EASISchool 3

SUPERCONDUCTING DIPOLES AND QUADRUPOLES FOR ACCELERATORS 1

October 5th, 2020

Stefania Farinon
Superconducting magnet design

- Superconducting magnet design is a true multiphysics problem involving several activities
  - Electromagnetic optimization (field quality, peak field on conductor, margin)
  - Choice of the conductor (transport properties)
  - Choice of the operating temperature and cryogenic design
  - Design of the mechanical support structure
  - Analysis of stability and quench protection
  - Manufacturing techniques
  - Cost analysis
Outline

- Overview of superconducting magnets for particle accelerators (dipoles and quadrupoles)
  - Conductor
  - Magnetic design
  - Mechanical design
    - Force, stress and pre-load
    - Support structures
  - Quench protection

Stefania Farinon
Susana Izquierdo Bermudez
Practical conductors for accelerator magnets
Comparison of critical current densities @ 4.2 K
https://fs.magnet.fsu.edu/~lee/plot/plot.htm
NbTi

- NbTi is the most widely used superconductor
- In High Energy Physics, NbTi has been used for Tevatron (Fermilab), HERA (DESY), RHIC (BNL), LHC (CERN)
  \[ J_c (B,T) = \frac{C}{B} \left[ \frac{B}{B_{c2}(T)} \right]^\alpha \left[ 1 - \frac{B}{B_{c2}(T)} \right]^\beta \left[ 1 - \left( \frac{T}{T_{c0}} \right)^{\gamma} \right] \]
  with \( T_{c0} = 9.2 \, \text{K} \)  \( B_{c20} = 14.5 \, \text{T} \)  \( B_{c2}(T) = B_{c20} \left[ 1 - \left( \frac{T}{T_{c0}} \right)^{1.7} \right] \)
- Fitting parameters for LHC wires \( J_c (5 \, \text{T}, 4.2 \, \text{K}) = 3000 \, \text{A/mm}^2 \):
  \( C = 92.1 \, \text{T} \cdot \text{kA/mm}^2, \alpha = 0.63, \beta = 1.0, \gamma = 2.3 \)
- Practical limit for accelerator magnets:
  - \( B_{c2} (1.9 \, \text{K}) = 13.5 \, \text{T} \), but \( J_c (13.5 \, \text{T}, 1.9 \, \text{K}) = 0 \)
  - To have reasonable current density \( B \lesssim 10 \, \text{T}, J_c (10 \, \text{T}, 1.9 \, \text{K}) = 1680 \, \text{A/mm}^2 \)
  - Taking some margins (see next slides) \( B \lesssim 8 \, \text{T} \)
**Nb$_3$Sn**

- Nb3Sn is the choice to go beyond the NbTi limits in accelerator magnets
  - 11 T dipoles and triplet quadrupoles in High Luminosity LHC
  \[
  J_C(B, T) = \frac{C}{B} \left[ \frac{B}{B_{c2}(T)} \right]^{\alpha} \left[ 1 - \frac{B}{B_{c2}(T)} \right]^{\beta} \left[ 1 - \left( \frac{T}{T_{c0}} \right) \right]^{1.527} \left[ 1 - \left( \frac{T}{T_{c0}} \right)^2 \right]^{\gamma}
  \]
  with $T_{c0} = 16$ K, $B_{c20} = 29$ T, $B_{c2}(T) = B_{c20} \left[ 1 - \left( \frac{T}{T_{c0}} \right)^{1.7} \right]$
- Fitting parameters for target FCC wires ($J_C(16 T, 4.2 K) = 1500$ A/mm$^2$):
  - $C = 267.845$ T · kA/mm$^2$, $\alpha = 0.5$, $\beta = 2$, $\gamma = 0.96$
- Practical limit for accelerator magnets:
  - $B_{c2}(1.9 K) = 28$ T
  - To have reasonable current density $B \leq 18$ T, $J_C(18$ T, $1.9 K) = 1480$ A/mm$^2$
  - Taking some margins (see next slides) $B \leq 16$ T, that double the performance WRT NbTi
Multifilament wires

