From Twistor Theory to Gravitational Waves

Twistor Theory and Beyond, 2021

Zvi Bern

ZB, C. Cheung, R. Roiban, C. H. Shen, M. Solon, M. Zeng, arXiv:1901.04424 and arXiv:1908.01493.
ZB, A. Luna, R. Roiban, C. H. Shen, M. Zeng, arXiv:2005.03071
ZB, J. Parra-Martinez, R. Roiban, E. Sawyer, C.-H. Shen, arXiv 2010.08559
ZB, J. Parra-Martinez, R. Roiban, M. Ruf. C.-H. Shen, M. Solon, M. Zeng, arXiv:2101.07254

UCLA Mani L. Bhaumik Institute for Theoretical Physics

Outline

1. **Comments on impact of twistors on scattering amplitudes, with a little history.**

2. **Current research on gravitational waves.**

Happy birthday to Lionel!

- A joy to know Lionel. Brilliant person we all admire. Delightful conversations over the years.

- We come from different worlds. Glue that binds us are twistors, $N = 4$ sYM and $N = 8$ supergravity.

One of the joys of working in amplitudes is we connect to people in other fields, whether mathematical or phenomenological

My first encounter with twistors

Around 1995 I was contacted by Warren Siegel:

Why are we using the name "spinor helicity" when we should be calling it "twistors"?

Another 10 years before I understood what Warren meant.

That understanding came in the form of Witten's 2004 twistor revolution, which continues to reverberate.

Twistor Revolution

Commun. Math. Phys. 252, 189–258 (2004)
Digital Object Identifier (DOI) 10.1007/s00220-004-1187-3

Communications in Mathematical Physics

Perturbative Gauge Theory as a String Theory in Twistor Space

Edward Witten

Institute for Advanced Study, Princeton, NJ 08540, USA

Received: 22 April 2004 / Accepted: 3 June 2004
Published online: 7 October 2004 – © Springer-Verlag 2004

Precursor from Nair
Sigma model on CP^1

Penrose twistor transform takes us from spinors to twistors

$$\widetilde{A}(\lambda_i, \mu_i) = \int \prod_i \frac{d^2\widetilde{\lambda}_i}{(2\pi)^2} \exp\left(\sum_j \mu_j^{\dot{a}} \widetilde{\lambda}_{j\dot{a}}\right) A(\lambda_i, \widetilde{\lambda}_i)$$

Gauge theory scattering amplitudes ⟷ Topological String Theory

David Kosower told me Ed was working on something related to amplitudes.

KITP collider physics workshop. Roiban, Spradlin and Volovich postdocs at KITP. Paper landed like a meteorite, sending out shock waves to this day.

Warren was right: I had been using twistors for many years without knowing it.

Lionel!

After the twistor revolution Lionel became the go to person for twistors.

PUBLISHED BY INSTITUTE OF PHYSICS PUBLISHING FOR SISSA

RECEIVED: *December 18, 2006*

Supersymmetric gauge theories in twistor space

Rutger Boels, Lionel Mason and David Skinner

The Mathematical Institute, University of Oxford
24-29 St. Giles, Oxford OX1 3LP, United Kingdom

Twistor rock star in amplitudes community

PUBLISHED FOR SISSA BY SPRINGER

RECEIVED: *October 8, 2010*

The complete planar S-matrix of $\mathcal{N} = 4$ SYM as a Wilson loop in twistor space

Lionel Mason[a] and David Skinner[b]

[a] *The Mathematical Institute,*
24-29 St. Giles', Oxford, OX1 3LB, United Kingdom
[a] *Perimeter Institute for Theoretical Physics,*
31 Caroline St., Waterloo, ON, N2L 2Y5, Canada

Lionel!

Lionel is the one who brought the twistor world together with scattering .

2005 Oxford conference

Twistor String Theory

Conference Poster

Link to program and transparencies

Link to `From Twistors to Amplitudes,' a QMUL workshop.

THE MATHEMATICAL INSTITUTE
University of Oxford

London Mathematical Society Workshop

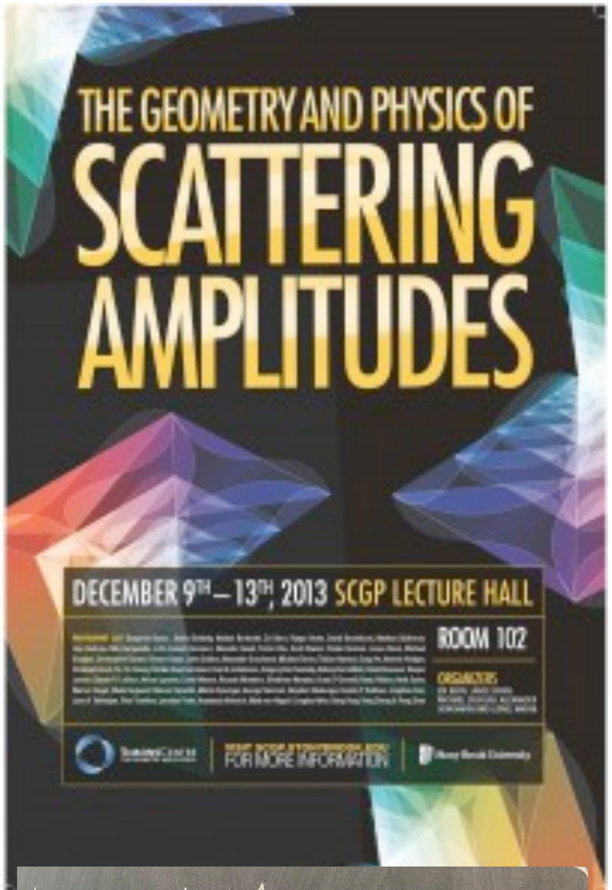
10-14 January 2005

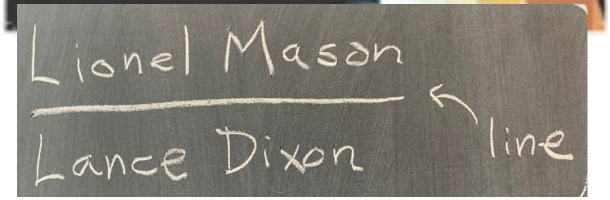
Abstract

This meeting was organised to take stock of the rapid progress being made on twistor-string theory and to encourage further cross-fertilization between string-theory, twistor theory and perturbative gauge theory. Twistor string theory was introduced by Witten in hep-th/0312171 as a string theory in twistor space that makes contact with N=4 super Yang-Mills theory on space-time via a generalization of the Penrose-Ward transform augmented by certain D-instanton corrections. It promises to combine many of the most attractive features of string theory and twistor theory and has implications not only for Yang-Mills but also for (conformal) gravity. It has in particular led to major advances in the calculations of Yang-Mills scattering amplitudes with applications to collider physics.

Z. Bern, P. Candelas, X. de la Ossa, S. Huggett, L. Mason

2013 Simons Center





This is where I learned about the twistor world.

It's time for another twistor/amplitude workshop

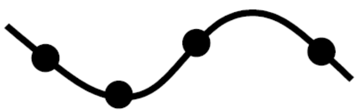
Amazing Structure

Witten (2004)

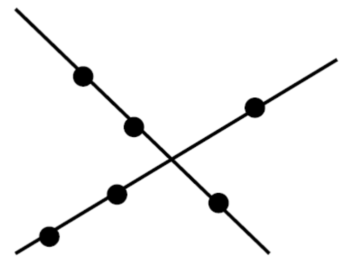
To this day I'm still amazed by structures uncovered by twistors.

Witten conjectured that in twistor–space gauge theory amplitudes have delta-function support on curves of degree:

$$d = q - 1 + L, \quad q = \#\ \text{negative helicities}, \quad L = \#\ \text{loops},$$



Connected picture



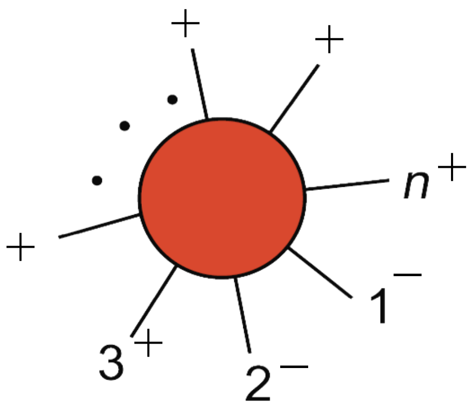
Disconnected picture

Witten
Roiban, Spradlin and Volovich
Cachazo, Svrcek and Witten
Gukov, Motl and Neitzke
Bena. Bern and Kosower

Remarkable structures in gauge theory scattering amplitudes.

Twistor Transform of MHV Amplitude

Witten (2003)



$$A^{\rm MHV}(\lambda_i) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

$$\lambda_1 = |1\rangle$$

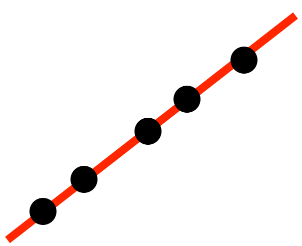
$$\tilde\lambda_1 = |1]$$

$$A^{\rm MHV}(\lambda_i)\delta(\sum_i k_i) = \int d^4x A^{\rm MHV}(\lambda_i)\exp(ix\lambda_i\tilde\lambda_i)$$

Apply twistor transform:

$$\int dx \exp(ikx) = \delta(k)$$

$$\int d\tilde\lambda = \exp(u\mu\tilde\lambda)\exp(ix\lambda\tilde\lambda) \;\Rightarrow\; A(\lambda_i,\mu_j) \propto \prod_j \delta(\mu_j + x\lambda_j)$$



The MHV amplitudes (2 minus rest positive)
Only has support on lines.

While trivial in this case, highly non-trivial way that it generalizes.

