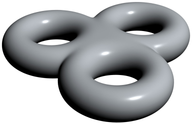
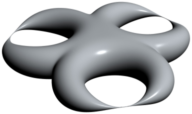
Loop Integrands from the Ambitwistor String

Yvonne Geyer

Chulalongkorn University
Bangkok





Twistor Theory and Beyond

arXiv:1507.00321, 1511.06315, 1607.08887, 1805.05344
with Lionel Mason, Ricardo Monteiro and Piotr Tourkine

arXiv:2106.03968
with Ricardo Monteiro and Ricardo Stark-Muchão

arXiv:1311.2564v1 [hep-th] 11 Nov 2013

Ambitwistor strings and the scattering equations

Lionel Mason* **and David Skinner†**

*The Mathematical Institute,
Andrew Wiles Building,
Woodstock Road, Oxford OX2
United Kingdom

†Department of Applied Mathe
Wilberforce Road, Cambridge
United Kingdom

ABSTRACT: We show that s
which only the massless part

16 Mar 2016

One-loop amplitudes on the Riemann sphere

Yvonne Geyer[1], Lionel Mason[1], Ricardo Monteiro[2], Piotr Tourkine[3]

[1]**Mathematical Institute, University of Oxford, Woodstock Road, Oxford OX2 6GG, UK**
[2]**CERN, Theory Group, Geneva, Switzerland**
sity of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

he scattering equations provide a powerful framework for the study
plitudes in a variety of theories. Their derivation from ambitwistor
l to proposals for formulae at one loop on a torus for 10 dimensional
l we recently showed how these can be reduced to the Riemann sphere
her theories
sions at one
ons. Firstly,
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, we give a

Loop Integrands for Scattering Amplitudes from the Riemann Sphere

Yvonne Geyer†, Lionel Mason†, Ricardo Monteiro†, Piotr Tourkine‡

†Mathematical Institute, University of Oxford, Woodstock Road, Oxford OX2 6GG, UK
‡DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

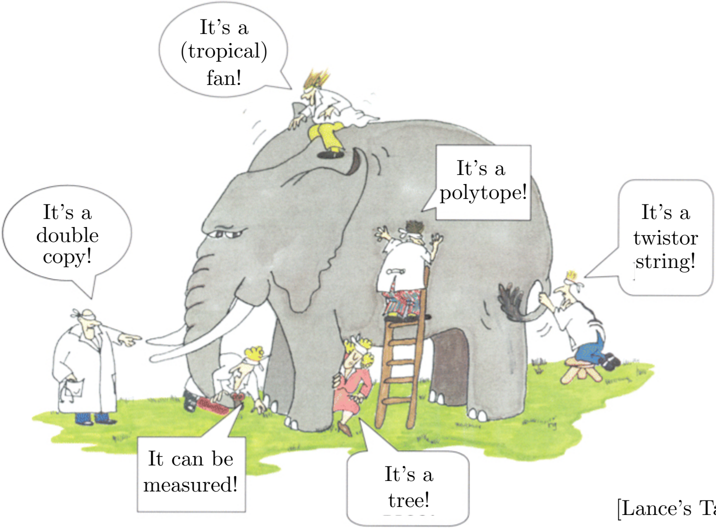
The scattering equations on the Riemann sphere gi
gauge theory and gravity amplitudes. Adamo, Casali
for supergravity amplitudes based on scattering equa
to transform this into a formula on the Riemann sph
integrands on the Riemann sphere that promises to hav
ing equations that depend on the loop momentum. W
low points, for supergravity and super-Yang-Mills amp
Finally, we show that the off-shell scattering equations
we give a proposal for the all-loop integrands for super

Two-Loop Scattering Amplitudes from the Riemann Sphere

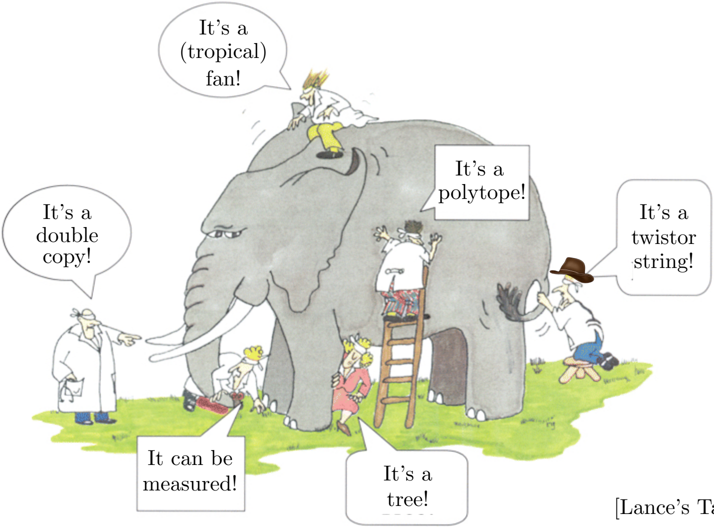
Yvonne Geyer[1], Lionel Mason[1], Ricardo Monteiro[2], Piotr Tourkine[3]

[1]Mathematical Institute, University of Oxford, Woodstock Road, Oxford OX2 6GG, UK
[2]Theoretical Physics Department, CERN, Geneva, Switzerland
[3]DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

The scattering equations give striking formulae for massless scattering amplitudes at tree level and, as shown recently, at one loop. The progress at loop level was based on ambitwistor string theory, which naturally yields the scattering equations. We proposed that, for ambitwistor strings, the standard loop expansion in terms of the genus of the worldsheet is equivalent to an expansion in terms of nodes of a Riemann sphere, with the nodes carrying the loop momenta. In this paper, we show how to obtain two-loop scattering equations with the correct factorization properties. We adapt genus-two integrands from the ambitwistor string to the nodal Riemann sphere and show that these yield correct answers, by matching standard results for the four-point two-loop amplitudes of maximal supergravity and super-Yang-Mills theory. In the Yang-Mills case, this requires the loop analogue of the Parke-Taylor factor carrying the colour dependence, which includes non-planar contributions.


It's a (tropical) fan!
It's a polytope!
It's a double copy!
It's a twistor string!
It can be measured!
It's a tree!

[Lance's Talk]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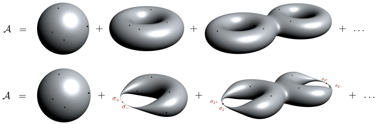
Outline

- Worldsheet models for Field Theory
  - sugra amplitudes
    $=$ ambitwistor string correlators
  - simplification via residue theorem



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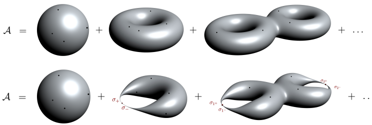

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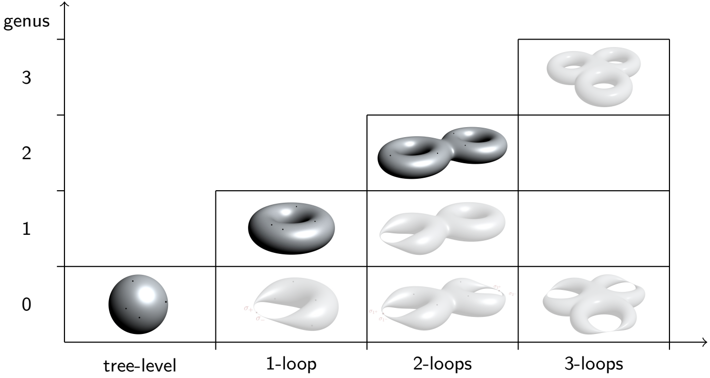

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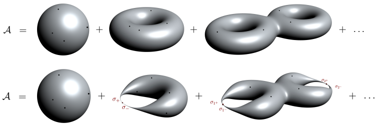

