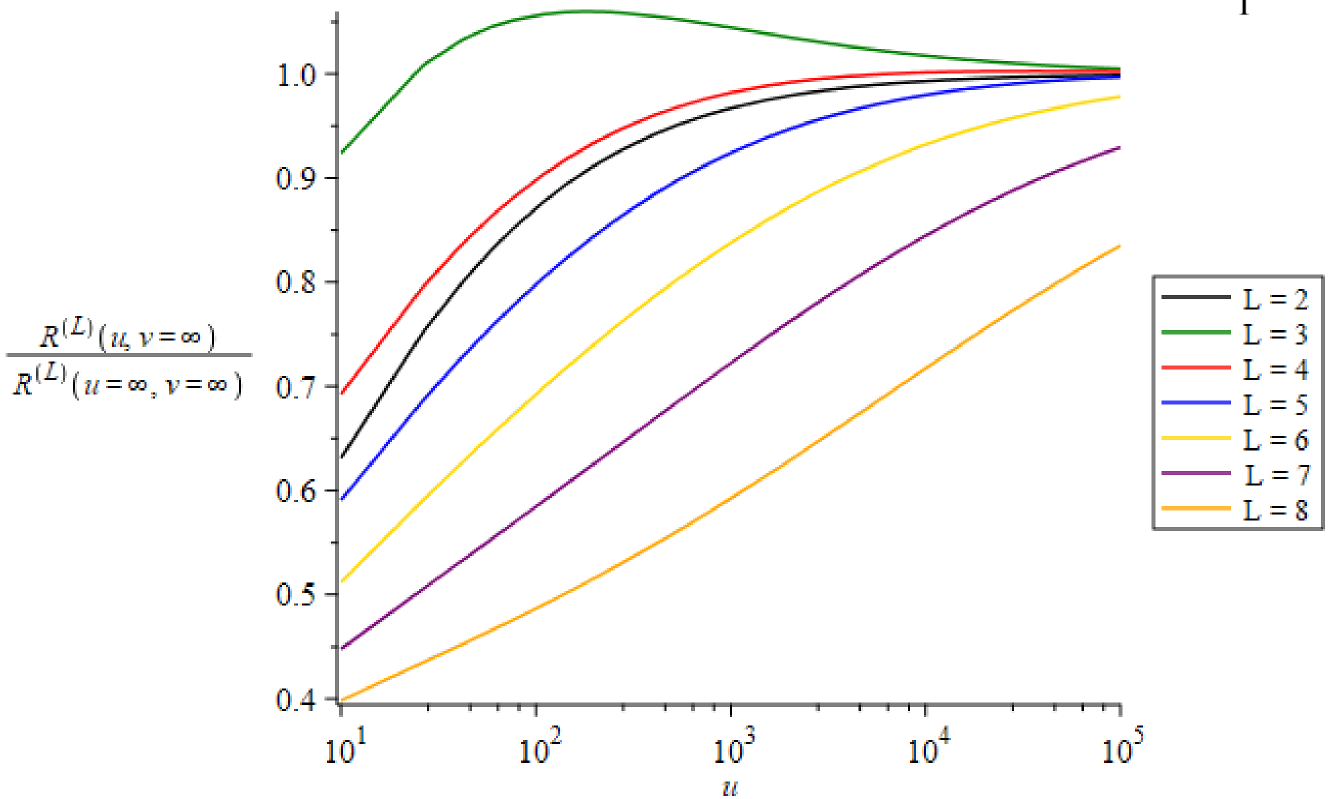
# Euclidean Region



# Real "impact factor" appears in space-like Regge limit, $v \to \infty$

Remainder function $R$ is nontrivial function of $u = \frac{s_{12}}{m_H^2}$ as $s_{23} \to \infty$