RSV Formula

The following formula encapsulates the entire tree-level S-matrix of $\mathcal{N} = 4$ super-Yang-Mills:

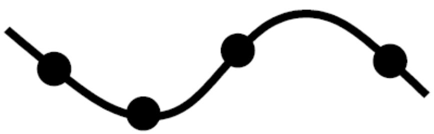
Roiban, Spradlin and Volovich

Integral over the moduli and curves

$$A_n = i(2\pi)^4 g_{\mathrm{YM}}^{n-2} \sum_{d=1}^{n-3} \int d\mathcal{M}_{n,d} \prod_{i=1}^{n} \delta^2\left(\lambda_i^{\alpha} - \xi_i P_i^{\alpha}\right) \prod_{k=0}^{d} \delta^2\left(\sum_{i=1}^{n} \xi_i \sigma_i^k \widetilde{\lambda}_i^{\dot\alpha}\right) \delta^4\left(\sum_{i=1}^{n} \xi_i \sigma_i^k \eta_{iA}\right)$$

$$P_i^{\alpha} = \sum_{k=0}^{d} a_k^{\alpha} \sigma_i^k$$

Degree *d* polynomial in the moduli



A very strange formula from Feynman diagram viewpoint.

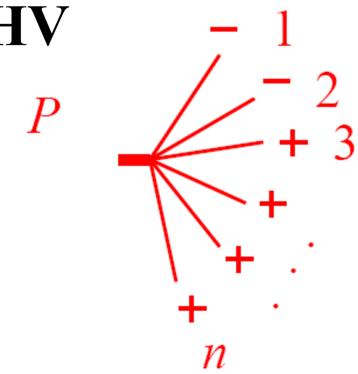
But it's true: impressive checks by Roiban, Spradlin and Volovich

- **An example of an amazingly beautiful formula whose practical value is still unclear. Planted seeds for CHY.**
- **Maybe in the future applications will become clear**

MHV Rules

Cachazo, Svrcek and Witten

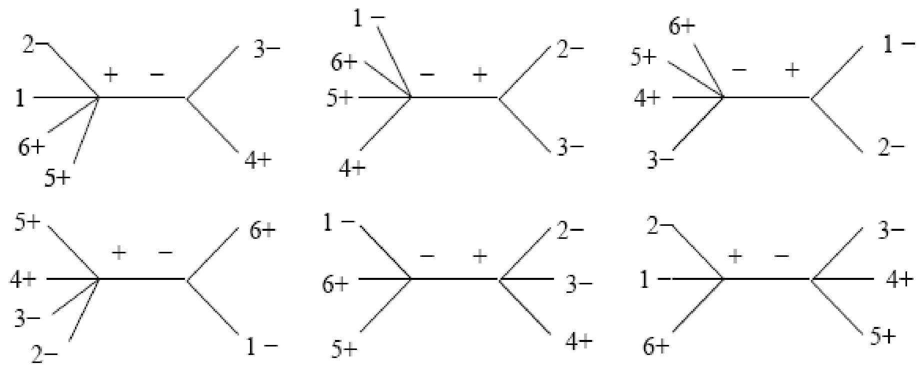
Disconnected picture suggests that in momentum space MHV amplitudes are vertices for building new amplitudes.



MHV amplitudes as vertices

QCD gluon scattering amplitude

$- - - + + +$



$$A_6(1^-,2^-,3^-,4^+,5^+,6^+) = \frac{\langle 1\,2\rangle^3}{\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|\,5+6+1\,|q\rangle\,\langle 5|\,6+1+2\,|q\rangle} \times \frac{1}{s_{34}} \times \frac{\langle 3|\,4\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4|\,3\,|q\rangle}$$

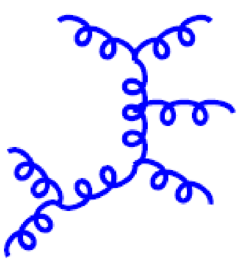
$$+ \frac{\langle 1|\,4+5+6\,|q\rangle^3}{\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 4|\,5+6+1\,|q\rangle} \times \frac{1}{s_{23}} \times \frac{\langle 2\,3\rangle^3}{\langle 3|\,2\,|q\rangle\,\langle 2|\,3\,|q\rangle}$$

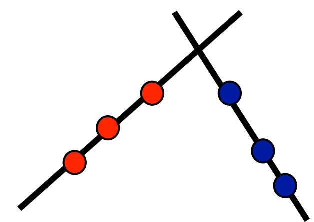
$$+ \frac{\langle 3|\,4+5+6\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6|\,3+4+5\,|q\rangle} \times \frac{1}{s_{12}} \times \frac{\langle 1\,2\rangle^3}{\langle 2|\,1\,|q\rangle\,\langle 1|\,2\,|q\rangle}$$

$$+ \frac{\langle 2\,3\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5|\,2+3+4\,|q\rangle\,\langle 2|\,3+4+5\,|q\rangle} \times \frac{1}{s_{61}} \times \frac{\langle 1|\,6\,|q\rangle^3}{\langle 6\,1\rangle\,\langle 6|\,1\,|q\rangle}$$

$$+ \frac{\langle 1|\,5+6\,|q\rangle^3}{\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 5|\,6+1\,|q\rangle} \times \frac{1}{s_{561}} \times \frac{\langle 2\,3\rangle^3}{\langle 3\,4\rangle\,\langle 4|\,2+3\,|q\rangle\,\langle 2|\,3+4\,|q\rangle}$$

$$+ \frac{\langle 1\,2\rangle^3}{\langle 6\,1\rangle\,\langle 2|\,6+1\,|q\rangle\,\langle 6|\,1+2\,|q\rangle} \times \frac{1}{s_{612}} \times \frac{\langle 3|\,4+5\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5|\,3+4\,|q\rangle}$$





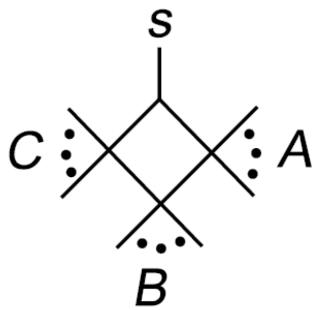
Easy to use

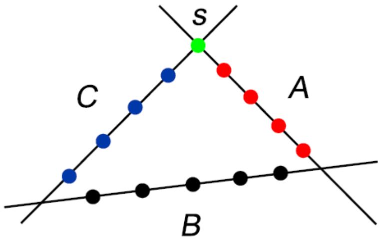
Twistor Structure at One Loop

At one-loop the coefficients of all integral functions have beautiful twistor space interpretations

Box integral | **Twistor space support**

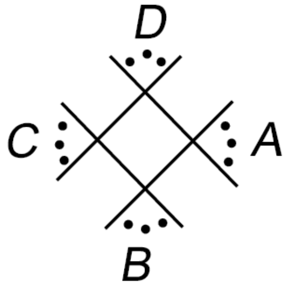
Three negative helicities

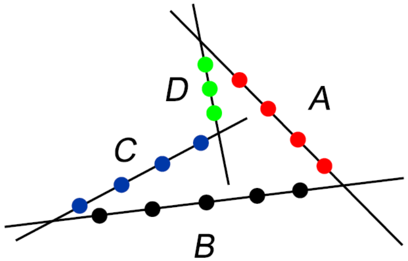




Bern, Dixon and Kosower
Britto, Cachazo and Feng

Four negative helicities





The existence of such twistor structures connected with loop-level simplicity.

Ambitwistor String

(2015)

See Yvonne Geyer's talk

JHEP
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RECEIVED: *July 28, 2015*
ACCEPTED: *October 12, 2015*
PUBLISHED: *November 5, 2015*

New ambitwistor string theories

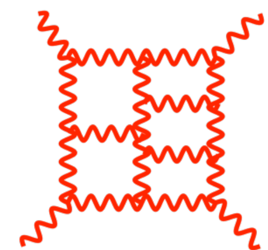
Eduardo Casali,[b] Yvonne Geyer,[a] Lionel Mason,[a] Ricardo Monteiro[a] and Kai A. Roehrig[b]

[a] *Mathematical Institute, University of Oxford, Woodstock Road, Oxford OX2 6GG, U.K.*
[b] *DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, U.K.*

At the time we were having serious trouble obtaining 5 loop $N = 8$ supergravity amplitude.

-- pushing the limits of such calculations

-- wanting to decisively understand UV properties of the theory.



Met Lionel at one of the conferences explaining that we could really use some help in $N = 8$ supergravity. Needed a fresh approach.

Might ambitwistor strings help with multiloop?

Ambitwistor String

See Yvonne Geyer's talk

PRL **115,** 121603 (2015) PHYSICAL REVIEW LETTERS week ending 18 SEPTEMBER 2015

Loop Integrands for Scattering Amplitudes from the Riemann Sphere

Yvonne Geyer,[1] Lionel Mason,[1] Ricardo Monteiro,[2] and Piotr Tourkine[2]
[1]*Mathematical Institute, University of Oxford, Woodstock Road, Oxford OX2 6GG, United Kingdom*
[2]*DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom*
(Received 28 July 2015; published 16 September 2015)

The scattering equations on the Riemann sphere give rise to remarkable formulas for tree-level gauge theory and gravity amplitudes. Adamo, Casali, and Skinner conjectured a one-loop formula for supergravity amplitudes based on scattering equations on a torus. We use a residue theorem to transform this into a formula on the Riemann sphere. What emerges is a framework for loop integrands on the Riemann sphere that promises to have a wide application, based on off-shell scattering equations that depend on the loop momentum. We present new formulas, checked explicitly at low points, for supergravity and super-Yang-Mills amplitudes and for n-gon integrands at one loop. Finally, we show that the off-shell scattering equations naturally extend to arbitrary loop order, and we give a proposal for the all-loop integrands for supergravity and planar super-Yang-Mills theory.

- **Loops are possible.**
- **Could this help us?**
- **Hard to get to high loops**

Q cuts

Help arrived in a repackaged form directly useful in field theory.