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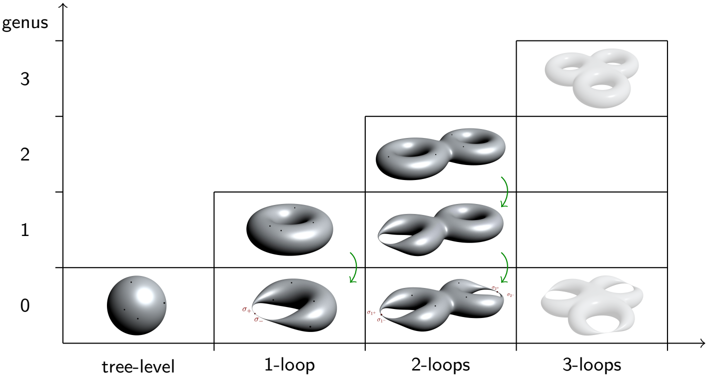

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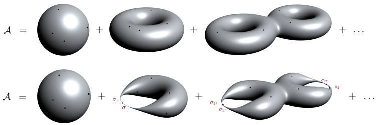

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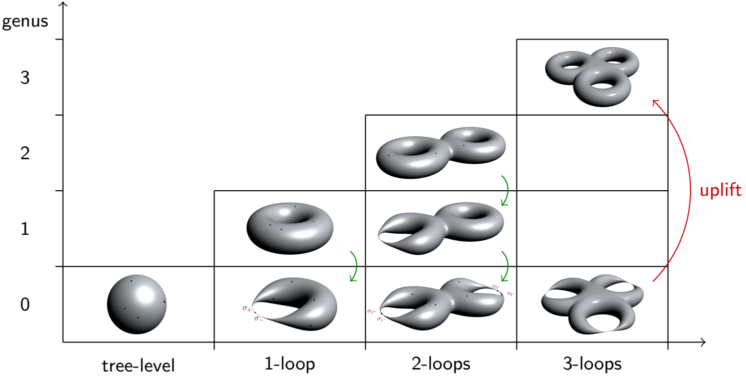


Worldsheet Model: Ambitwistor String

Worldsheet models for Field Theory

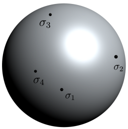
Ambitwistor string

[Mason,Skinner '13; c.f. Berkovits]

$\overline{D} = \bar{\partial} + e\partial$

no α'!

$$S_{\mathbb{A}} = \frac{1}{2\pi}\int_{\Sigma} P\cdot \bar{D}X - \frac{\tilde{e}}{2}P^2 + S_M$$



- *chiral* worldsheet theory: $X^\mu \in \Omega^0(\Sigma)$, $P_\mu \in \Omega^0(K_\Sigma)$
- 'RNS' model: $S_M = S_{\psi_1} + S_{\psi_2}$ (others possible)
  - action: $S_\psi = \int \psi\cdot\overline{D}\psi + \chi\, P\cdot\psi$ with $\psi^\mu_{r=1,2} \in \Pi\Omega^0(K_\Sigma^{1/2})$
  - BRST: free, linear CFTs with $d_{\text{crit}} = 10$

Worldsheet models for Field Theory

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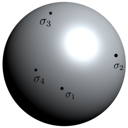

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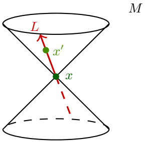
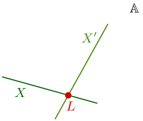
- target space: $\mathbb{A}$ = phase space of complexified null geodesics





Vector vs twistor representations of $\mathbb{A}$

- **Vector representation** [Mason,Skinner '13; c.f. Berkovits]

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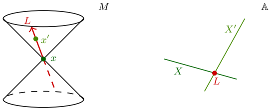

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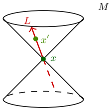

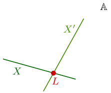

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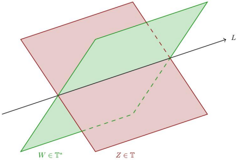
- **Twistor representation** [Witten, Berkovits '04, RSVW]

$$S = \int_{\Sigma} W\cdot\overline{D}Z - Z\cdot\overline{D}W + a\,Z\cdot W$$



  - 4d twistor / ambitwistor string
  - $Z\in\Omega^0(K_\Sigma^{1/2}\otimes\mathbb{T})$, $W\in\Omega^0(K_\Sigma^{1/2}\otimes\mathbb{T}^*)$
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Vector vs twistor representations of $\mathbb{A}$

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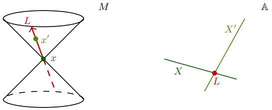

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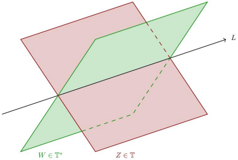

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- related models in $d=5,6$ [YG, Mason, Skinner '20; Albonico, YG, Mason WiP]

Spectrum and correlators

- Spectrum: type II supergravity

$$V_{\mathsf{NS}} = c\tilde{c}\,\delta(\gamma_1)\delta(\gamma_2)\,\epsilon_{\mu\nu}\psi_1^\mu\psi_2^\nu\,e^{ik\cdot X} \qquad \text{with} \quad k^2 = \epsilon_{\mu\nu}k^\nu = \epsilon_{\mu\nu}k^\mu = 0$$

$\Rightarrow$ worldsheet theory for QFT amplitudes

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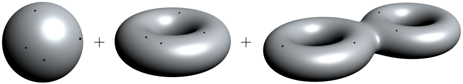
$\mathcal{A} =$  $+ \ \ldots$

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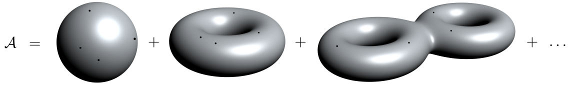

- tree-level = CHY amplitude [Cachazo, He, Yuan '13]

$$\mathcal{A}_n^{(0)} = \left\langle \prod_{i=1}^{n} V(\sigma_i) \right\rangle = \int_{\mathfrak{M}_{0,n}} \frac{d^n\sigma}{\text{vol}\,\mathsf{SL}(2,\mathbb{C})} \prod_i{}' \bar{\delta}\,(\mathcal{E}_i)\,\mathcal{I}_n$$

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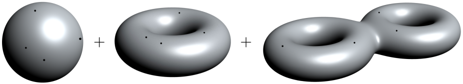
$\mathcal{A} =$  $+ \ \dots$

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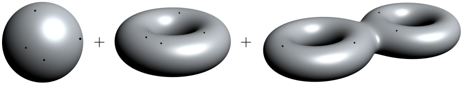
$\mathcal{A}$ =  + …

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  - P localizes onto EoM: $\bar{\partial}P_\mu = \sum_i k_{i\,\mu}\bar{\delta}(\sigma-\sigma_i)\,d\sigma$
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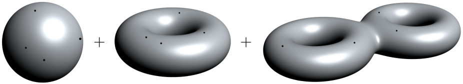

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  - $P^2 = 0 \quad \leftrightarrow \quad$ scattering equations $\mathcal{E}_i = \text{Res}_{\sigma_i}P^2 = 2k_i\cdot P(\sigma_i)$

CHY amplitudes [Cachazo, He, Yuan '13]

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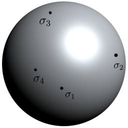

- <u>Measure</u>
  - integral over $\mathfrak{M}_{0,n}$

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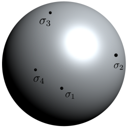