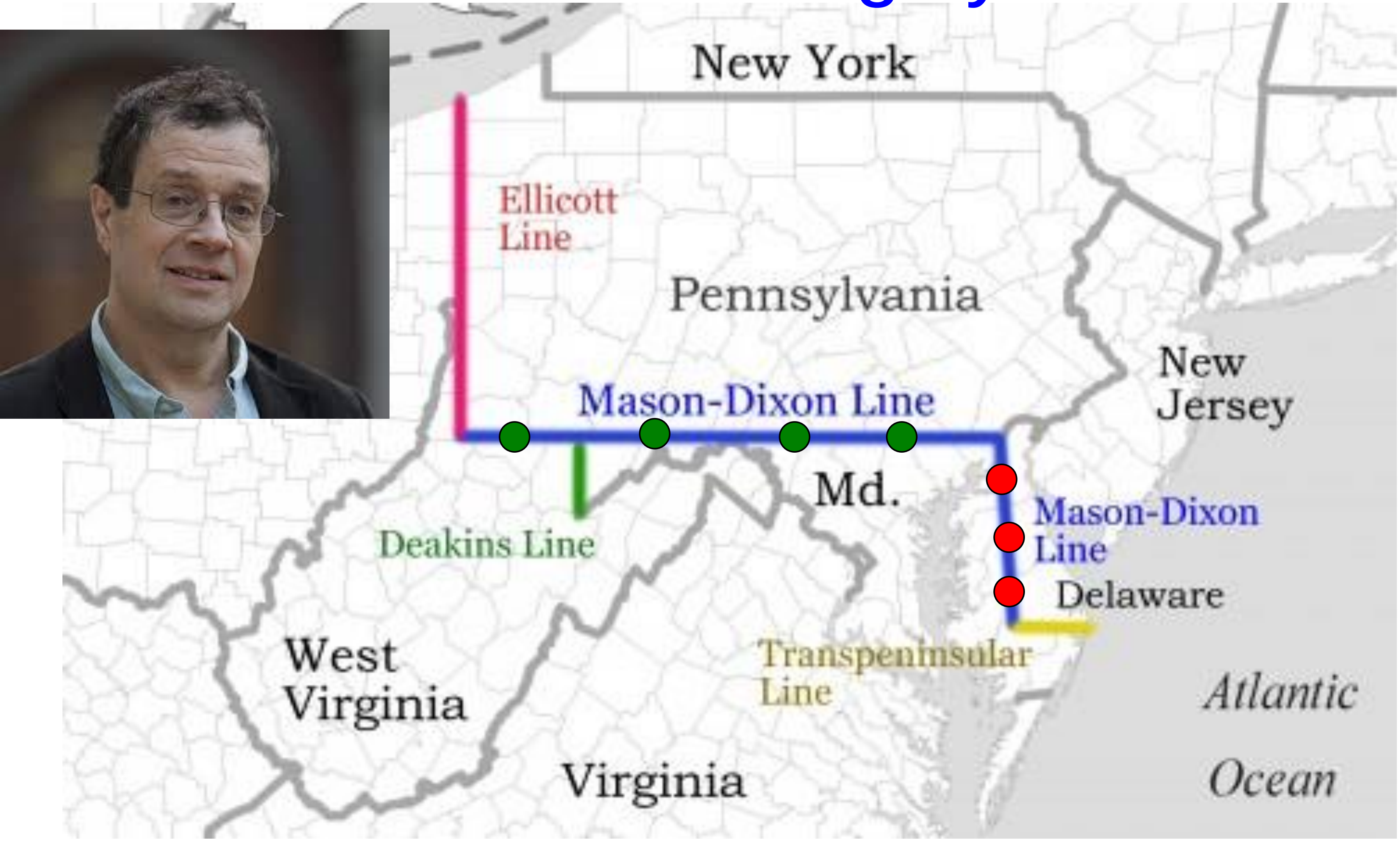


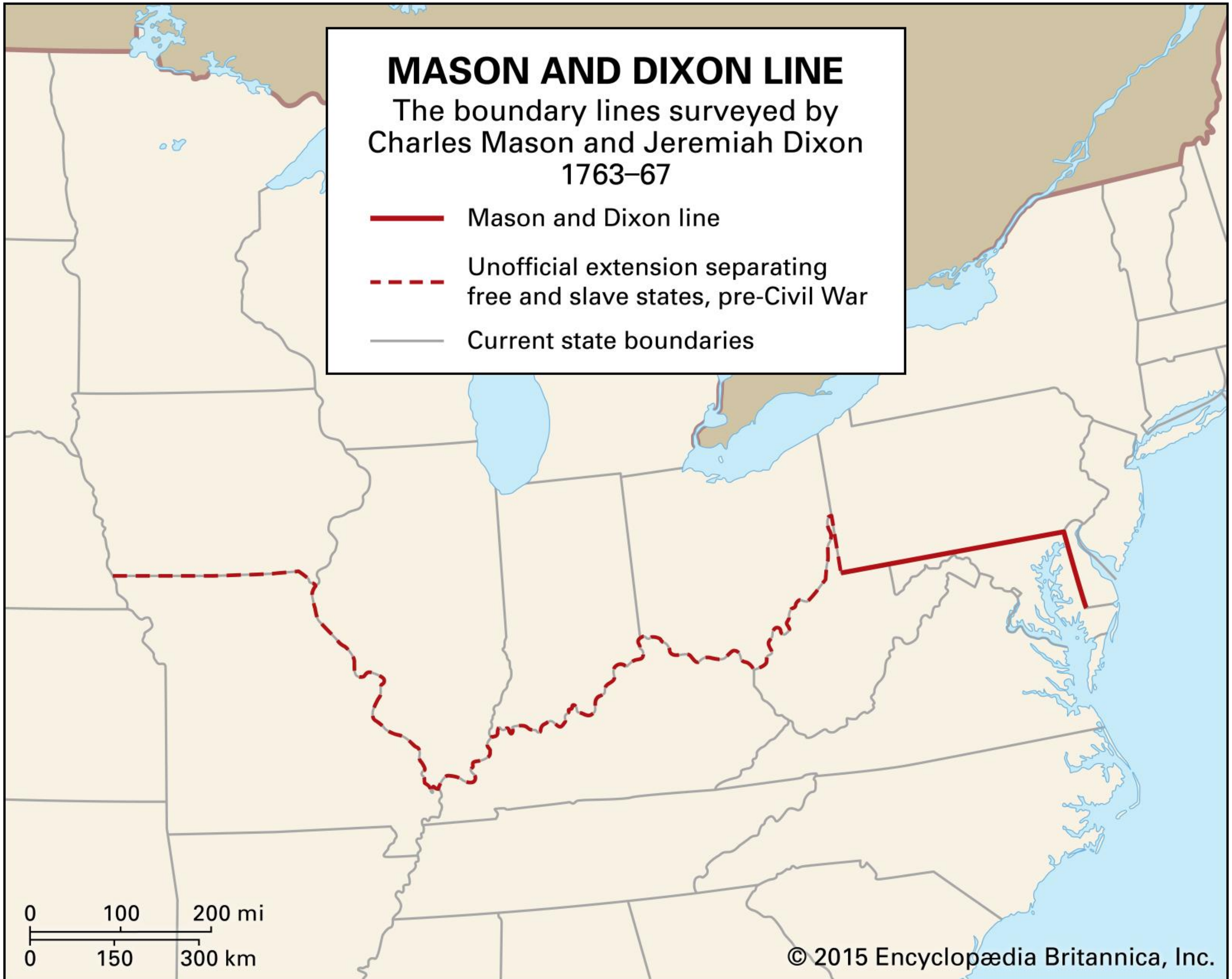
# Physics Summary & Outlook

- Form factors as well as scattering amplitudes in planar N=4 SYM can now be **bootstrapped** to high loop order
- Rich information about many different kinematic limits
- Great synergy with pentagon/FFOPE methods: our perturbative information also aided in the construction of the form factor transition $\mathcal{F}$ beyond leading order
- Can we go to finite coupling for generic kinematics? What are the right finite-coupling functions? Clues from OPE/integrability?
- Lessons for QCD?

# Lionel Summary:
# MHV = Mason Highly Virtuous



# Linear Extrapolations Can Be Dangerous



# Happy Birthday Lionel!