PRL **116,** 061601 (2016) PHYSICAL REVIEW LETTERS week ending 12 FEBRUARY 2016

New Representations of the Perturbative *S* Matrix

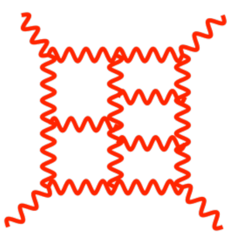
Christian Baadsgaard,[1,2] N. E. J. Bjerrum-Bohr,[2] Jacob L. Bourjaily,[2] Simon Caron-Huot,[2] Poul H. Damgaard,[2] and Bo Feng[3]

Motivated by structure of amplitude from ambitwistor string.

$$\frac{1}{D_1 \cdots D_m} = \sum_{i=1}^{m} \frac{1}{D_i}\left[\prod_{j\neq i} \frac{1}{D_j - D_i}\right].$$

Rearrange according to partial fractioning.
Linear in loop momentum

- **Algorithmic construction of integrand from unitarity cuts.**
- **Double copy visible.**
- **UV properties worsened, but key is to get an integrand.**
- **Looked to be very promising approach. Worked out 2 loops.**
- **Ultimately, we put this aside in favor of generalized double copy.** ZB, Carrasco, Chen, Edison, Johansson, Parra-Martinez, Roiban, Zeng



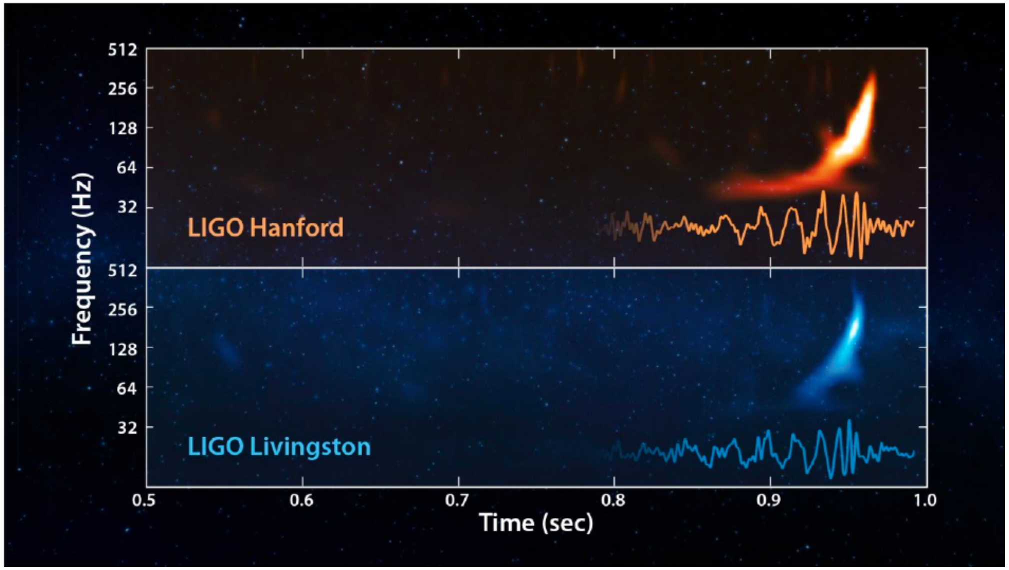
Q cuts a fresh approach that is certainly worth revisiting

Part 2: Recent Work on Gravitational Waves

(not much connection to twistors, except that we still use spinor helicity)

Outline

Era of gravitational-wave astronomy has begun.



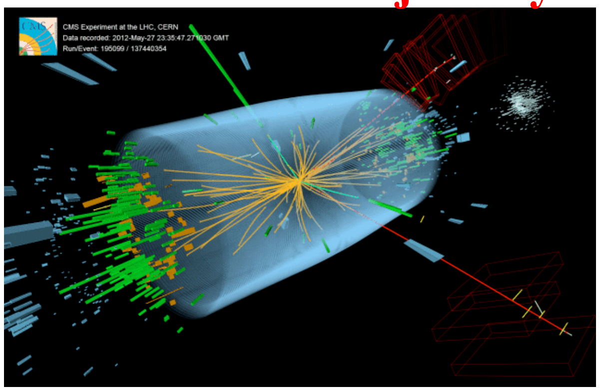
For an instant brighter in gravitational radiation than all the stars in the visible universe are in EM radiation!

How can we in scattering amplitudes community, help out with core mission of LIGO/Virgo?

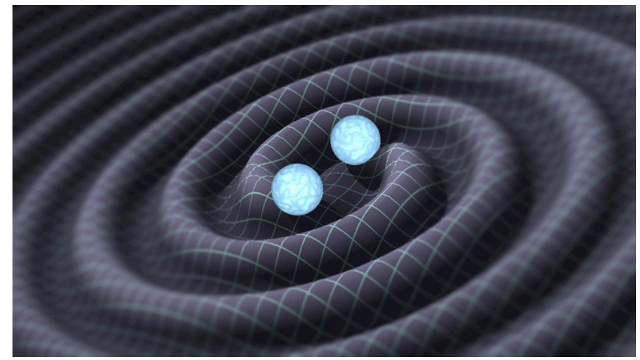
Can Particle Theory Help with Gravitational Waves?

What does particle physics have to do with classical dynamics of astrophysical objects?

unbounded trajectory



bounded orbit



gauge theories, QCD, electroweak
quantum field theory

General Relativity
classical physics

Black holes and neutron stars are point particles as far as long wavelength radiation is concerned.

Iwasaki (1971); Goldberger, Rothstein (2006), Porto; Vaydia, Foffa , Porto, Rothstein, Sturani; Kol; Bjerrum-Bohr, Donoghue, Holstein, Plante, Pierre Vanhove; Levi, Steinhoff; Vines etc

Will explain that scattering amplitudes well suited to push state-of-the-art perturbative calculations for gravitational-wave physics.

Can Quantum Scattering Help with Gravitational Waves?

We are very good at gravitational perturbation theory for scattering amplitudes.

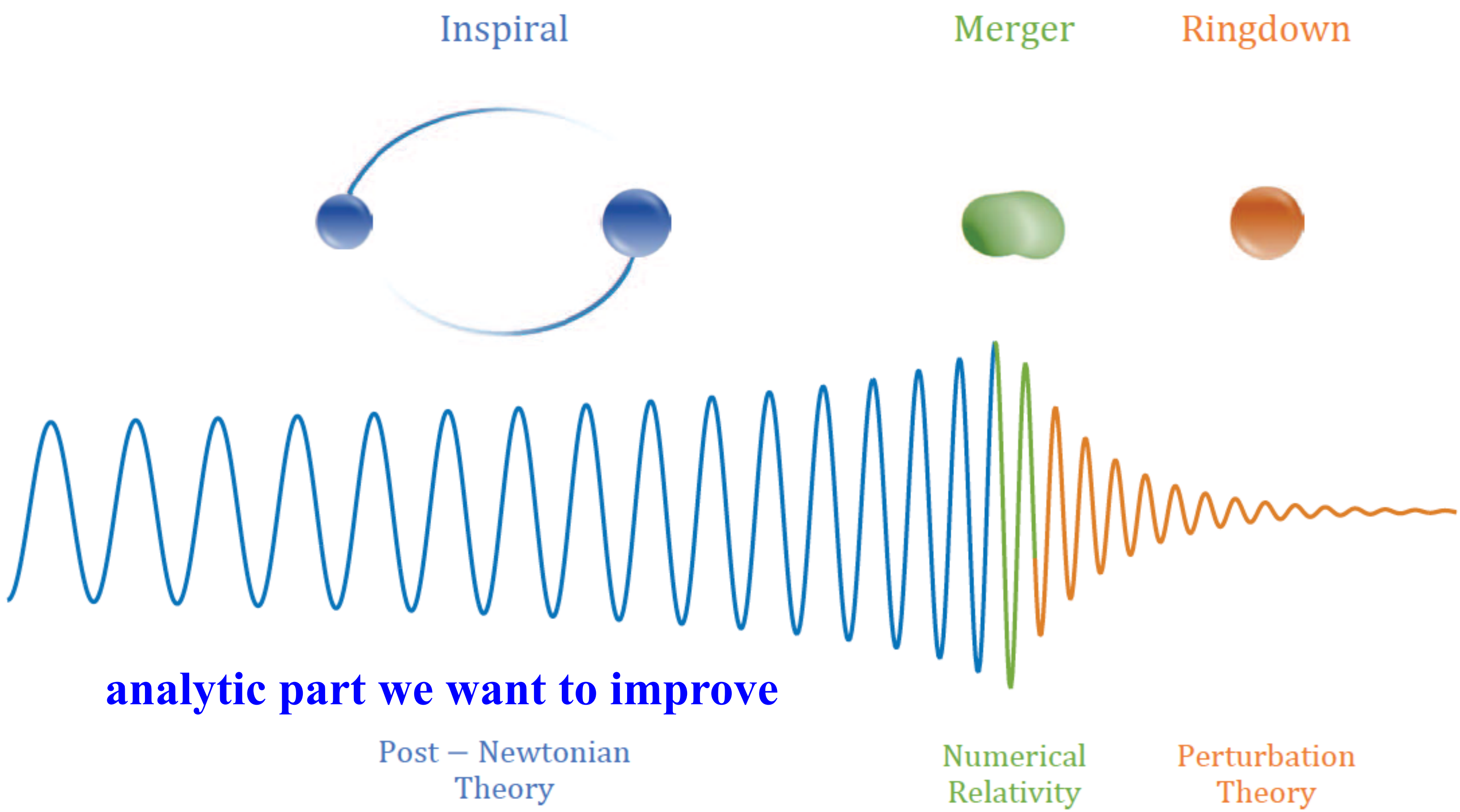
However, two serious issues for applying this to gravitational waves:

1. **We do quantum *not* classical perturbation theory.**
2. **Scattering process unbounded orbit. Want bounded one for binary black hole gravitational wave emission.**

Two key topics:

- **Modern approach to perturbative gravity.**
- **How do we effectively deal with the above annoying issues?**

