# One-sided type-D vacuum metrics are integrable

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In 1995 Lionel pointed out to me that, for an ASD Einstein solution with a Killing vector $K$, the SD part of $dK$, once normalised, was an integrable complex structure. From that I showed that such a solution was determined by a solution of the so-called $SU(\infty)$ *Toda field equation*, called the Toda equation for brevity and also known as the *Boyer-Finley equation*. I didn't know at the time that Maciej Przanowski had already discovered this but I went a bit beyond him by showing that all such solutions arose this way (see arxiv:hep-th/0609071).

# Introduction:2

More recently I found a related result, that one-sided type-D vacuum solutions, ASD or not, necessarily have a symmetry and then are also all determined by the Toda equation (arxiv:2003.03234).
As a corollary, if there is a second symmetry commuting with the first then Richard Ward's trick (Ward 1990) linearises the Toda equation, so that these metrics are determined by an axisymmetric solution of the flat-space Laplacian. This is potentially of interest since, as was pointed out to me by Steffen Aksteiner, the Chen-Teo metric ("A new AF gravitational instanton", arxiv:1107.0763) is type-D and has two commuting symmetries.

# ASD Einstein ($\Lambda \neq 0$) with a Killing vector

To review how the first case works, suppose $K^a$ is a KV in an ASD Einstein metric. Decompose $dK$:

$$\nabla_a K_b = \phi_{AB}\epsilon_{A'B'} + \psi_{A'B'}\epsilon_{AB},$$

then Lionel (see also Pontecorvo 1992) tells us that the following is an integrable complex structure:

$$J_a{}^b := \psi^{-1}\delta_A{}^B\psi_{A'}{}^{B'} \text{ when } 2\psi^2 = \psi_{A'B'}\psi^{A'B'}.$$

Furthermore

$$J_{ab}K^b = w^{-2}\nabla_a w \text{ with } w = \Lambda\psi^{-1},$$

and then introduce $P$ by

$$(Pw^2)^{-1} = g(K, K) = g(dw/w^2, dw/w^2).$$

We're closing in on the metric!

# ASD Einstein ($\Lambda \neq 0$) with a Killing vector, cont.

Follow Claude LeBrun (1991): introduce a 'time'-coordinate $\tau$ with $K\tau = 1$ and a complex coordinate $\zeta = x + iy$ on the 2-planes orthogonal to $K$ and $JK$, then the metric is

$$g = \frac{P}{w^2}\left(e^u(dx^2 + dy^2) + dw^2\right) + \frac{1}{Pw^2}(d\tau + \theta)^2$$

for some $u$ and $\theta$, after which a glance (!) at the Einstein equations gives $u$ as a solution of the Toda equation, an (integrable) equation for $d\theta$ and the expression

$$P = (wu_w - 2)/4\Lambda,$$

and that's it!

# One-sided type-D vacuum

In terms of the curvature spinors (assuming Riemannian) by this I mean

$$\Phi_{ABA'B'} = 0 = \Lambda, \quad \Psi_{ABCD} = \Psi o_{(A} o_B o^\dagger_C o^\dagger_{D)},$$

with no condition on $\Psi_{A'B'C'D'}$ (or vice versa, interchanging primed and unprimed). Here $\Psi$ is real, and we'll assume it nonzero, and $o_A o^{\dagger A} = 1$. From Penrose and Walker 1970 (adjusted for conventions) one knows that the following is a real Killing spinor:

$$\omega_{AB} := i\Psi^{-1/3} o_{(A} o^\dagger_{B)}$$

which therefore defines a real Killing vector $K^a$ via

$$\nabla_{AA'}\omega_{BC} = \epsilon_{A(B} K_{C)A'},$$

and from Pontecorvo 1992 again we know that $\omega_{AB}$ defines an integrable complex structure via

$$J_a{}^b := \chi_A{}^B \delta_{A'}{}^{B'} \text{ when } \chi_{AB} = \Psi^{1/3}\omega_{AB}.$$

Now we play the same game again to get the metric...