- For practical applications, superconducting materials are produced in small filaments and surrounded by a stabilizer (typically copper) to form a multifilament wire or strand.
  - Typical filament diameter is in the range $3-10 \mu m$ for NbTi and $\leq 50 \mu m$ for Nb$_3$Sn; typical strand diameter is $\leq 1 \text{ mm}$.
  - Fine filaments to:
    - Reduce effects due to magnetization.
    - Limit flux jumps.
  - Copper matrix:
    - For protection and stability (see Susana presentation).
  - Twisting:
    - To reduce interfilament coupling and AC losses.
Multistrand cables

- Most of the superconducting coils for particle accelerators are wound from a multi-strand Rutherford cable.

- Main advantages:
  - Rutherford cables
    - high density compaction of strands
    - rectangular or trapezoidal shape (to stack arc-shaped coils)
  - multi-strand cables
    - large current density with small number of turns
    - smaller coil inductance
    - current redistribution in case of problem in a portion of a strand
  - strand twisting
    - to reduce inter-strand coupling and AC losses
  - strand transposition
    - to eliminate the flux enclosed
    - to increase the mechanical stability
In SC magnets what really matters is the overall 'engineering' current density $J_{\text{eng}}$

$$J_{\text{eng}} = \frac{N_{\text{strand}} I}{A_{\text{cable}}} = J_C \lambda_{\text{strand}} \lambda_{\text{cable}}$$

- Typical $R_{\text{Cu-non Cu}}$ ranges from 1 to 2, then $\lambda_{\text{strand}}$ ranges from 0.33 to 0.5 ($\lambda_{\text{strand}} = \frac{1}{1 + R_{\text{Cu-non Cu}}}$)

- $\lambda_{\text{cable}}$ takes into account the total space occupied by each turn, and is typically 0.7 to 0.8

- So typically $J_{\text{eng}}$ is only 20% to 40% of $J_C$
Margin on the loadline

- The margin of a magnet is defined with respect to its weakest point, i.e. the peak field.
- \( J_{SS} \) (Short Sample) corresponds to the intersection of the loadline for the peak field and the critical current density curve: ideally is the maximum performance of the magnet.
- The loadline fraction is the ratio \( J_{op}/J_{SS} \).
- The margin on the loadline is \( 1 - J_{op}/J_{SS} \).
The temperature margin is a physical quantity related to the energy which can be released before crossing the critical surface (order of few K).

It is the temperature rise necessary for the $J_C$ curve to intersect the loadline at operative field.
Dipole and quadrupole definition
Dipole and quadrupole definition

- Dipole magnets generate constant and uniform field $B$:

- Quadrupole magnets generate constant and uniform gradient $G$:
What is the effect of a dipole on a travelling particle?

- A particle of charge $q$ travelling in a uniform magnetic field $B$ at speed $v$ is subjected to the Lorentz force $F_L = qv \times B$.
- The Lorentz force is balanced by the centrifugal force $F_C = mv^2/r$.
- The result is that the Lorentz force keeps particles in a circular orbit:
  $$\frac{mv^2}{\rho} = qvB \quad \Rightarrow \quad \rho = \frac{mv}{qB} = \frac{p}{qB}$$
What is the effect of a quadrupole on a travelling particle?

Since the only force is magnetic:
\[ \vec{F} = m \frac{d^2 \vec{R}}{dt^2} = q \vec{v} \times \vec{B}, \quad \vec{R} = (\rho + x)\hat{x} + y\hat{y} \]

If \( v \sim v_s \gg v_x, v_y \)
\[ \vec{v} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & v \\ B_x & B_y & 0 \end{vmatrix} = -vB_y\hat{x} + vB_x\hat{y} \]

Along s direction, \( s = vt \) then \( \frac{d}{dt} = \frac{d}{ds} \frac{ds}{dt} = \nu \frac{d}{ds} \):
\[ v^2 \frac{d^2 x}{ds^2} = -\frac{1}{m} qvB_y \]
\[ v^2 \frac{d^2 y}{ds^2} = \frac{1}{m} qvB_x \]

Time is replaced by \( s \), which is the reference orbit given by the bending magnets and is moving with the beam.

\[ \frac{d^2 x}{ds^2} + \frac{q}{p} B_y = 0 \quad \text{and} \quad \frac{d^2 y}{ds^2} - \frac{q}{p} B_x = 0 \]

where \( p = mv \)
What is the effect of a quadrupole on a travelling particle?