Two Body Problem

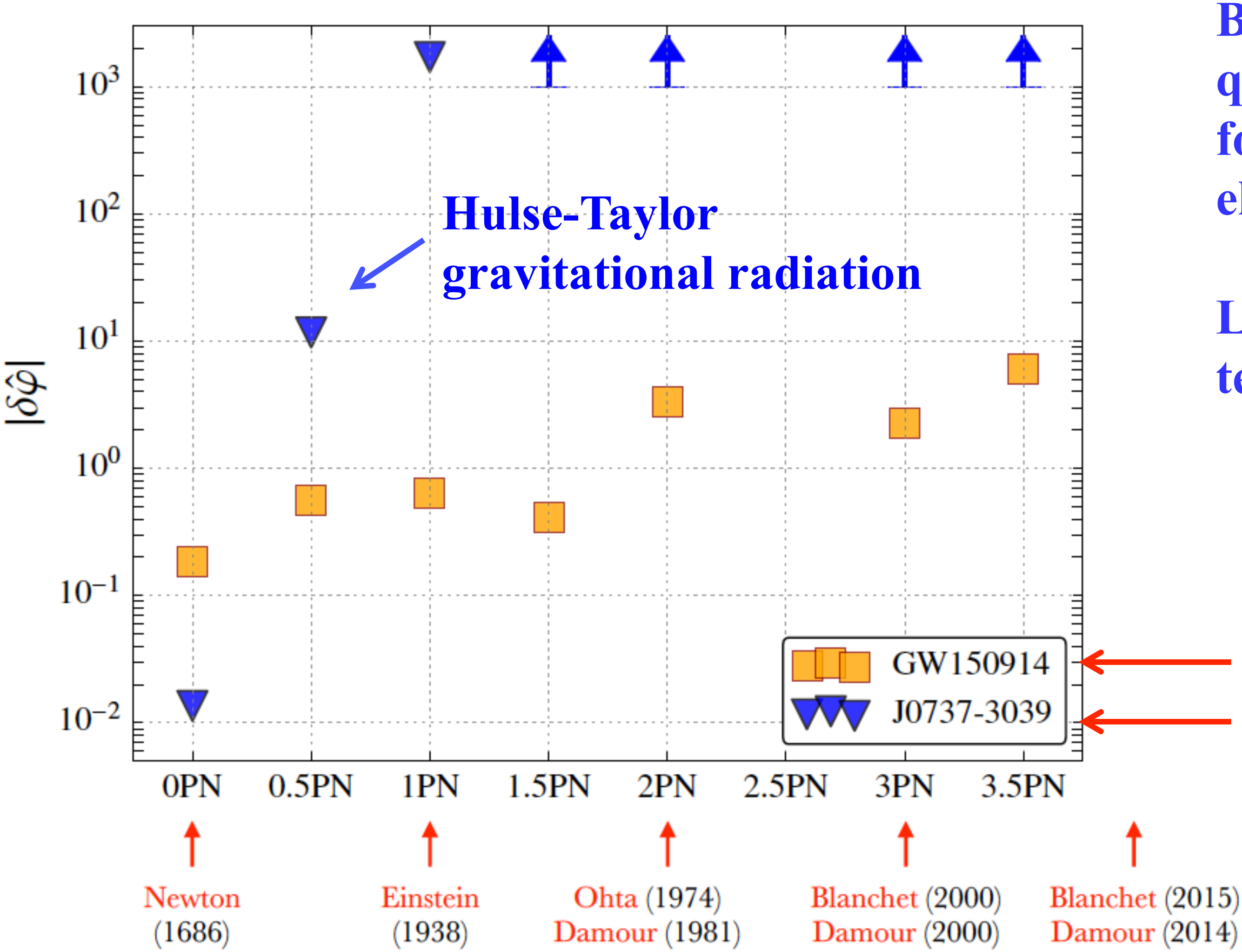


Small errors accumulate. Need for high precision.

From Antelis and Moreno, arXiv:1610.03567

Importance of higher orders for LIGO/Virgo

LIGO/Virgo Collaboration arXiv:1602.03841



LIGO

Binary pulsar

Binary pulsar confirms quadrupole radiation formula and not much else.

LIGO/Virgo tests PN terms from GR

LIGO/Virgo sensitive to high PN orders.

Post Newtonian Approximation

For orbital mechanics:

$$v^2 \sim \frac{GM}{R} \ll 1$$

Expand in *G* and v^2

virial theorem

In center of mass frame:

$$m = m_A + m_B, \quad \nu = \mu/M,$$

$$\mu = m_A m_B/m, \quad P_R = P \cdot \hat{R}$$

$$\frac{H}{\mu} = \frac{P^2}{2} - \frac{Gm}{R}$$

Newton

$$+ \frac{1}{c^2}\left\{ -\frac{P^4}{8} + \frac{3\,\nu\,P^4}{8} + \frac{Gm}{R}\left(-\frac{{P_R}^2\,\nu}{2} - \frac{3\,P^2}{2} - \frac{\nu\,P^2}{2}\right) + \frac{G^2 m^2}{2R^2} \right\}$$

$$+ \ldots$$

1PN: Einstein, Infeld, Hoffmann; Droste, Lorentz

Hamiltonian known to 4PN order.

2PN: Ohta, Okamura, Kimura and Hiida.

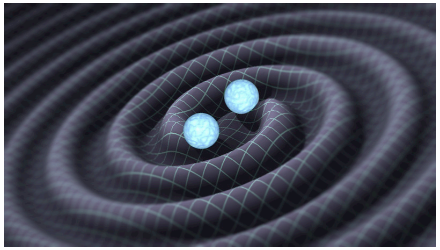
3PN: Damour, Jaranowski and Schaefer; L. Blanchet and G. Faye.

4PN: Damour, Jaranowski and Schaefer; Foffa (2017), Porto, Rothstein, Sturani (2019).

Which problem to solve?

ZB, Cheung, Roiban, Shen, Solon, Zeng

Some problems for (analytic) theorists:



1. Spin.
2. Finite size effects.
3. New physics effects.
4. Radiation.
5. High orders in perturbation theory.

Which problem should we solve?

- Needs to be extremely difficult using standard methods.
- Needs to be of direct importance to LIGO theorists.
- Needs to be in a form that can in principle enter LIGO analysis pipeline.

2-body Hamiltonian at 3rd post-Minkowskian order

PN versus PM expansion for conservative two-body dynamics

From Buonanno
Amplitudes 2018

$$\mathcal{L} = -Mc^2 + \underbrace{\frac{\mu v^2}{2} + \frac{GM\mu}{r}}_{\downarrow} + \frac{1}{c^2}\left[\dots\right] + \frac{1}{c^4}\left[\dots\right] + \dots$$

$$E(v) = -\frac{\mu}{2}v^2 + \cdots$$

non-spinning compact objects

		0PN	1PN	2PN	3PN	4PN	5PN	...
0PM:	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	...
1PM:		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	v^{10}/r	...
2PM:			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	v^8/r^2	...
3PM:				$1/r^3$	v^2/r^3	v^4/r^3	v^6/r^3	...
4PM:					$1/r^4$	v^2/r^4	v^4/r^4	...
...						...	...	

(credit: Justin Vines)

$$1 \to Mc^2, \qquad v^2 \to \frac{v^2}{c^2}, \qquad \frac{1}{r} \to \frac{GM}{rc^2}.$$

current known PN results — current known PM results — overlap between PN & PM results — unknown

- **PM results** (Westfahl 79, Westfahl & Goller 80, Portilla 79-80, Bel et al. 81, Ledvinka et al. 10, Damour 16-17, Guevara 17, Vines 17, Bini & Damour 17-18, Vines in prep)

High-energy gravitational scattering and the general relativistic two-body problem

Thibault Damour*
Institut des Hautes Etudes Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France
(Dated: October 31, 2017)

A technique for translating the classical scattering function of two gravitationally interacting bod-
...
tum gra... ...mplitude of two particles, and we urge amplitude experts to use their novel techniques to compute the 2-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian

"… and we urge amplitude experts to use their novel techniques to compute the 2-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian."

Hard to resist an invitation with this kind of clarity!

The recent observation [1–4] of gravitational wave sig-nals from inspiralling and coalescing binary black holes has been significantly helped, from the theoretical side, by the availability of a large bank of waveform templates, defined [5, 6] within the analytical effective one-body (EOB) formalism [7–11]. The EOB formalism combines in a su... results ... with s... numer... (see [1... system... of bina... sults u... mostly based on the post-Newtonian (PN) approach to the general relativistic two-body interaction. The con-servative two-body dynamics was derived, successively, at the second post-Newtonian (2PN) [14, 15], third post-

...ntly intro-... to derive ... from the (gauge-invariant) *scattering function* Φ linking (half) the center of mass (c.m.) classical gravitational scattering angle χ to the total energy, $E_{\rm real} \equiv \sqrt{s}$, and the total angular momentum, J, of the system[1]

$$\frac{1}{2} \ldots$$

$$\ldots \frac{\ldots}{M} \ldots \frac{\ldots}{Gm_1m_2} \ldots \frac{\ldots}{G\mu M} \ldots$$

with

$$M \equiv m_1 + m_2;\ \mu \equiv \frac{m_1m_2}{\ldots};\ \nu \equiv \frac{\mu}{M} = \frac{m_1m_2}{(\ldots)^2}.$$

- **Difficult using standard methods.**
- **Of direct importance to LIGO/Virgo theorists.**
- **Can in principle enter LIGO/Virgo analysis pipeline.**

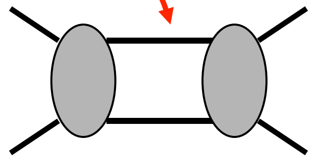
From Tree to Loops: Generalized Unitarity Method

Use tree amplitudes to build higher-order (loop) amplitudes.