$\bar{\delta}(z) = \bar{\partial}\left(\frac{1}{2\pi i\, z}\right)$

- **Measure**
  - integral over $\mathfrak{M}_{0,n}$
  - fully localized on scattering equations

momenta $k_i \in \mathbb{R}^d$
$k_i^2 = 0$

$$\mathcal{E}_i = \mathrm{Res}_{\sigma_i} P^2 = \sum_{j \neq i} \frac{2k_i \cdot k_j}{\sigma_i - \sigma_j} \qquad \text{with} \quad P_\mu(\sigma) = \sum_i \frac{k_{i\,\mu}}{\sigma - \sigma_i} d\sigma$$

$\sigma_i \in \mathbb{CP}^1$

CHY amplitudes

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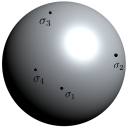

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- Integrand $\mathcal{I}_n^{(0)}$
  - specifies theory

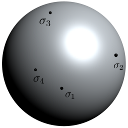
Yang-Mills, $\epsilon_\mu\, t^{\mathfrak{a}}\, e^{ik\cdot X}$: $\mathcal{I}_{\text{YM}} = \mathcal{I}_{\text{kin}}(\sigma_i, k_i, \epsilon_i) \times \mathcal{C}(\sigma_i, \mathfrak{a}_i)$

Gravity, $\epsilon_\mu\, \tilde{\epsilon}_\nu\, e^{ik\cdot X}$: $\mathcal{I}_{\text{grav}} = \mathcal{I}_{\text{kin}}(\sigma_i, k_i, \epsilon_i) \times \mathcal{I}_{\text{kin}}(\sigma_i, k_i, \tilde{\epsilon}_i)$

CHY amplitudes

[Cachazo, He, Yuan '13]

$$\mathcal{A}_n^{(0)} = \int_{\mathfrak{M}_{0,n}} \frac{d^n \sigma}{\text{vol}\, \mathsf{SL}(2,\mathbb{C})} \prod_i{}' \bar{\delta}\left(\mathcal{E}_i\right) \boxed{\mathcal{I}_n^{(0)}}$$



- Measure
  - integral over $\mathfrak{M}_{0,n}$
  - fully localized on scattering equations

$$\mathcal{E}_i = \text{Res}_{\sigma_i} P^2 = \sum_{j \neq i} \frac{2k_i \cdot k_j}{\sigma_i - \sigma_j} \qquad \text{with} \quad P_\mu(\sigma) = \sum_i \frac{k_{i\,\mu}}{\sigma - \sigma_i} d\sigma$$

- Integrand $\mathcal{I}_n^{(0)}$
  - specifies theory

$$\text{Yang-Mills,} \quad \epsilon_\mu\, t^{\mathfrak{a}}\, e^{ik\cdot X} : \qquad \mathcal{I}_{\mathsf{YM}} = \mathcal{I}_{\mathsf{kin}}(\sigma_i, k_i, \epsilon_i) \times \mathcal{C}(\sigma_i, \mathfrak{a}_i)$$

$$\text{Gravity,} \quad \epsilon_\mu\, \tilde{\epsilon}_\nu\, e^{ik\cdot X} : \qquad \mathcal{I}_{\mathsf{grav}} = \mathcal{I}_{\mathsf{kin}}(\sigma_i, k_i, \epsilon_i) \times \mathcal{I}_{\mathsf{kin}}(\sigma_i, k_i, \tilde{\epsilon}_i)$$

  - 'woldsheet double copy' c.f [Kawai,Lewellen,Tye '86; Bern,Carrasco,Johansson '08]

Gravity $\sim$ YM^2

Integrand and the Colour-kinematics duality

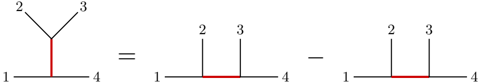
- Colour-kinematics duality

[Bern, Carrasco, Johansson '08]

$f^{\mathfrak{a}_1\mathfrak{a}_2\cdot} f^{\cdot\mathfrak{a}_3\cdot}\ldots$

$$\mathcal{A}_{\mathrm{YM}} = \sum_{\alpha\in\Gamma_n} \frac{N_\alpha(\epsilon)\, C_\alpha(\mathfrak{a})}{D_\alpha} \qquad \mathcal{A}_{\mathrm{grav}} = \sum_{\alpha\in\Gamma_n} \frac{N_\alpha(\epsilon)\, N_\alpha(\tilde{\epsilon})}{D_\alpha}$$

Kinematic numerators N_α satisfying same Jacobi's as C_α:



Integrand and the Colour-kinematics duality

- Colour-kinematics duality

  [Bern,Carrasco,Johansson '08]

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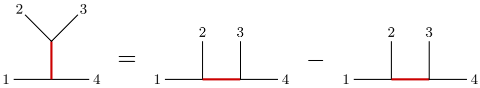

- CHY integrands

  [CHY '13; Bjerrum-Bohr et.al. '16, ...]

Integrand and the Colour-kinematics duality

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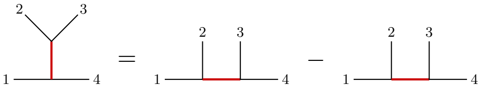

- **CHY integrands**

  [CHY '13; Bjerrum-Bohr et.al. '16, ...]

  - Connection to BCJ:

$$\mathcal{C}(\mathfrak{a}) = \sum_{\alpha\in S_{n-2}} \frac{C_{\alpha}(\mathfrak{a})}{(1\,\alpha\,n)} \qquad\qquad \mathcal{I}_{\text{kin}}(\epsilon) \overset{\text{SE}}{=} \sum_{\alpha\in S_{n-2}} \frac{N_{\alpha}(\epsilon)}{(1\,\alpha\,n)}$$

Parke-Taylor factor

$(12\ldots n) := \sigma_{12}\sigma_{23}\ldots\sigma_{n1}$

Integrand and the Colour-kinematics duality

- **Colour-kinematics duality**

[Bern,Carrasco,Johansson '08]

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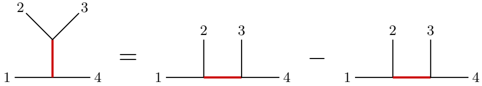

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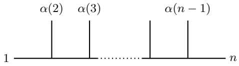
  - Colour C_α and BCJ numerators N_α for 'half-ladder' master diagrams



Integrand and the Colour-kinematics duality

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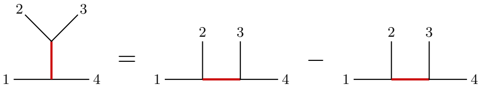

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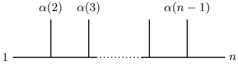
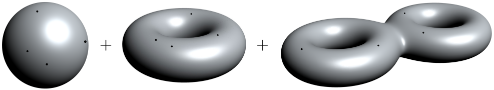


genus-g correlators = loop integrands

$\mathcal{A} =$



$+ \dots$

Genus-g correlator = loop int's

[Adamo, Casali, Skinner, Tourkine, YG, Mason, Monteiro '13-'18]

$g \leq 2$

$$\mathcal{A}_n^{(g)} = \left\langle \prod_{i=1}^{n} V(\sigma_i) \right\rangle_{\Sigma_g}$$

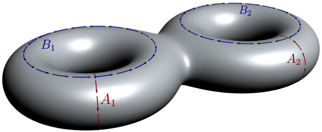
Genus-g correlator = loop int's

$g \leq 2$

$$\mathcal{A}_n^{(g)} = \left\langle \prod_{i=1}^{n} V(\sigma_i) \right\rangle_{\Sigma_g} = \int d^{10}\ell^I \int_{\mathfrak{M}_{g,n}} \prod_{I \leq J} d\Omega_{IJ}\, \bar{\delta}(u^{IJ}) \prod_i \bar{\delta}(\mathcal{E}_i)\, \mathcal{I}_n^{(g)}$$