# One-sided type-D vacuum, cont.

Introduce $W = (K^a K_a)^{-1}$ and a new coordinate $z := -\Psi^{-1/3}$ for then

$$J_{ab}K^b = \nabla_a z,$$

and introduce the complex coordinate $\zeta = x + iy$ as before, then the metric is

$$g = W\left(e^u(dx^2 + dy^2) + dz^2\right) + W^{-1}(d\tau + \theta)^2,$$

for some $u$ and $\theta$.
Now it's a bit more work to impose the vacuum equations and obtain that $u$ is a solution of the Toda equation, an (integrable) equation for $d\theta$ and the expression

$$W = z(2 - zu_z)/2.$$

I should say again that Maciej Przanowski and collaborators had these metrics before me (1984, 1987 and 1991) but this approach clearly gives all of them.

# Adding another symmetry to one-sided type-D vacuum

If we add a second symmetry commuting with the first in the case of one-sided type-D vacuum then w.l.o.g. (but w.s.w.) it is $L = \partial/\partial y$ and $Lu = 0$. Now there is a hodograph-like transformation due to Richard Ward 1990 which linearises the Toda equation: if

$$u_{xx} + (e^u)_{zz} = 0$$

then eliminate $(x, z, u)$ in favour of $(R, Z, V)$ via

$$x = V_Z, \quad z = \frac{1}{2} R V_R, \quad u = \log(R^2/4),$$

when

$$V_{ZZ} + R^{-1}(R V_R)_R = 0.$$

Thus $u(x, y, z)$ is determined by an axisymmetric, harmonic $V(R, Z)$. Furthermore the metric can be given explicitly in terms of coordinates $(\tau, y, R, Z)$ and the function $V$ and its harmonic conjugate.

# The metric explicitly

$$g = \begin{pmatrix} dt & dy \end{pmatrix} \begin{pmatrix} \widetilde{W} & F\widetilde{W} \\ F\widetilde{W} & F^2\widetilde{W} + \frac{1}{4}R^2\widetilde{W}^{-1} \end{pmatrix} \begin{pmatrix} dt \\ dy \end{pmatrix} + \Omega^2(dR^2 + dZ^2)$$

with $V(R,Z)$ harmonic and axisymmetric, $H$ conjugate to $V$ and

$$\widetilde{W} = \frac{2}{V_R}\left(\frac{((V_{RZ})^2 + (V_{ZZ})^2)}{R((V_{RZ})^2 + (V_{ZZ})^2) + V_R V_{ZZ}}\right),$$

$$F = \frac{1}{2}\left(H + \frac{R(V_R)^2 V_{RZ} - R^2 V_Z((V_{RZ})^2 + (V_{ZZ})^2)}{2((V_{RZ})^2 + (V_{ZZ})^2)}\right),$$

$$\Omega^2 = \frac{1}{8}R^2 V_R \left(R((V_{RZ})^2 + (V_{ZZ})^2) + V_R V_{ZZ}\right).$$

# ASD Einstein with two commuting symmetries

These are also determined by a solution of the Toda equation, and w.l.o.g. (but w.s.w.) a second symmetry commuting with the first can be taken to be $L = \partial/\partial y$ with $Lu = 0$. So this case also linearises.

As a simple example, the axisymmetric harmonic function $V = R^2 - 2Z^2$ leads to the metric

$$g = \frac{4}{R^4}(d\tau - Z dY)^2 + \frac{1}{R^2}(dR^2 + dY^2 + dZ^2),$$

which is the Bergman metric (in disguise).

# what about one-sided type-D Einstein?

With $\Lambda \neq 0$ and assuming $\psi$ not constant (since that is Einstein-Kähler), this still has a Killing spinor and therefore a Killing vector, an integrable complex structure and the same metric ansatz BUT the function $u$ satisfies a modified Toda equation:

$$u_{xx} + u_{yy} + (e^u)_{zz} + e^u(A(z)u_z + B(z)) = 0,$$

with

$$A = \frac{72\Lambda z^2}{1 - 12\Lambda z^3}, \quad B = -\frac{144\Lambda z}{1 - 12\Lambda z^3}.$$

Now this modified Toda is *not* integrable (see e.g. arXiv:2003.03234 v2; the constants in $A, B$ can be checked by finding the Schwarzschild-de Sitter metric) – allowing nonzero $\Lambda$ in this case has removed the integrability.