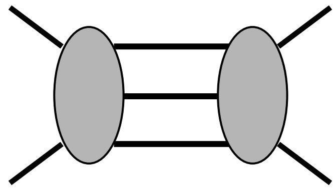
ZB, Dixon, Dunbar and Kosower (1994)

$$E^2 = \vec{p}^{\,2} + m^2$$ ← **on-shell**

Two-particle cut:

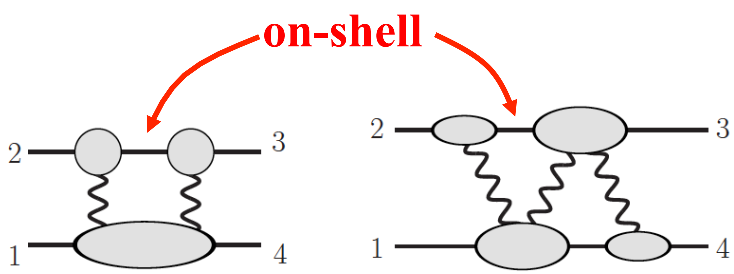


Three-particle cut:



- **Systematic assembly of complete loop amplitudes from tree amplitudes.**
- **Works for any number of particles or loops.**

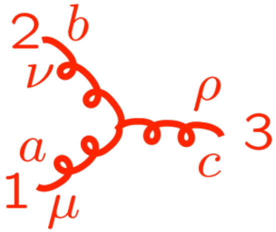
Generalized unitarity as a practical tool for loops.



ZB, Dixon and Kosower;
ZB, Morgan;
Britto, Cachazo, Feng;
Ossala,Pittau,Papadopoulos;
Ellis, Kunszt, Melnikov;
Forde; Badger;
ZB, Carrasco, Johansson, Kosower
and many others

Idea used in the "NLO revolution" in QCD collider physics.
No gauge fixing in the formalism.

Standard perturbative approach:



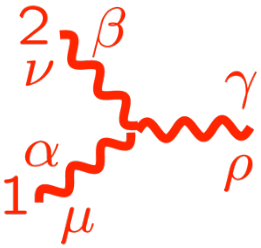
Three-gluon vertex:

$$V_{3\,\mu\nu\sigma}^{abc} = -g f^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho+\eta_{\nu\rho}(k_1-k_2)_\mu+\eta_{\rho\mu}(k_1-k_2)_\nu)$$

Three-graviton vertex:

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

$$\begin{aligned}
&G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \\
&\quad \mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\
&\qquad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\
&\qquad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\
&\qquad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]
\end{aligned}$$



About 100 terms in three vertex

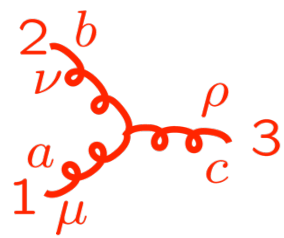
Naïve conclusion: Gravity more complicated than gauge theory.

Simplicity of Gravity Amplitudes

People were looking at gravity amplitudes the wrong way.

***On-shell* three vertices contains all information:** $E_i^2 - \vec{k}_i^2 = 0$

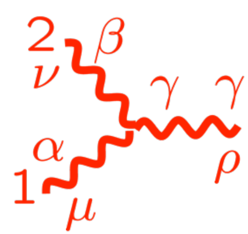
Yang-Mills (QCD) gauge theory:



color factor (→ f^{abc})

$$-g f^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

Einstein gravity:



$$i\kappa(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$
$$\times (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$

"square" of Yang-Mills vertex.

Starting from this on-shell vertex any multi-loop amplitude can be constructed via modern methods

Gravitons are like two gluons!

KLT Relation Between Gravity and Gauge Theory

KLT (1985)

Kawai-Lewellen-Tye string relations in low-energy limit:

gravity **gauge-theory color ordered**

$$M_4^{\rm tree}(1,2,3,4) = -i s_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,,$$

$$M_5^{\rm tree}(1,2,3,4,5) = i s_{12} s_{34} A_5^{\rm tree}(1,2,3,4,5)\, A_5^{\rm tree}(2,1,4,3,5)$$

$$+\, i s_{13} s_{24} A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5)$$

Inherently gauge invariant!

Generalizes to explicit all-leg form. ZB, Dixon, Perelstein, Rozowsky

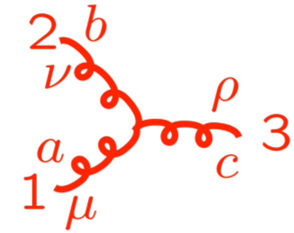
1. **Gravity amplitudes derivable from gauge theory.**
2. **Once gauge-theory amplitude is simplified, so is gravity.**
3. **Standard Lagrangian methods offer no hint why this is possible.**

Duality Between Color and Kinematics

ZB, Carrasco, Johansson (2007)

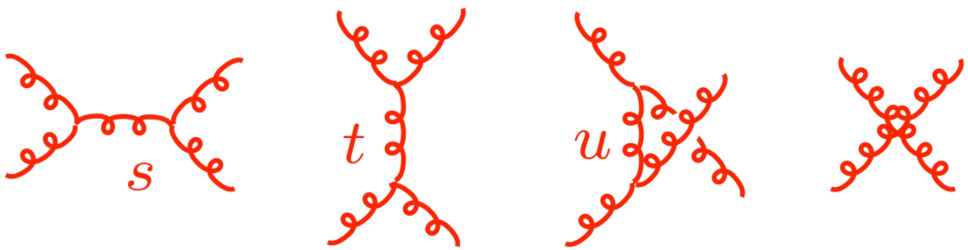
coupling constant → color factor ← momentum dependent kinematic factor

$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = if^{abc}T^c$

Jacobi Identity $f^{a_1 a_2 b} f^{b a_4 a_3} + f^{a_4 a_2 b} f^{b a_3 a_1} + f^{a_4 a_1 b} f^{b a_2 a_3} = 0$



Use 1 = *s/s* = *t/t* = *u/u* to assign 4-point diagram to others.

$s = (k_1 + k_2)^2 \quad t = (k_1 + k_4)^2$

$u = (k_1 + k_3)^2$

$$\mathcal{A}_4^{\text{tree}} = g^2 \left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right)$$

Color factors satisfy Jacobi identity: $c_u = c_s - c_t$

Numerator factors satisfy similar identity: $n_u = n_s - n_t$

Duality Between Color and Kinematics

Consider five-point tree amplitude:

ZB, Carrasco, Johansson (BCJ)

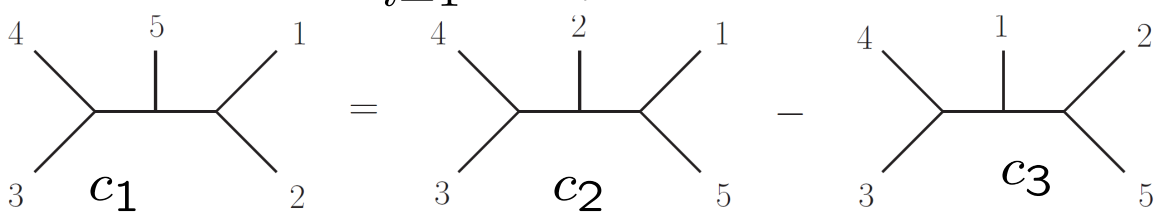
gauge theory

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

color factor

kinematic numerator factor

Feynman propagators



$$c_1 = f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2} \qquad c_2 = f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5} \qquad c_3 = f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$n_i \sim k_4 \cdot k_5 \, k_2 \cdot \varepsilon_1 \, \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

$$c_1 + c_2 + c_3 = 0 \;\Leftrightarrow\; n_1 + n_2 + n_3 = 0$$

Claim: We can always find a rearrangement so color and kinematics satisfy the *same* algebraic constraint equations.

Proven at tree level

ZB, Carrasco, Johansson; Kiermaier; Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove; Cachazo, etc

Gravity from Gauge Theory

ZB, Carrasco, Johansson

gauge theory (QCD): $$\mathcal{A}_n^{\text{tree}} = i g^{n-2} \sum_i \frac{c_i\, n_i}{D_i}$$

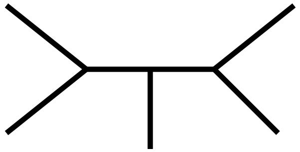
color factor

kinematic numerator factor

Feynman propagators

$$c_k = c_i - c_j$$
$$n_k = n_i - n_j$$

$$c_i \to n_i$$



sum over diagrams with only 3 vertices

Einstein gravity: $$\mathcal{M}_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i^2}{D_i}$$

$$n_i \sim k_4 \cdot k_5\, k_2 \cdot \varepsilon_1\, \varepsilon_2 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

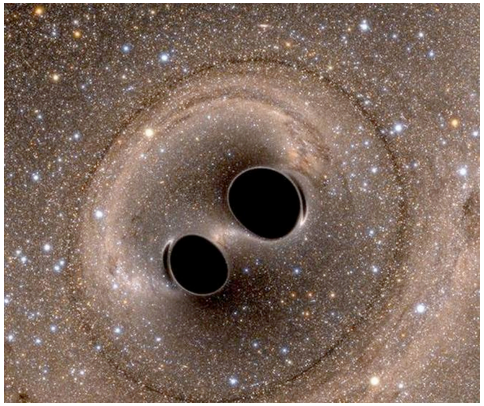
Gravity and gauge theory kinematic numerators are the same!

We use this form of double copy in latest calculations.

Double Copy for Classical Solutions

Goal is to formulate gravity solutions directly in terms of gauge theory

Variety of special cases:



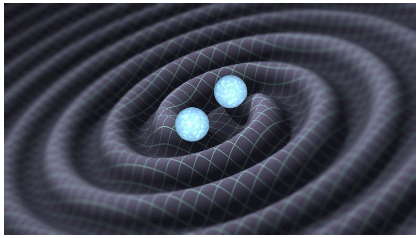
- **Schwarzschild and Kerr (spinning) black holes.**
- **Solutions with cosmological constant.**
- **Radiation from accelerating black hole.**
- **Maximally symmetric space times.**
- **Plane wave background.**
- **Gravitational radiation.**

Luna, Monteiro, O'Connell and White;
Luna, Monteiro, Nicholsen, O'Connell and White;
Ridgway and Wise; Carrillo González, Penco, Trodden;
Adamo, Casali, **Mason,** Nekovar; Adamo, **Mason**, Sharma;
Adamo, Liderton; Adamo, Kol; Goldberger and Ridgway; Chen;
Luna, Monteiro, Nicholson, Ochirov; Bjerrum-Bohr, Donoghue, Vanhove;
O'Connell, Westerberg, White; Luna, Monteiro, Nicholson, O'Connell;
Godazgar, Momteiro, Veiga, Pope; Chacón, García-Compeán, Luna,
Monteiro, White; Kosower, Maybee, O'Connell; Kim, Lee, Monteiro,
Nicholson, Veiga; etc

Still no general understanding.
But plenty of examples.