- Moduli space $\mathfrak{M}_{g,n}$
  - homology basis: $\#(A_I, B_J) = \delta_{IJ}$

    modular group: $\mathrm{Sp}(4,\mathbb{Z})_{\#}$
  - holomorphic differentials ω_I



$$\delta_{IJ} = \oint_{A_I} \omega_J \qquad \Omega_{IJ} = \oint_{B_I} \omega_J$$

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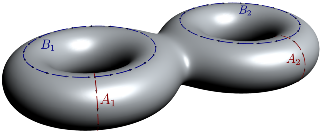

$$\delta_{IJ} = \oint_{A_I} \omega_J \qquad\qquad \Omega_{IJ} = \oint_{B_I} \omega_J$$

- Scattering equations
  - P determined by $\bar{\partial} P = \sum_i k_i\, \bar{\delta}(z - z_i)\, dz$

$$P_\mu(z) = 2\pi i\, \ell_\mu^I\, \omega_I(z) + \sum_i k_{i\,\mu}\, \omega_{i,*}(z)$$

hom. solution
loop momenta

merom. diff's $\omega_{[ij]}$
$\mathsf{Res}_{z_i} \omega_{ij} = 1$

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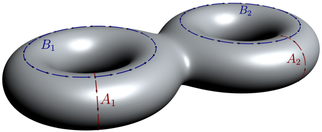

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  - scattering equations enforce $P^2(z) = 0$:

$$\boxed{\mathcal{E}_i = \mathrm{Res}_{z_i} P^2} \qquad \boxed{P^2\Big|_{\mathcal{E}_i = 0} = u^{IJ} \, \omega_I \omega_J}$$

Higher genus amplitude formulae

$$\mathcal{A}_n^{(g)} = \int d^{10}\ell^I \int_{\mathfrak{M}_{g,n}} \prod_{I \leq J} d\Omega_{IJ}\, \bar{\delta}(u^{IJ}) \prod_i \bar{\delta}(\mathcal{E}_i)\, \mathcal{I}_n^{(g)}$$

- Properties
  - modular invariance
  - localization on scattering equations

$$\dim \mathfrak{M}_{g,n} = \# \text{ SE's} = 3g - 3 + n$$

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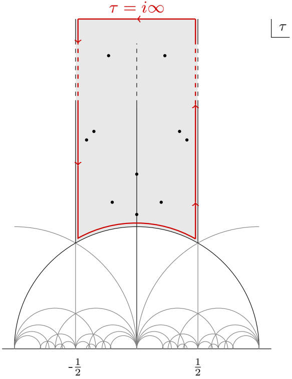
- Questions
  - loop integration UV divergent in $d=10$
  - calculation of loop integrand?

Field theory! How can we see that the integrand is rational?

Residue theorem to the nodal sphere

- Residue theorem on fundamental domain

  Look at $g = 1$:



solutions to $u = 0$
on $\mathcal{E}_i = 0$

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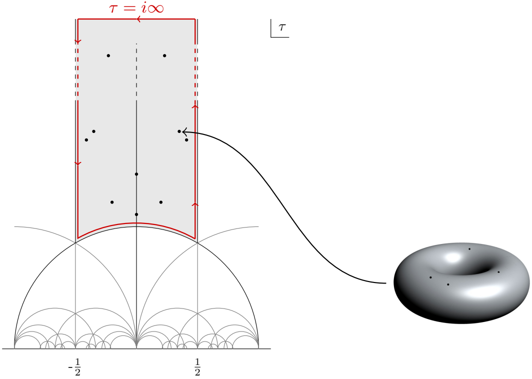

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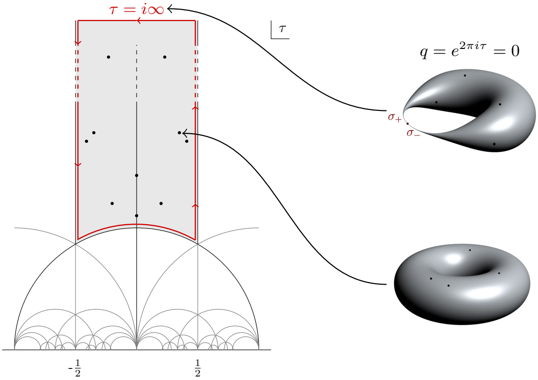

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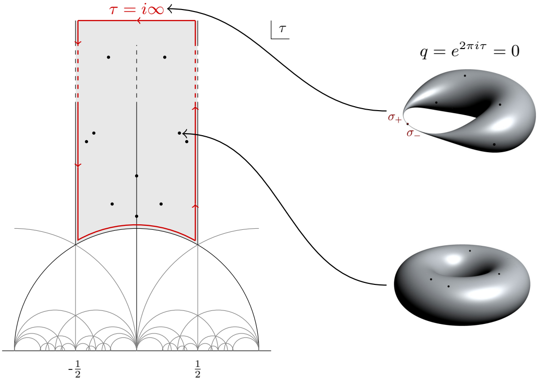
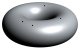


solutions to $u = 0$
on $\mathcal{E}_i = 0$

- Integrand localizes on nodal sphere

$$\mathfrak{I}_n^{(1)} = \int_{\mathfrak{M}_{1,n}} \frac{dq}{q}\, \bar{\delta}(u)\, \mathcal{I}^{(1)}(q)$$

$=$ 

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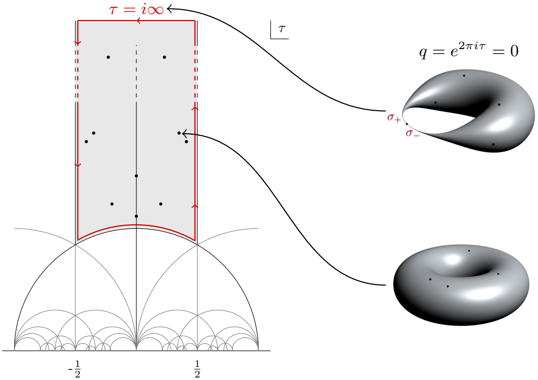

solutions to $u = 0$
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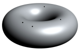
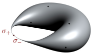
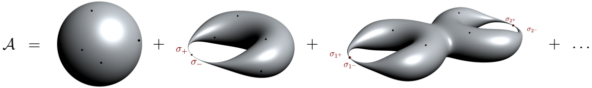
- Integrand localizes on nodal sphere

$$\mathfrak{I}_n^{(1)} = \int_{\mathfrak{M}_{1,n}} \frac{dq}{q}\,\bar{\delta}(u)\,\mathcal{I}^{(1)}(q) \overset{\text{res}}{=} -\int_{\mathfrak{M}_{1,n}} \frac{dq}{u}\,\bar{\delta}(q)\,\mathcal{I}^{(1)}(q) = -\frac{1}{\ell^2}\int_{\mathfrak{M}_{0,n+2}} \mathcal{I}^{(1)}(0)$$

$=$  $\overset{\text{res}}{=}$ 

loop expansion = nodal expansion



Loop amplitudes from the nodal sphere

$$\mathcal{A}_n^{(g)} = \int \frac{d^{10}\ell^I}{\prod(\ell^I)^2} \int_{\mathfrak{M}_{0,n+2g}} c^{(g)} \left(\mathcal{J}^{(g)}\,\mathcal{I}_L^{(g)}\right)\left(\mathcal{J}^{(g)}\,\mathcal{I}_R^{(g)}\right) \prod_{A=1}^{n+2g}{}' \bar{\delta}\,(\mathcal{E}_A)$$

- From residue theorem
  - traded localization on $P^2 = 0$ for $q_{II} = e^{i\pi\Omega_{II}} = 0$