- Quadrupole field: \( B_x = G_y, \quad B_y = Gx, \quad G \) is the field gradient

\[
\frac{d^2x}{ds^2} + \frac{q}{p} B_y = 0 \quad \Rightarrow \quad \frac{d^2x}{ds^2} + \frac{qG}{p} x = 0
\]

- this is a (mass-spring) harmonic oscillator; the motion can be described by the function:

\[
x(s) = A \cos(\psi(s))
\]

- The global effect is focusing in the \( x \) direction and defocusing in the \( y \) direction (QF)

- If fields have the opposite sign, we get focusing in the \( y \) direction and defocusing in the \( x \) direction (QD)

- In an accelerator, quadrupoles give the force necessary to stabilize the linear motion
Magnetic field of a current line

- From the Maxwell equation:
  \[
  \nabla \times \vec{B} = \mu_0 \vec{J} \quad \oint \vec{B} \, d\ell = \mu_0 I
  \]

- It’s easy to find that
  \[
  B(r) = \frac{\mu_0 I}{2\pi r}
  \]
  lying on a plane perpendicular to the current line and tangent to the circumference of radius \( r \)
Basics of complex numbers

- By definition, the complex number $i$ is the solution of the equation $i^2 = -1$
- A general complex number is identified by 2 components: $z = a + ib$
  - where $a$ is the real part
  - and $b$ is the imaginary part
- It can be also written in the exponential form $z = re^{i\theta} = r(\cos \theta + i \sin \theta)$
Magnetic field of a current line

- In a more general coordinate system, using complex notation:

  \[ \mathbf{B}(z) = \frac{\mu_0 I}{2\pi(z-a)}, \text{ con } z = \rho e^{i\phi} \text{ e } a = r e^{i\theta} \]

- In facts:

  \[ \mathbf{B}(z) = \frac{\mu_0 I}{2\pi(\rho e^{i\phi} - r e^{i\theta})} = \frac{\mu_0 I}{2\pi[(\rho \cos \phi - r \cos \theta) + i(\rho \sin \phi - r \sin \theta)]} \]

  \[ = \frac{\mu_0 I}{2\pi} \frac{(r \cos \theta - \rho \cos \phi) + i(\rho \sin \phi - r \sin \theta)}{r^2 + \rho^2 - 2r \rho \cos(\theta - \phi)} \]

  \[ = \frac{\mu_0 I}{2\pi R} \frac{r \cos \theta - \rho \cos \phi}{R} \]

  \[ = \frac{\mu_0 I}{2\pi R} (\sin \gamma + i \cos \gamma) = B_y + iB_x \]
Magnetic field in \( z = \rho e^{i\varphi} \) of a current line in \( a = re^{i\vartheta} \) if \( \rho < r \)

- \( \mathbf{B}(z) = \frac{\mu_0 I}{2\pi(z-a)} = \frac{\mu_0 I}{2\pi(\rho e^{i\varphi} - re^{i\vartheta})} = -\frac{\mu_0 I}{2\pi re^{i\vartheta}} \frac{1}{1 - \frac{\rho}{r} e^{i(\varphi - \vartheta)}} \)

- Reminding that if \( \epsilon < 1 \) :
  \[
  \frac{1}{1-\epsilon} = \sum_{n=1}^{\infty} \epsilon^{n-1}
  \]

- \( \mathbf{B}(z) = \frac{\mu_0 I}{2\pi r} e^{-i\vartheta} \sum_{n=1}^{\infty} \left[ \frac{\rho}{r} e^{i(\varphi - \vartheta)} \right]^{n-1} = -\frac{\mu_0 I}{2\pi r} \sum_{n=1}^{\infty} e^{-in\vartheta} \left( \frac{\rho e^{i\varphi}}{r} \right)^{n-1} = -\frac{\mu_0 I}{2\pi r} \sum_{n=1}^{\infty} e^{-in\vartheta} \left( \frac{R_{ref}}{r} \right)^{n-1} \left( \frac{\rho e^{i\varphi}}{R_{ref}} \right)^{n-1} \)