Someone smart needs to come and clean this up!

Scattering Amplitudes and Gravitational Radiation



A small industry has developed to study this.

- **Connection to scattering amplitudes.**

  Bjerrum-Bohr, Donoghue, Holstein, Plante, Pierre Vanhove; Luna, Nicholson, O'Connell, White; Guevara; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon; Damour; Bautista, Guevara; Kosower, Maybee, O'Connell; Plefka, Steinhoff, Wormsbecher; Foffa, Mastrolia, Sturani, Sturm; Guevara, Ochirov, Vines; Chung, Huang, Kim, Lee; etc.

- **Worldline approach for radiation and double copy.**

  Goldberger and Ridgway; Goldberger, Li, Prabhu, Thompson; Chester; Shen.

- **Technical issues having to do with keeping right physical states.**

  Luna, Nicholson, O'Connell, White; Johansson, Ochirov; Johansson, Kalin; Henrik Johansson, Gregor Kälin, Mogull.

Key Question: Can we calculate something of direct interest to LIGO/Virgo, decisively *beyond* previous state of the art?

Effective Field Theory Approach

ZB, Cheung, Roiban, Shen, Solon, Zeng

Cheung, Rothstein, Solon (2018)

Amplitudes community

Gravitational Scattering Amplitudes

Effective Field Theory Methods

EFT community

Kawai, Lewellen, Tye
ZB, Dixon, Dunbar and Kosower
ZB, Dixon, Dunbar, Perelstein, Rozowsky
ZB, Carrasco, Johansson; Etc

Beneke, Smirnov (Method of regions)
Goldberger, Rothstein;
Porto; Neill, Rothstein;
Vaydia, Foffa , Porto, Rothstein, Sturani;
Kol, Smolkin, Levi, Steinhoff, etc.

Post Minkowskian Potentials

In a form useful for bound state problem

The EFT directly gives us a two-body Hamiltonian of a form appropriate to enter LIGO analysis pipeline (after importing into EOB or pheno models).

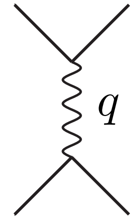
We prefer the EFT matching when pushing into new territory.

Potentials and Amplitudes

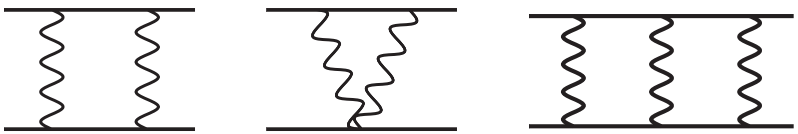
Iwasaki; Bjerrum-Bohr, Donoghue, Vanhove; Neill, Rothstein
Bjerrum-Bohr, Damgaard, Festuccia, Planté. Vanhove, etc

Tree-level: Fourier transform gives classical potential.

$$V(r) \sim \int \frac{d^3 q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} A^{\text{tree}}(\mathbf{q})$$



At higher orders things quickly become less obvious:



- **What I learned in grad school on $\hbar$ counting is wrong. Loops have classical pieces.**
- **Double counting and iteration.**
- **$1/\hbar$ scaling of loop amplitudes.**
- **Non-uniqueness of potential.**
- **Cross terms between $1/\hbar$ and $\hbar$**

$$e^{iS_{\text{classical}}/\hbar}$$

$1/\hbar^L$ at L loops

Piece of loops are classical: Our task is to extract these pieces.

We harness EFT to clean up confusion

EFT is a Clean Approach

No need to re-invent the wheel.
Build EFT from which we can read off potential.

Goldberger and Rothstein
Neill, Rothstein
Cheung, Rothstein, Solon (2018)

$$L_{\rm kin} = \int_{\boldsymbol{k}} A^\dagger(-\boldsymbol{k})\left(i\partial_t + \sqrt{\boldsymbol{k}^2 + m_A^2}\right)A(\boldsymbol{k}) + \int_{\boldsymbol{k}} B^\dagger(-\boldsymbol{k})\left(i\partial_t + \sqrt{\boldsymbol{k}^2 + m_B^2}\right)B(\boldsymbol{k})$$

***A*, *B* scalars represents spinless black holes**

$$L_{\rm int} = -\int_{\boldsymbol{k},\boldsymbol{k}'} V(\boldsymbol{k},\boldsymbol{k}')\, A^\dagger(\boldsymbol{k}')A(\boldsymbol{k})B^\dagger(-\boldsymbol{k}')B(-\boldsymbol{k})$$

Match amplitudes of this theory to the full theory in classical limit to extract a potential which can then be directly used for bound state.

The EFT is used to define the potential and Hamiltonian

EFT Matching

full general relativity
(complicated)

Amplitude methods
double copy

↓

tree amplitude

$\hbar \to 0$

generalized
unitarity

↓

loop integrand

Loop integration
Method of regions

↓

GR loop amplitude

effective theory
(simpler)

build
ansatz

↓

potential

Feynman
diagrams

↓

loop integrand

loop
integration

↓

EFT loop amplitude

identical
physics

GR loop amplitude $=$ **EFT loop amplitude**

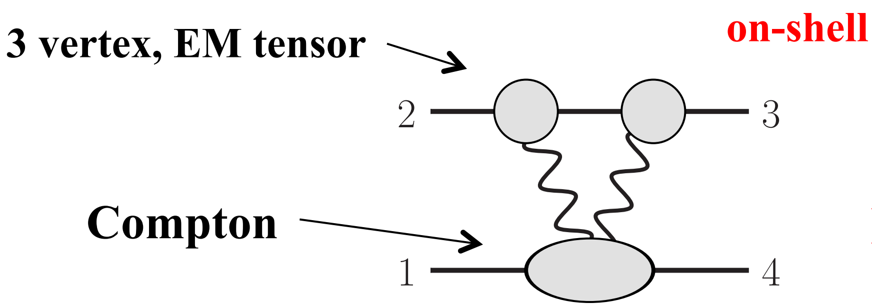
Roundabout but efficiently determines 2 body potential

General Relativity: Unitarity + Double Copy

- **Long-range force: Two matter lines must be separated by graviton propagators**
- **Classical potential: 1 matter line per loop is cut (on-shell).**

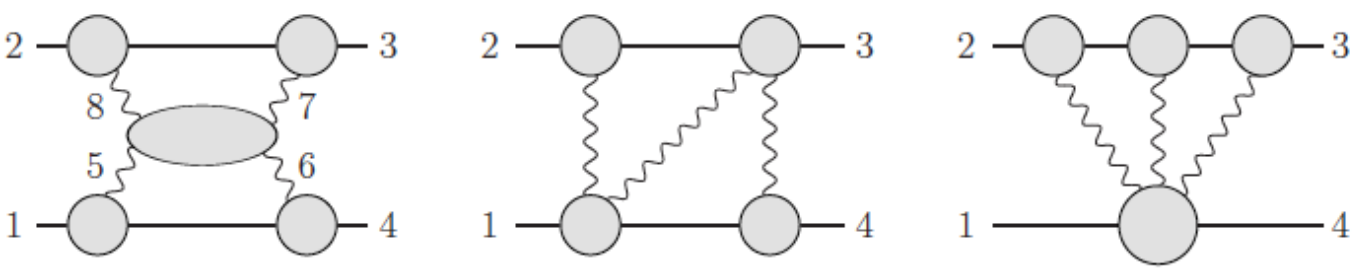
Neill and Rothstein ; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon

Only independent unitarity cut for 2 PM 2 body Hamiltonian.

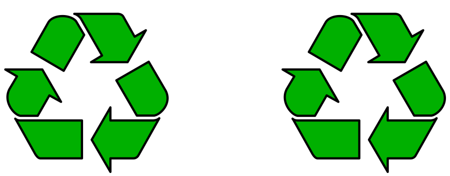


Treat exposed lines on-shell (long range).
Pieces we want are simple!

Independent generalized unitarity cuts for 3 PM.

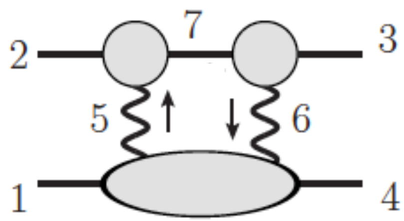


Our amplitude tools fit perfectly with extracting pieces we want.