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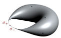
$$\prod_{I<J} \frac{dq_{IJ}}{q_{IJ}} = \frac{\mathcal{J}^{(g)}}{\mathrm{vol}\ \mathrm{SL}(2,\mathbb{C})} \qquad \mathcal{J}^{(g)} = J^{(g)} \prod_{I\pm} d\sigma_{I\pm}$$



$$J^{(1)} = \left(\sigma_{+-}\right)^{-2} \quad **$$

$$J^{(2)} = \left(\sigma_{1+2+}\sigma_{1+2-}\sigma_{1-2+}\sigma_{1-2-}\right)^{-1}$$

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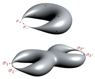
  - $c^{(g)}$ remnant of fundamental domain



$$c^{(1)} = 1$$

$$c^{(2)} = \frac{\sigma_{1+2-}\,\sigma_{1-2+}}{\sigma_{1+1-}\,\sigma_{2+2-}}$$

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  - $c^{(g)}$ remnant of fundamental domain
  - Scattering equations

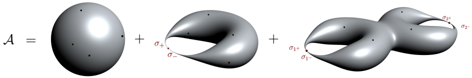
$$\boxed{\mathcal{E}_A = \text{Res}_{\sigma_A} \mathfrak{P}^{(g)}} \qquad \mathfrak{P}^{(g)} = P^2 - \left(\ell^I \omega_{I+I-}\right)^2 + L_{(g)}^{IJ}\, \omega_{I+I-}\omega_{J+J-}$$

$L_{(1)}^{IJ} = 0$

$L_{(2)}^{12} = \ell_1^2 + \ell_2^2$

Comments

- Different theories possible
  - dim. red. to $d \leq 10$
  - sugra and sYM (next slide)



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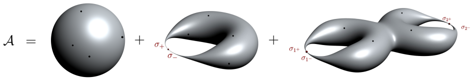

- Unorthodox integrand representation
  - 'linear' propagator factors of form $2\ell_I \cdot K + K^2$
  - related to standard representation by residue theorem

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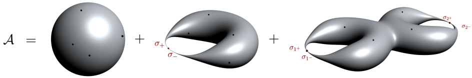

- **Unorthodox integrand representation**
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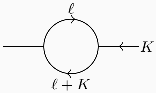
Example

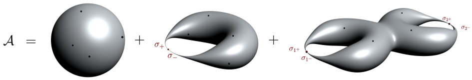


$$\frac{1}{\ell^2(\ell+K)^2} = \frac{1}{\ell^2(2\ell\cdot K + K^2)} + \frac{1}{(\ell+K)^2(-2\ell\cdot K - K^2)}$$

$$\xrightarrow{\text{shift}} \frac{1}{\ell^2}\left(\frac{1}{2\ell\cdot K + K^2} + \frac{1}{-2\ell\cdot K + K^2}\right)$$

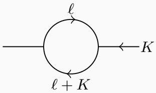
Comments

- **Different theories possible**
  - dim. red. to $d \leq 10$
  - sugra and sYM (next slide)



- **Unorthodox integrand representation**
  - 'linear' propagator factors of form $2\ell_I \cdot K + K^2$
  - related to standard representation by residue theorem

Example



$$\frac{1}{\ell^2(\ell+K)^2} = \frac{1}{\ell^2(2\ell\cdot K + K^2)} + \frac{1}{(\ell+K)^2(-2\ell\cdot K - K^2)}$$

$$\xrightarrow{\text{shift}} \frac{1}{\ell^2}\left(\frac{1}{2\ell\cdot K + K^2} + \frac{1}{-2\ell\cdot K + K^2}\right)$$

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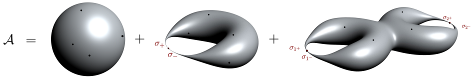

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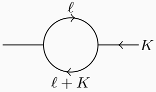

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- Physical interpretation of $\mathfrak{P}^{(g)}$ and $c^{(g)}$
  - $\mathfrak{P}^{(g)}$: correct poles in 'linear' representation
  - $c^{(g)}$: no unphysical poles

Colour-kinematics at loop level

[Bern, Carrasco, Johansson '10]

- BCJ double copy at g loops
  State-of-the-art: 5 loops
  [Bern, Carrasco, Chen, Edison, Johansson, Parra-Martinez, Roiban, Zeng '17-18]

$$\mathcal{A}^{(g)}_{\text{YM}} = \sum_{\alpha\in\Gamma^{(g)}_n} \int \prod_{I=1}^{g} d^D\ell^I \, \frac{N_{\alpha}(\epsilon)\, C_{\alpha}(\mathfrak{a})}{S_\alpha \, D_\alpha} \qquad \mathcal{A}^{(g)}_{\text{grav}} = \sum_{\alpha\in\Gamma^{(g)}_n} \int \prod_{I=1}^{g} d^D\ell^I \, \frac{N_{\alpha}(\epsilon)\, N_{\alpha}(\tilde{\epsilon})}{S_\alpha \, D_\alpha}$$

symmetry factor

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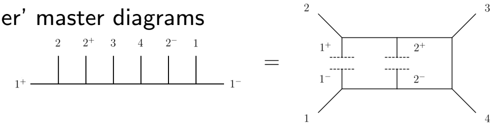
- **Nodal sphere**
  [He,Schlotterer,Zhang '16-'17; YG,Monteiro '17-19'; ...]
  - sYM from single copy

$$\mathcal{I}^{(g)}_{\text{YM}} = \mathcal{C}^{(g)} \left(\mathcal{J}^{(g)}\mathcal{I}^{(g)}_{\text{kin}}(\epsilon)\right) \qquad \mathcal{I}^{(g)}_{\text{grav}} = \left(\mathcal{J}^{(g)}\mathcal{I}^{(g)}_{\text{kin}}(\epsilon)\right)\left(\mathcal{J}^{(g)}\mathcal{I}^{(g)}_{\text{kin}}(\tilde\epsilon)\right)$$

  - Half-integrands in BCJ representation:

$$\mathcal{C}^{(g)} = \sum_{\alpha\,\in\, S_{n+2g-2}} \frac{C^{(g)}(1^+\alpha\,1^-)}{(1^+\,\alpha\,1^-)} \qquad \mathcal{J}^{(g)}\mathcal{I}^{(g)}_{\text{kin}} = \sum_{\alpha\,\in\, S_{n+2g-2}} \frac{N^{(g)}(1^+\alpha,1^-)}{(1^+\,\alpha\,1^-)}$$

  - 'half-ladder' master diagrams



Beyond two loops?

↓

Hard! Need new strategy.

Status of loop amplitudes: Superstring vs Supergravity

- Superstring

  4-pt amplitude, massless external states

  - tree-level and 1-loop: [Green, Schwarz '82]
  - 2-loops: [D'Hoker, Phong; Berkovits '05]
  - 3-loops: partial work [D'Hoker, Phong; Cacciatori, d.Piazza, v.Greemen] [Gomez, Mafra '13]

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GOAL: sugra advances ⟶ superstring

Tools: modern amplitudes techniques

- colour-kinematics duality
- ambitwistor string

4-pt amplitudes for $g \leq 2$

- Supergravity
  from ambitwistor string, higher genus and nodal sphere

$$\mathcal{A}_{\mathbb{A}}^{(g)} = \mathcal{R}^4 \int d^{10}\ell^I \quad \int_{\mathfrak{M}_{g,4}} \prod_{I \leq J} d\Omega_{IJ} \, \left(\mathcal{Y}_{\mathbb{A}}^{(g)}\right)^2 \prod_{i=1}^{4} \bar{\delta}(\mathcal{E}_i) \prod_{I \leq J} \bar{\delta}(u^{IJ})$$

$$= \mathcal{R}^4 \int \frac{d^{10}\ell^I}{\prod_I (\ell^I)^2} \int_{\mathfrak{M}_{0,4+2g}} c^{(g)} \left(\mathcal{J}^{(g)} \, \mathcal{Y}^{(g)}\right)^2 \prod_{A=1}^{4+2g}{}' \bar{\delta}(\mathcal{E}_A)$$