- \[
  = \sum_{n=1}^{\infty} (B_n + iA_n)(\cos(n-1)\varphi + i\sin(n-1)\varphi) \left( \frac{\rho}{R_{ref}} \right)^{n-1}
  \]

With

- \( B_n = -\frac{\mu_0 I}{2\pi r} \left( \frac{R_{ref}}{r} \right)^{n-1} \cos n\vartheta \)
- \( A_n = \frac{\mu_0 I}{2\pi r} \left( \frac{R_{ref}}{r} \right)^{n-1} \sin n\vartheta \)

\( A_n \) and \( B_n \) are in T
Harmonic components of magnetic field

- The magnetic field can be expanded in series as
  \[ B_x + i B_y = \sum_{n=1}^{\infty} (B_n + i A_n)(\cos(n-1)\varphi + i \sin(n-1)\varphi) \left( \frac{\rho}{R_{\text{ref}}} \right)^{n-1} \]

- where \( B_n \) coefficients are the normal multipoles and \( A_n \) coefficients are the skew multipoles: dipole (\( n=1 \)), quadrupole (\( n=2 \)), sextupole (\( n=3 \)), octupole (\( n=4 \)), ....

- To get them in practical adimensioned units, harmonics are often normalized:
  \[ B_x + i B_y = 10^{-4} B_{\text{norm}} \sum_{n=1}^{\infty} (b_n + i a_n)(\cos(n-1)\varphi + i \sin(n-1)\varphi) \left( \frac{r}{R_{\text{ref}}} \right)^{n-1} \]
  with \( b_n = -\frac{10^4 \mu_0 I}{B_{\text{norm}} 2\pi R_{\text{ref}} r} \left( \frac{R_{\text{ref}}}{r} \right)^n \cos n\vartheta \)
  \( a_n = \frac{10^4 \mu_0 I}{B_{\text{norm}} 2\pi R_{\text{ref}} r} \left( \frac{R_{\text{ref}}}{r} \right)^n \sin n\vartheta \)

- \( B_{\text{norm}} \) [T] is the normalization field, \( B_{\text{norm}} = B_1 \) for dipoles, \( B_{\text{norm}} = B_2 = G R_{\text{ref}} \) for quadrupoles, etc.
Dipoles

4.5T

5.3T

3.5T

8.3T

HERA,
9 m, 75 mm
416 dipoles

RHIC,
9 m, 80 mm
264 dipoles

LHC,
15 m, 56 mm
1276 dipoles

Tevatron,
6 m, 76 mm
774 dipoles
Perfect dipole #1: «wall-dipole»

- A uniform current density flowing in two parallel walls of infinite height generates a pure dipolar field
  - winding and mechanical structure are not particularly complicated
  - the coil is theoretically infinite
  - coil truncation results in an acceptable field quality only for large dimensions
  - simply applying the Biot-Savart law $B_y = -\frac{\mu_0 J w}{2}$
Perfect dipole #2: intersecting circles

- Within a cylinder carrying uniform $J$, the field is $B(r) = \frac{\mu_0 J r}{2}$, directed tangentially.

- Combining the effect of the two cylinders:
  
  $B_y = \frac{\mu_0 J}{2} (-r_1 \cos \theta_1 + r_2 \cos \theta_2) = -\frac{\mu_0 J s}{2}$
  
  $B_x = \frac{\mu_0 J}{2} (+r_1 \sin \theta_1 - r_2 \sin \theta_2) = 0$

From “Superconducting Magnets”, M.N.Wilson
Perfect dipole #2: intersecting ellipses

- Analogously, two intersecting ellipses of semi-axes $b$ and $c$ generate a pure dipolar field given by:

$$B_y = -\mu_0 J s \frac{c}{(b + c)}$$

- The shape of intersecting circles and ellipses is not particularly favourable to winding:
  - central aperture is not circular
  - an inner mechanical support could be needed (further reducing available aperture)
Perfect dipole #3: $J \cos \vartheta$ distribution

- Let us consider a current density distribution $J \cos \vartheta$ in a shell of inner radius $R$ and thickness $w$.