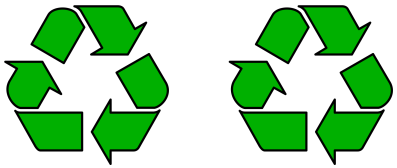


gravity **loops**

Generalized Unitarity Cuts



2nd post-Minkowkian order



KLT relations

$$C_{\rm GR} = \sum_{h_5,h_6=\pm} M_3^{\rm tree}(3^s, 6^{h_6}, -7^s)\, M_3^{\rm tree}(7^s, -5^{h_5}, 2^s)\, M_4^{\rm tree}(1^s, 5^{-h_5}, -6^{-h_6}, 4^s)$$

$$= \sum_{h_5,h_6=\pm} it[A_3^{\rm tree}(3^s, 6^{h_6}, -7^s)\, A_3^{\rm tree}(7^s, -5^{h_5}, 2^s)\, A_4^{\rm tree}(1^s, 5^{-h_5}, -6^{-h_6}, 4^s)]$$

$$\times\, [A_3^{\rm tree}(3^s, 6^{h_6}, -7^s)\, A_3^{\rm tree}(7^s, -5^{h_5}, 2^s)\, A_4^{\rm tree}(4^s, 5^{-h_5}, -6^{-h_6}, 1^s)]$$

Problem of computing the generalized cuts in gravity is reduced to multiplying and summing gauge-theory tree amplitudes.

$$A_4^{\rm tree}(1^s, 2^+, 3^+, 4^s) = i\frac{m^2\,[2\,3]}{\langle 2\,3\rangle\, \tau_{12}} \qquad A_4^{\rm tree}(1^s, 2^+, 3^-, 4^s) = i\frac{\langle 3|\,1\,|2]^2}{s_{23}\tau_{12}}$$

$$\tau_{12} = 2p_1\cdot p_2 \qquad s_{23} = (p_1+p_2)^2$$

- **For spinless case, same logic works to all orders: KLT and BCJ work for massless *n*-point in *D*-dimension. Dimensional reduction gives massive case**
- **Unwanted states (dilaton) easy to remove with physical state projectors.**

Amplitude in Conservative Classical Potential Limit

ZB, Cheung, Roiban, Shen, Solon, Zeng (BCRSSZ)

To make story short. The O(G^3) or 3PM conservative terms are:

$$\mathcal{M}_3 = \frac{\pi G^3 \nu^2 m^4 \log \boldsymbol{q}^2}{6\gamma^2\xi}\Bigg[3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3 - \frac{48\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} - \frac{18\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{(1+\gamma)(1+\sigma)}\Bigg] + \frac{8\pi^3 G^3 \nu^4 m^6}{\gamma^4\xi}\Big[3\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)F_1 - 32 m^2\nu^2\left(1-2\sigma^2\right)^3 F_2\Big]$$

$$m = m_1 + m_2 \qquad \mu = m_A m_B/m, \qquad \nu = \mu/m, \qquad \gamma = E/m,$$

$$\xi = E_1 E_2/E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1 \cdot p_2/m_1 m_2,$$

- **Amplitude remarkably compact.**
- **Arcsinh and the appearance of a mass singularity is new and robust feature. Cancels mass singularity of real radiation, as expected from KLN theorem.**
  Di Vecchia, Heissenberg, Russo, Veneziano; Damour
- **IR finite parts of amplitude directly connected to scattering angle.**
  Expanded on by Kälin, Porto; Bjerrum-Bohr, Cristofoli, Damgaard
- **Derived conservative scattering angle has simple mass dependence.**
  Observed by Antonelli, Buonanno, Steinhoff, van de Meent, Vines (1901.07102)
  Comprehensive understanding: Damour

Conservative O(G^3) 2-body Hamiltonian

BCRSSZ

The O(G^3) 3PM Hamiltonian:

$$H(\boldsymbol{p},\boldsymbol{r}) = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + V(\boldsymbol{p},\boldsymbol{r})$$

$$V(\boldsymbol{p},\boldsymbol{r}) = \sum_{i=1}^{3} c_i(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)^i,$$

Newton in here

$$c_1 = \frac{\nu^2 m^2}{\gamma^2 \xi}\left(1 - 2\sigma^2\right), \qquad c_2 = \frac{\nu^2 m^3}{\gamma^2 \xi}\left[\frac{3}{4}\left(1 - 5\sigma^2\right) - \frac{4\nu\sigma\left(1 - 2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1 - 2\sigma^2\right)^2}{2\gamma^3\xi^2}\right],$$

$$c_3 = \frac{\nu^2 m^4}{\gamma^2 \xi}\left[\frac{1}{12}\left(3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3\right) - \frac{4\nu\left(3 + 12\sigma^2 - 4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2 - 1}}\right.$$
$$- \frac{3\nu\gamma\left(1 - 2\sigma^2\right)\left(1 - 5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} - \frac{3\nu\sigma\left(7 - 20\sigma^2\right)}{2\gamma\xi} - \frac{\nu^2\left(3 + 8\gamma - 3\xi - 15\sigma^2 - 80\gamma\sigma^2 + 15\xi\sigma^2\right)\left(1 - 2\sigma^2\right)}{4\gamma^3\xi^2}$$
$$\left. + \frac{2\nu^3(3 - 4\xi)\sigma\left(1 - 2\sigma^2\right)^2}{\gamma^4\xi^3} + \frac{\nu^4(1 - 2\xi)\left(1 - 2\sigma^2\right)^3}{2\gamma^6\xi^4}\right],$$

$$m = m_1 + m_2 \qquad \mu = m_A m_B/m, \qquad \nu = \mu/m, \qquad \gamma = E/m,$$
$$\xi = E_1 E_2/E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1 \cdot p_2/m_1 m_2,$$

- **Expanding in velocity gives infinite sequence of terms in PN expansion.**
- **Can be put into EOB form.** Antonelli, Buonannom Steinhoff,van de Meent, Vines

How do we know it is right?

Original check:

Compared to 4PN Hamiltonians after canonical transformation

Damour, Jaranowski, Schäfer; Bernard, Blanchet, Bohé, Faye, Marsat

Thibault Damour seriously questioned correctness.
Specific corrections proposed. Damour, arXiv:1912.02139v1

Saint Julien 2011

Subsequent calculations confirm our 3PM result:

1. **Papers confirming our result in 6PN overlap.**

   Blümlein, Maier, Marquard, Schäfer;
   Bini, Damour, Geralico

2. **Subsequent calculations reproducing our 3PM result.**

   Cheung and Solon; Kälin, Liu, Porto

3. **Scattering angle checks.** ZB, Ita, Parra-Martinez, Ruf

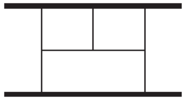
4. **Adding real radiation removes mass singularity.**

   Di Vecchia, Heissenberg, Russo, Veneziano; Damour

3PM results have passed highly nontrivial checks and careful scrutiny.

Conservative Potential-Mode Contribution O(G^4)

ZB, Parra-Martinez, Roiban, Ruf. Shen, Solon, Zeng



O(G^4) amplitude

test particle — 1st self force — Iteration. No need to compute — tail effect (IR divergent)

$$\mathcal{M}_4(\boldsymbol{q}) = G^4 M^7 \nu^2 |\boldsymbol{q}| \left(\frac{\boldsymbol{q}^2}{4^{\frac{1}{3}}\tilde{\mu}^2}\right)^{-3\epsilon} \pi^2 \left[\mathcal{M}_4^{\rm p} + \nu\left(\frac{\mathcal{M}_4^{\rm t}}{\epsilon} + \mathcal{M}_4^{\rm f}\right)\right] + \int_\ell \frac{\tilde{I}_{r,1}^4}{Z_1 Z_2 Z_3} + \int_\ell \frac{\tilde{I}_{r,1}^2 \tilde{I}_{r,2}}{Z_1 Z_2} + \int_\ell \frac{\tilde{I}_{r,1}\tilde{I}_{r,3}}{Z_1} + \int_\ell \frac{\tilde{I}_{r,2}^2}{Z_1}$$

$$\mathcal{M}_4^{\rm p} = -\frac{35\left(1 - 18\sigma^2 + 33\sigma^4\right)}{8\left(\sigma^2-1\right)}, \qquad \mathcal{M}_4^{\rm t} = h_1 + h_2 \log\left(\tfrac{\sigma+1}{2}\right) + h_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}},$$

$$\begin{aligned}
\mathcal{M}_4^{\rm f} = {} & h_4 + h_5 \log\left(\tfrac{\sigma+1}{2}\right) + h_6 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + h_7 \log(\sigma) - h_2 \frac{2\pi^2}{3} + h_8 \frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1} + h_9 \left[\operatorname{Li}_2\left(\tfrac{1-\sigma}{2}\right) + \frac{1}{2}\log^2\left(\tfrac{\sigma+1}{2}\right)\right] \\
& + h_{10}\left[\operatorname{Li}_2\left(\tfrac{1-\sigma}{2}\right) - \frac{\pi^2}{6}\right] + h_{11}\left[\operatorname{Li}_2\left(\tfrac{1-\sigma}{1+\sigma}\right) - \operatorname{Li}_2\left(\tfrac{\sigma-1}{\sigma+1}\right) + \frac{\pi^2}{3}\right] + h_2 \frac{2\sigma(2\sigma^2-3)}{(\sigma^2-1)^{3/2}}\left[\operatorname{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - \operatorname{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right] \\
& + \frac{2h_3}{\sqrt{\sigma^2-1}}\left[\operatorname{Li}_2\left(1-\sigma-\sqrt{\sigma^2-1}\right) - \operatorname{Li}_2\left(1-\sigma+\sqrt{\sigma^2-1}\right) + 5\operatorname{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - 5\operatorname{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) + 2\log\left(\tfrac{\sigma+1}{2}\right)\operatorname{arccosh}(\sigma)\right] \\
& + h_{12}\mathrm{K}^2\left(\tfrac{\sigma-1}{\sigma+1}\right) + h_{13}\mathrm{K}\left(\tfrac{\sigma-1}{\sigma+1}\right)\mathrm{E}\left(\tfrac{\sigma-1}{\sigma+1}\right) + h_{14}\mathrm{E}^2\left(\tfrac{\sigma-1}{\sigma+1}\right)
\end{aligned}$$