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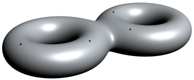
- <u>Type II superstring</u>
  chiral splitting form [D'Hoker,Phong '88, '05]



$$\mathcal{A}^{(g)}_{\mathbb{S}} = \mathcal{R}^4 \int_{\mathfrak{M}_{g,4}} \left|\prod_{I\leq J} d\Omega_{IJ}\right|^2 \int d^{10}\ell^I \; \left|\mathcal{Y}^{(g)}_{\mathbb{S}}\right|^2$$

$$\times \prod_{i<j} \left|E(z_i,z_j)\right|^{\alpha' s_{ij}/2} \left| e^{\frac{\alpha'}{2}\left(i\pi\, \Omega_{IJ}\, \ell^I \ell^J + 2\pi i \sum_j \ell^I k_j \int_{z_0}^{z_j} \omega_I\right)} \right|^2$$

Chiral integrands

- Observation 1:
  $\exists$ representations s.t.

$$\mathcal{Y}_{\mathbb{S}}^{(g)} \cong \mathcal{Y}_{\mathbb{A}}^{(g)} \qquad \text{mod} \;\; (d\text{-exact}, (\mathcal{E}, u))$$

  - superstring: mod d-exact terms, $\mathcal{Y}_{\mathbb{S}}^{(g)}$ independent of α'
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  Direct equality for BCJ representation

  $$\mathcal{Y}_{\mathbb{S}}^{(g)} = \mathcal{Y}_{\mathbb{A}}^{(g)} \qquad \text{s.t.} \qquad (2\pi i)^4 \mathcal{J}^{(g)} \mathcal{Y}^{(g)} = \sum_{\alpha \in S_{2+2g}} \frac{N_{\text{BCJ}}^{(g)}(1^+ \alpha \, 1^-)}{(1^+ \alpha \, 1^-)}$$

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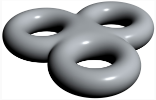
Assumptions

- straightforward extension of $\mathcal{A}_{\mathbb{S}}^{(g)}$ and $\mathcal{A}_{\mathbb{A}}^{(g)}$ to $g = 3$
  - Schottky problem for $g \geq 4$
  - non-projectedness of supermoduli space for $g \geq 5$
    [Donagi,Witten '13; Witten '15]
  - scattering equations on nodal sphere for $g \geq 4$?



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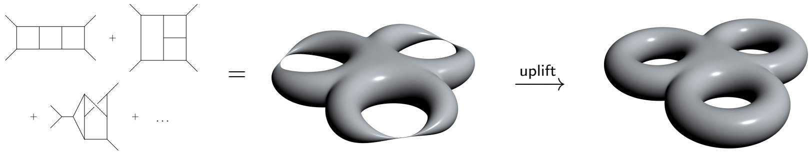
Strategy

[YG, Monteiro, Stark-Muchão '21]



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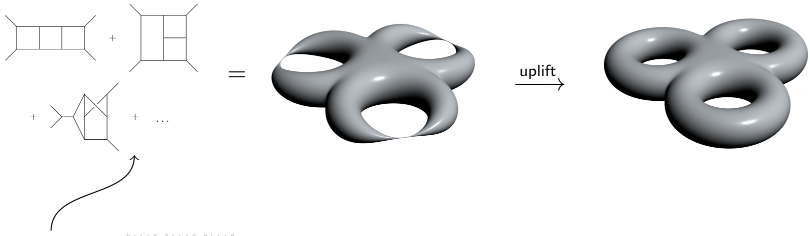
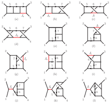


colour-kinematics
& BCJ numerators

Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s(-\tau_{35}+\tau_{45}+t)-t(\tau_{25}+\tau_{45})+u(\tau_{25}+\tau_{35})-s^2\right)/3$
(h)	$\left(s(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u)\right.$ $\left.+t(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17})+s^2\right)/3$
(i)	$\left(s(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t)\right.$ $\left.+t(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46})+u\,\tau_{25}+s^2\right)/3$
(j)–(l)	$s(t-u)/3$



(i) start with supergravity loop integrand in a BCJ representation, $N^{(g)}$

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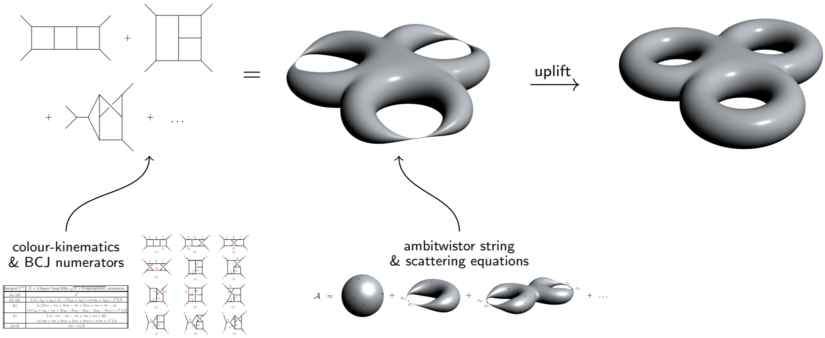

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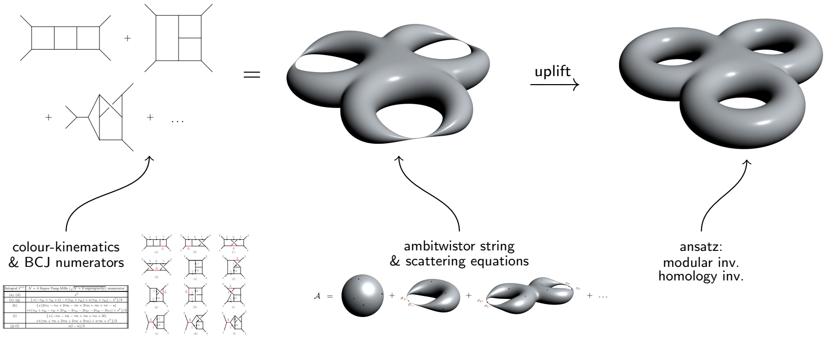

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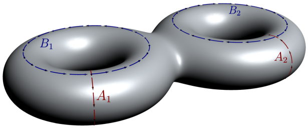
(iii) uplift to higher genus: $\boxed{\mathcal{Y}^{(g)}_{\mathbb{S}} = \mathcal{Y}^{(g)}_{\mathbb{A}}}$

- $\mathcal{Y}^{(g)}_{\mathbb{A}}\big|_{\text{nodal}} = \mathcal{Y}^{(g)}$
- modular invariance

Proof of concept:
2-loop integrand



2-loop superstring amplitude from field theory

Upshot: reproduces known $\mathcal{Y}^{(2)}_{\mathbb{S}}$
[D'Hoker,Phong '05; Berkovits '05]

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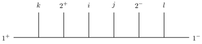
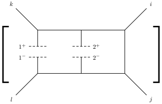
Numerators: $N^{(2)}\Big[$ 

$\Big] = N^{(2)}\Big[$ 

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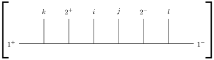
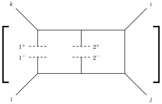
Numerators: $N^{(2)}$ 

$= N^{(2)}$ 

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(ii) Translate to nodal sphere
Using colour-kinematics / BCJ numerator relation on WS

$$(2\pi i)^4 \mathcal{J}^{(2)} \mathcal{Y}^{(2)} = \sum_{\alpha \in S_{4+2}} \frac{N^{(2)}(1^+ \alpha\, 1^-)}{(1^+ \alpha\, 1^-)}$$