- I remind that the normal harmonic component of a line current in $(r, \vartheta)$ is given by:

$$B_n(\rho, \vartheta) = -\frac{\mu_0 I}{2\pi R_{ref}} \left(\frac{R_{ref}}{r}\right)^n \cos n \vartheta$$

- To get the total contribution we replace $I$ with $JdS = J \cos \vartheta \cdot rdrd\vartheta$ and integrate from 0 to $2\pi$. 
Perfect dipole #3: $J \cos \vartheta$ distribution

\[
B_n = -\frac{\mu_0 J}{2\pi R_{ref}} \int_{R}^{R+w} \left( \frac{R_{ref}}{r} \right)^n r \int_0^{2\pi} \cos \vartheta \cos n \vartheta d\vartheta \\
\int_0^{2\pi} \cos \vartheta \cos n \vartheta d\vartheta = \begin{cases} 
\pi & \text{if } n = 1 \\
0 & \text{if } n \neq 1
\end{cases}
\]

- The only surviving term is $B_1$, i.e. the dipole field:

\[
B_1 = -\frac{\mu_0 J_0}{2\pi R_{ref}} \int_{R}^{R+w} \left( \frac{R_{ref}}{r} \right) r \int_0^{2\pi} \cos \vartheta \cos \vartheta d\vartheta = -\frac{\mu_0 J w}{2}
\]

- How can we approach this distribution using real conductors?

\[
\begin{align*}
B_1 \propto \text{current density (obvious)} \\
B_1 \propto \text{coil width } w \text{ (less obvious)} \\
B_1 \text{ is independent of the aperture } R \text{ (surprising)}
\end{align*}
\]
Perfect dipole vs real dipole

- Using real conductors, current density need to be uniform

- The perfect $J \cos \vartheta$ distribution is approached accumulating turn close to the midplane (where $\cos \vartheta \sim 1$) and reducing them at $90^\circ$ (where $\cos \vartheta \to 0$)
  - the aperture is circular
  - the winding is self-supporting (roman arc)
Sector dipoles

- The simplest approach is the sector dipole
- To get the dipole field $B_1$ we start again from the general expression for a current line
  \[ B_n(r, \theta) = -\frac{\mu_0 I}{2\pi R_{ref}} \left( \frac{R_{ref}}{r} \right)^n \cos n \theta \]
- Replacing $I \rightarrow J dS = J \cdot r dr d\theta$ and integrating for $n = 1$ we find:
  \[ B_1 = -2 \frac{\mu_0 J}{2\pi R_{ref}} \int_R^{R+w} \left( \frac{R_{ref}}{r} \right) r dr \int_{-\alpha}^{\alpha} \cos \theta d\theta \]
  \[ = -\frac{2\mu_0 J w \sin \alpha}{\pi} \]
Symmetrical line currents

\[ B_n = -\frac{\mu_0 I}{2\pi R_{\text{ref}}} \left( \frac{R_{\text{ref}}}{r} \right)^n \cos n \vartheta - \frac{\mu_0 (-I)}{2\pi R_{\text{ref}}} \left( \frac{R_{\text{ref}}}{r} \right)^n \cos n (\pi - \vartheta) \]

\[ B_n = -\frac{\mu_0 I}{2\pi R_{\text{ref}}} \left( \frac{R_{\text{ref}}}{r} \right)^n \left[ \cos n \vartheta - \cos n (\pi - \vartheta) \right] \]

\[ B_n = -\frac{\mu_0 I}{2\pi R_{\text{ref}}} \left( \frac{R_{\text{ref}}}{r} \right)^n \cos n \vartheta [1 - \cos n \pi] \]

\[ = \begin{cases} 
-2 & \text{for odd } n \\
0 & \text{for even } n 
\end{cases} \]
Multipoles of a sector dipole