← elliptic

$$\nu = m_1 m_2/(m_1+m_2)^2 \qquad \sigma = p_1 \cdot p_2 / m_1 m_2,$$

$$h_1 = \frac{1151 - 3336\sigma + 3148\sigma^2 - 912\sigma^3 + 339\sigma^4 - 552\sigma^5 + 210\sigma^6}{12(\sigma^2-1)}$$

$$h_2 = \frac{1}{2}\left(5 - 76\sigma + 150\sigma^2 - 60\sigma^3 - 35\sigma^4\right)$$

$$h_3 = \sigma\frac{\left(-3+2\sigma^2\right)}{4(\sigma^2-1)}\left(11 - 30\sigma^2 + 35\sigma^4\right)$$

$$\begin{aligned} h_4 = {} & \frac{1}{144(\sigma^2-1)^2\sigma^7}(-45 + 207\sigma^2 - 1471\sigma^4 + 13349\sigma^6 \\ & - 37566\sigma^7 + 104753\sigma^8 - 12312\sigma^9 - 102759\sigma^{10} - 105498\sigma^{11} \\ & + 134745\sigma^{12} + 83844\sigma^{13} - 101979\sigma^{14} + 13644\sigma^{15} + 10800\sigma^{16}) \end{aligned}$$

$$\begin{aligned} h_5 = {} & \frac{1}{4(\sigma^2-1)}(1759 - 4768\sigma + 3407\sigma^2 - 1316\sigma^3 + 957\sigma^4 \\ & - 672\sigma^5 + 341\sigma^6 + 100\sigma^7) \end{aligned}$$

$$\begin{aligned} h_6 = {} & \frac{1}{24(\sigma^2-1)^2}(1237 + 7959\sigma - 25183\sigma^2 + 12915\sigma^3 + 18102\sigma^4 \\ & - 12105\sigma^5 - 9572\sigma^6 + 2973\sigma^7 + 5816\sigma^8 - 2046\sigma^9) \end{aligned}$$

$$h_7 = 2\sigma\frac{\left(-852 - 283\sigma^2 - 140\sigma^4 + 75\sigma^6\right)}{3(\sigma^2-1)}$$

$$\begin{aligned} h_8 = {} & \frac{\sigma}{8(\sigma^2-1)^2}(-304 - 99\sigma + 672\sigma^2 + 402\sigma^3 - 192\sigma^4 - 719\sigma^5 \\ & - 416\sigma^6 + 540\sigma^7 + 240\sigma^8 - 140\sigma^9) \end{aligned}$$

$$h_9 = \frac{1}{2}\left(52 - 532\sigma + 351\sigma^2 - 420\sigma^3 + 30\sigma^4 - 25\sigma^6\right)$$

$$h_{10} = 2\left(27 + 90\sigma^2 + 35\sigma^4\right)$$

$$h_{11} = 20 + 111\sigma^2 + 30\sigma^4 - 25\sigma^6$$

$$h_{12} = \frac{834 + 2095\sigma + 1200\sigma^2}{2(\sigma^2-1)}$$

$$h_{13} = -\frac{1183 + 2929\sigma + 2660\sigma^2 + 1200\sigma^3}{2(\sigma^2-1)}$$

$$h_{14} = \frac{7\left(169 + 380\sigma^2\right)}{4(\sigma-1)}$$

Read O(G^4) radial action off directly from scattering amplitude:

$$I_{r,4}(J) = -\frac{G^4 M^7 \nu^2 \pi \boldsymbol{p}^2}{8EJ^3}\left(\frac{4\tilde{\mu}^2 e^{2\gamma_E} J^2}{\boldsymbol{p}^2}\right)^{4\epsilon}\left[\mathcal{M}_4^{\rm p} + \nu\left(\frac{\mathcal{M}_4^{\rm t}}{\epsilon} + \mathcal{M}_4^{\rm f} - 14\mathcal{M}_4^{\rm t}\right)\right]$$

$$\chi = -\frac{\partial I_r}{\partial J}$$

Scattering angle (χ); radial action (I_r); angular momentum (J)

- **Here radiation contributions to conservative tail effect not included. New to O(G^4).**
- **IR divergence will cancel once radiation is included. Working on this.**
- **High-energy limit has mass singularity. Presumably, cancels against real radiation.**

Conservative Potential Mode Contributions O(G^4)

The two body Hamiltonian:

$$H^{\text{iso}} = \sqrt{\mathbf{p}^2 + m_1} + \sqrt{\mathbf{p^2} + m_2} + \sum_{i=1}^{4} \frac{G^n}{r^n} c_n(\mathbf{p^2})$$

Isotropic gauge Hamiltonian

$$c_4 = \frac{M^7\nu^2}{4\xi E^2}\left[\mathcal{M}_4^{\text{p}} + \nu\left(\frac{\mathcal{M}_4^{\text{t}}}{\epsilon} + \mathcal{M}_4^{\text{f}} - 10\mathcal{M}_4^{\text{t}}\right)\right] + \mathcal{D}^3\left[\frac{E^3\xi^3}{3}c_1^4\right] + \mathcal{D}^2\left[\left(\frac{E^3\xi^3}{\boldsymbol{p}^2} + \frac{E\xi(3\xi-1)}{2}\right)c_1^4 - 2E^2\xi^2 c_1^2 c_2\right]$$
$$+\left(\mathcal{D} + \frac{1}{\boldsymbol{p}^2}\right)\left[E\xi(2c_1c_3 + c_2^2) + \left(\frac{4\xi-1}{4E} + \frac{2E^3\xi^3}{\boldsymbol{p}^4} + \frac{E\xi(3\xi-1)}{\boldsymbol{p}^2}\right)c_1^4 + \left((1-3\xi) - \frac{4E^2\xi^2}{\boldsymbol{p}^2}\right)c_1^2 c_2\right],$$

$$\xi = E_1E_2/E^2, \qquad E = E_1 + E_2, \qquad \nu = \mu/m, \qquad \mathcal{D} = \frac{d}{dp^2}$$

As for amplitude, radiation effects on conservative dynamics *not* included here.

Above divergent part of tail effect gives radiated energy.

Bini, Damour Geralico

$$\Delta E = \frac{G^3 M^7 \nu^3 \pi \boldsymbol{p}^2}{4E^2 J^3}\mathcal{M}_4^{\text{t}}$$

Matches direct calculation

Herrmann, Parra-Martinez, Ruf, Zeng

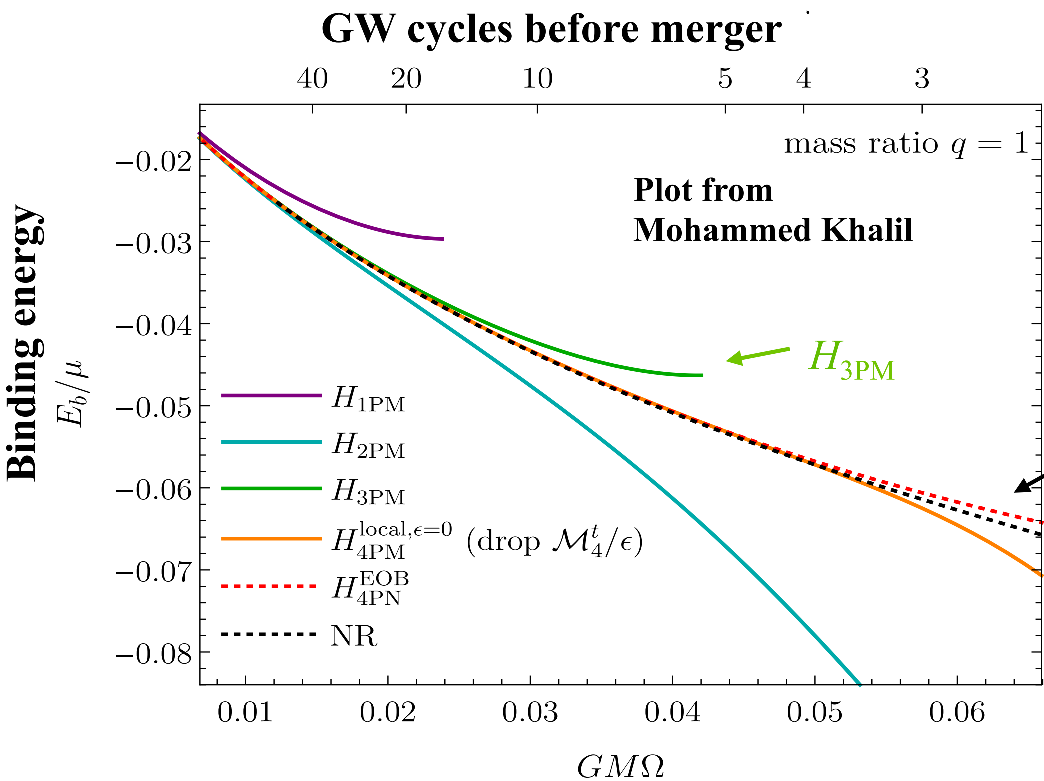
Results agree with all known overlap PN results through 6PN.

Blumlein, Maier, Marquard, Schafer; Bini, Damour and Geralico

Preliminary O(G^4) Binding Energy

Khalil, Buonanno, Vines, Steinhoff

Even though only local in time contributions included, good to look at binding energy to see if we are on a right track.



Even better by feeding 4PM through EOB

NR and 4PN EOB (profession grade!)

$H_{4\mathrm{PM}}$ missing part of tail terms

- ***Not* conclusive (missing piece), but very encouraging.**
- **Motivates us to finish missing radiation tail contributions!**

Outlook

Amplitude methods have a lot of promise and their use has already been tested for a variety of problems.

- **Pushing state of the art for high orders in G.** ZB, Cheung, Roiban, Parra-Martinez, Ruf, Shen, Solon, Zeng
- **Radiation.** Cristofoli, Gonzo, Kosower, O'Connell; Herrmann, Parra-Martinez, Ruf, Zeng; Di Vecchia, Heissenberg, Russo, Veneziano
- **Finite size effects.** Cheung and Solon; Haddad and Helset; Kälin, Liu, Porto; Cheung, Shah, Solon; ZB, Parra-Martinez, Roiban, Sawyer, Shen
- **Spin.** Vaidya; Geuvara, O'Connell, Vines; Chung, Huang, Kim, Lee; ZB, Luna, Roiban, Shen, Zeng; Kosmopoulos, Luna, etc

Expect many more state of the art calculations in coming years.

Summary

- Twistors had a huge impact on scattering amplitudes. Lionel is our twistor guru, always at forefront of developments.
- Amplitudes provide a useful way to think about problems in gravity.
- Amplitudes are independent of gauges, coordinates and field variables, making it simpler to identify useful new structures.
- Methods work on a variety of problems. High orders in two body potential in GR for gravitational wave problem.
- Most exciting part is that the methods not close to exhausted.

Happy birthday to Lionel!