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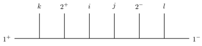
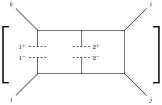
Numerators: $N^{(2)}\Bigg[$  $\Bigg] = N^{(2)}\Bigg[$  $\Bigg] = s_{ij}$

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(iii) Uplift to $g = 2$

Ansatz with correct modular weight:

- modular weight -2
- construct from $\Delta_{ij}^{(2)} = \varepsilon^{IJ} \omega_I(z_i) \omega_J(z_j)$

$\omega_I\big|_{\text{nodal}} = \frac{1}{2\pi i} \frac{\sigma_{I^+ I^-} \, d\sigma}{(\sigma - \sigma_{I^+})(\sigma - \sigma_{I^-})}$

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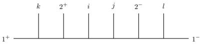
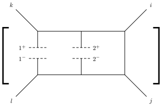
Numerators: $N^{(2)}\Big[$ 

$\Big] = N^{(2)}\Big[$ 

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Lessons from 2 loops: $g = 2$ ansatz

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- Properties
  - modular weight $\quad \mathrm{mod}(\mathcal{Y}_{\mathbb{S}}^{(g)}) = g - 4$
  - homology inv.
  - one-form in z_i

} Functional basis?

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- Objects on Σ_2
  - $\Delta_{i_1 \dots i_g}^{(g)}$ of weight $\quad \mathrm{mod}(\Delta^{(g)}) = -1$

    $$\Delta_{i_1 \dots i_g}^{(g)} = \det \omega_I(z_{i_J})$$
  - ring of mod forms $\quad \Psi_4, \Psi_6, \Psi_{10}, \Psi_{12}$

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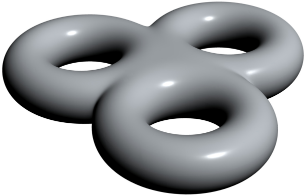
  - ring of mod forms $\quad \Psi_4, \Psi_6, \Psi_{10}, \Psi_{12}$
  - RNS superstring: $\quad \Xi_8[\delta]/\Psi_{10}$ chiral measure

    $S^\delta(z_i, z_j)$ Szegő kernels

$n \leq 3$pt: $\quad \sum_\delta \Xi_8[\delta]\,(S^\delta_{..})^n_{\text{cyc}} = 0$

4pt: $\quad \sum_\delta \frac{\Xi_8[\delta]}{\Psi_{10}}(S^\delta_{..})^4_{\text{cyc}} \cong \pi^4\left(\Delta_{12}^{(2)}\Delta_{34}^{(2)} - \Delta_{14}^{(2)}\Delta_{23}^{(2)}\right)$

New results:
3-loop integrand



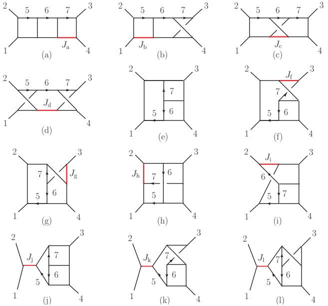
3 loops (i): BCJ representation

(i) Supergravity integrand in BCJ representation

[Bern,Carrasco,Johansson '10]

Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\big(s\,(-\tau_{35}+\tau_{45}+t) - t\,(\tau_{25}+\tau_{45}) + u\,(\tau_{25}+\tau_{35}) - s^2\big)/3$
(h)	$\big(s\,(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u)$ $+t\,(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}) + s^2\big)/3$
(i)	$\big(s\,(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t)$ $+t\,(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46}) + u\,\tau_{25} + s^2\big)/3$
(j)-(l)	$s(t-u)/3$

$(\tau_{ir} = 2k_i \cdot \ell_{r-4})$



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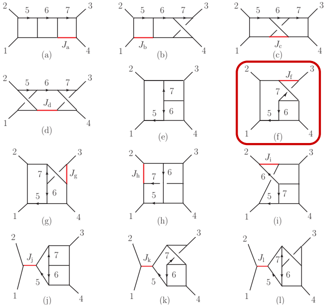
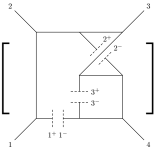


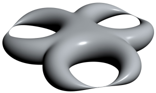


$$N^{(3)}\left[\;\cdot\;\right] = \frac{1}{3}\, s_{12}(s_{12}-s_{14}) + \frac{2}{3}\,\ell^1 \cdot k_2\,(s_{13}-s_{14})$$
$$+\frac{2}{3}\,\ell^1 \cdot \Big(k_3\,(s_{13}-s_{12}) + k_4\,(s_{12}-s_{14})\Big)$$

3 loops (ii): nodal sphere

(ii) Translate to nodal sphere
Use colour-kinematics on the worldsheet

$$(2\pi i)^4 \mathcal{J}^{(3)} \mathcal{Y}^{(3)} = \sum_{\alpha \in S_{6+2}} \frac{N^{(3)}(1^+ \alpha 1^-)}{(1^+ \alpha 1^-)}$$



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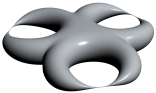

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- $\mathcal{J}^{(3)}$ from modular parameters

$$\prod_{I<J} \frac{dq_{IJ}}{q_{IJ}} = \frac{\mathcal{J}^{(g)}}{\text{vol SL}(2,\mathbb{C})} \qquad \mathcal{J}^{(g)} = J^{(g)} \prod_{I^\pm} d\sigma_{I^\pm}$$

$J^{(3)} \sim J_{\text{hyp}}$

$$J_{\text{hyp}} = \sigma_{1^+2^-}\sigma_{2^+3^-}\sigma_{3^+1^-} - \sigma_{1^+3^-}\sigma_{3^+2^-}\sigma_{2^+1^-}$$

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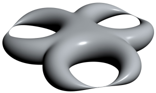

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$J^{(3)} \sim J_{\text{hyp}}$ and $\Psi_9\big|_{\text{nodal}} \sim J_{\text{hyp}}$

- Hyperelliptic locus $y^2 = \prod_{a=1}^{2g+2}(x - x_a)$:

$$\Psi_9 = 0 \quad \text{with} \quad {\Psi_9}^2 = -\prod_\delta \vartheta_\delta(0)$$

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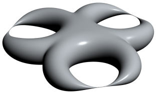

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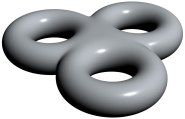
Take-away:
- $\mathcal{J}^{(3)} \mathcal{Y}^{(3)} \neq 0$ on hyperelliptic $J_{\text{hyp}} = 0$
- $\mathcal{Y}^{(3)}_{\mathbb{S}} \sim \dfrac{\chi_8(z_i)}{\Psi_9} + \ldots$

$$\mathcal{Y}_{\mathbb{S}}^{(g)} = \ell_\mu^I \, \mathcal{Y}_I^\mu + \frac{\mathcal{Y}_0}{2\pi i}$$



- Construction of Ansatz

  Requirements:

  - mod$(\mathcal{Y}_{\mathbb{S}}^{(g)}) = g - 4$
  - one-form in z_i
  - $\mathcal{Y}_{\mathbb{S}}^{(3)}\big|_{\text{nodal}} = \mathcal{Y}^{(3)}$

3 loops (iii): higher genus

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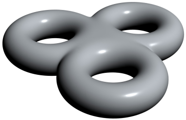

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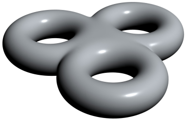

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Genus-3 tools:

- $\Delta^{(3)}_{i_1 i_2 i_3} = \det \omega_I(z_{i_J})$
- ring of mod forms

  34 generators [Tsuyumine '86]
- chiral measure $\Xi_8[\delta]/\Psi_9$

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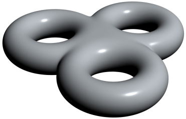

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- Result

$$\mathcal{Y}_I^\mu = \frac{2}{3}\left(\alpha_1^\mu \omega_I(z_1)\Delta^{(3)}_{234} + \text{cyc}(1234)\right)$$

$$\mathcal{Y}_0 = s_{13}s_{14}\,(\mathcal{D}_{12,34} - \mathcal{S}_{12,34}) + \text{cyc}(234)$$

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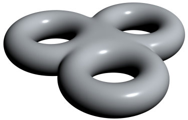

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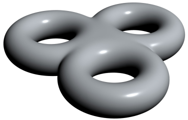

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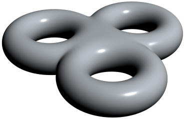

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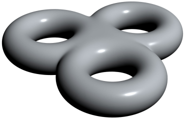

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- $\mathcal{D}_{12,34} = \frac{1}{3}\left(\omega_{34}(z_1)\Delta^{(3)}_{234} + (1 \leftrightarrow 2)\right) + (12 \leftrightarrow 34)$
- $\mathcal{S}_{12,34} = \frac{1}{15}\left(\sum_\delta \frac{\Xi_8[\delta]}{\Psi_9}\left(S^\delta_{12}S^\delta_{23}S^\delta_{34}S^\delta_{41} - \frac{1}{16}(S^\delta_{12})^2(S^\delta_{34})^2\right) + (1 \leftrightarrow 2)\right)$
  - sum over 36 even spin structures δ
  - chiral measure $\Xi_8[\delta]/\Psi_9$ [C,DP,vG '08]

Proposal for 3-loop 4-pt superstring integrand

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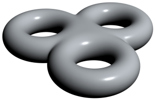

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  - modular invariance
  - field theory limit $\mathcal{Y}^{(3)}_{\mathbb{S}}\big|_{\text{nodal}} = \mathcal{Y}^{(3)}$

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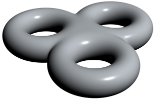

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    - move z_l around $\mathfrak{B}_L$ cycle:

$$z_i \to z_i + \delta_{il}\,\mathfrak{B}_L \qquad \ell^I \to \ell^I - \delta^I_L\, k_l$$

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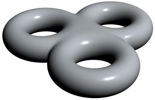

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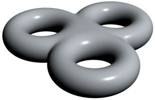
  - simplification of $\mathcal{S}_{12,34}$
  - RNS origin of measure unclear [Witten '15]
  - Functional basis? $\leftrightarrow$ Uniqueness?



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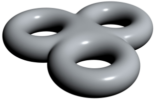
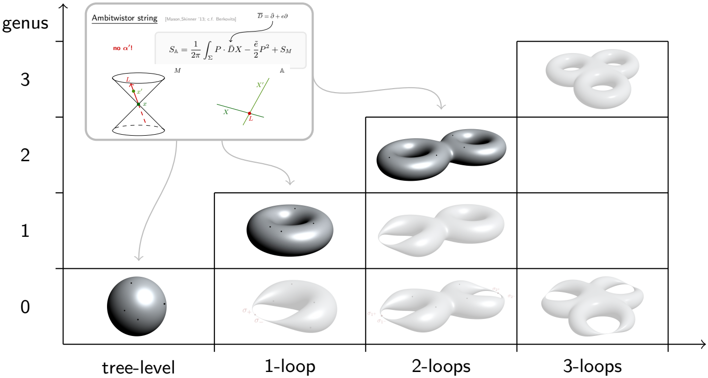


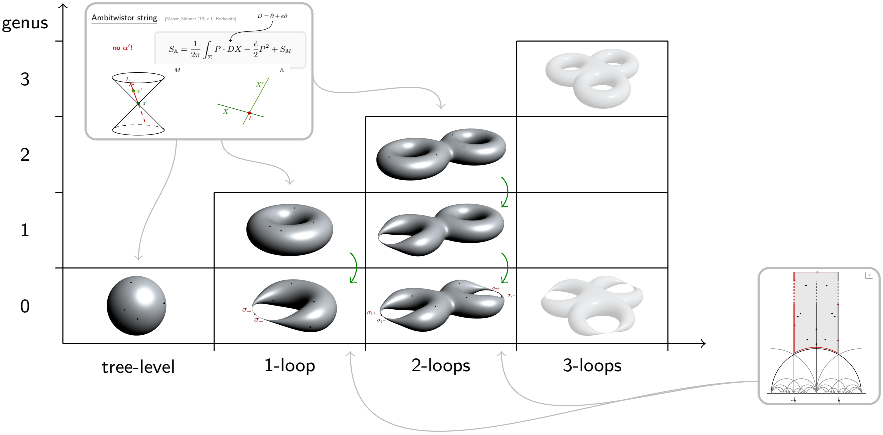
Outlook

- Summary



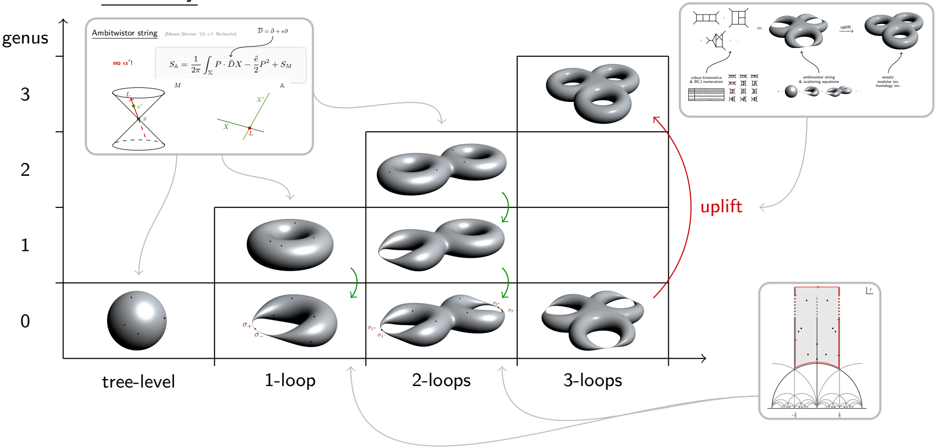
Outlook

▶ Summary



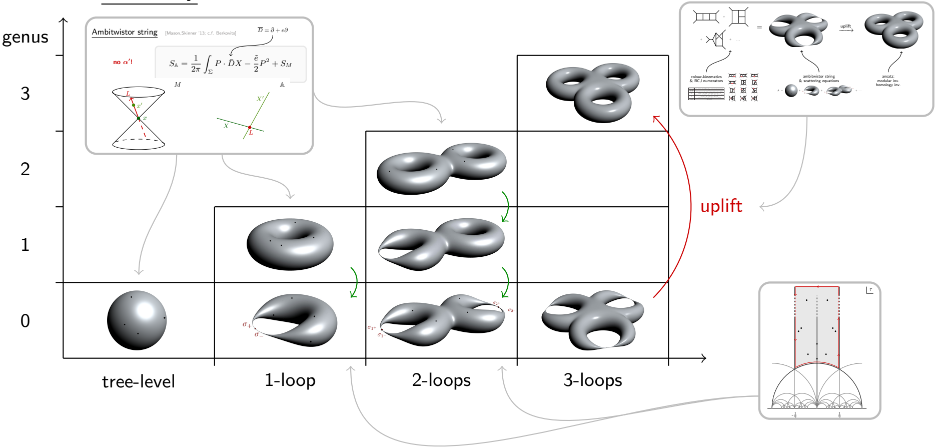
Outlook

- Summary



Outlook

- Summary



- Outlook
  - better understanding strings vs. ambitwistor strings
  - stronger evidence for 3-loop 4-pt superstring proposal
  - higher loops?

Happy Birthday Lionel!