- Following the result for symmetrical line currents, $B_n = 0$ for even $n$
- For odd $n$:

$$B_n = -2 \frac{\mu_0 J}{2 \pi R_{\text{ref}}} \int_{R}^{R+w} \left( \frac{R_{\text{ref}}}{r} \right)^n r dr \int_{-\alpha}^{\alpha} \cos n \vartheta d\vartheta$$

$$= - \frac{2}{n(n-2)} \frac{\mu_0 R_{\text{ref}}^{n-1}}{\pi} \sin n \alpha \left( \frac{1}{R^{n-2}} - \frac{1}{(R + w)^{n-2}} \right)$$

- Normalizing to the dipole field $B_1 = - \frac{2 \mu_0 J w \sin \alpha}{\pi}$

$$b_n = \frac{1}{n(n-2)} \frac{R_{\text{ref}}^{n-1}}{w \sin \alpha} \left( \frac{1}{R^{n-2}} - \frac{1}{(R + w)^{n-2}} \right) \cdot 10^4$$
Multipoles of a sector dipole

- Multipoles are proportional to $\sin n \alpha$
- The solution of the equation $\sin n \alpha = 0$ is $\alpha = k \frac{\pi}{n}$ with $k$ integer such that $0 < \alpha < \frac{\pi}{2}$
- With 1 sector we can set to zero only one multipole:
  - $b_3 = 0$ if $\alpha = \frac{\pi}{3}$
  - $b_5 = 0$ if $\alpha = \frac{\pi}{5}, \frac{2}{5}\pi$
  - $b_7 = 0$ if $\alpha = \frac{\pi}{7}, \frac{2}{7}\pi, \frac{3}{7}\pi$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$B_1$ (T)</th>
<th>$b_3$</th>
<th>$b_5$</th>
<th>$b_7$</th>
<th>$b_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{3}{7}\pi$</td>
<td>-5.9</td>
<td>-914</td>
<td>106</td>
<td>0</td>
<td>-8</td>
</tr>
<tr>
<td>$\frac{\pi}{3}$</td>
<td>-5.2</td>
<td>0</td>
<td>-239</td>
<td>61</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{2}{7}\pi$</td>
<td>-4.7</td>
<td>632</td>
<td>-298</td>
<td>0</td>
<td>22</td>
</tr>
<tr>
<td>$\frac{\pi}{5}$</td>
<td>-3.5</td>
<td>1844</td>
<td>0</td>
<td>-99</td>
<td>-17</td>
</tr>
<tr>
<td>$\frac{\pi}{7}$</td>
<td>-2.6</td>
<td>2560</td>
<td>431</td>
<td>0</td>
<td>-31</td>
</tr>
</tbody>
</table>

R=50 mm, w=15 mm, $J_0=5\cdot10^8$ A/m$^2$
2-sector dipole

\[
B_n = -4 \frac{\mu_0 J}{2\pi R_{\text{ref}}} \int_{R}^{R+w} \left( \frac{R_{\text{ref}}}{r} \right)^n r dr \left( \int_0^{\alpha_1} \cos n \theta d\theta + \int_{\alpha_2}^{\alpha_3} \cos n \theta d\theta \right) \quad \text{for odd } n
\]

\[
= - \frac{2}{n(n-2)} \frac{\mu_0 J R_{\text{ref}}^{n-1}}{\pi} \left( \sin n \alpha_1 - \sin n \alpha_2 + \sin n \alpha_3 \right) \left( \frac{1}{R^{n-2}} - \frac{1}{(R+w)^{n-2}} \right)
\]

- 3 free parameters, means that we can set to zero 3 multipoles at a time:

\[
\begin{align*}
\sin 3 \alpha_1 - \sin 3 \alpha_2 + \sin 3 \alpha_3 &= 0 \quad B_3 = 0 \\
\sin 5 \alpha_1 - \sin 5 \alpha_2 + \sin 5 \alpha_3 &= 0 \quad B_5 = 0 \\
\sin 7 \alpha_1 - \sin 7 \alpha_2 + \sin 7 \alpha_3 &= 0 \quad B_7 = 0
\end{align*}
\]

- A possible solution is nearly \( \alpha_1 = 43.2^\circ, \quad \alpha_2 = 52.2^\circ, \quad \alpha_3 = 67.3^\circ \)

<table>
<thead>
<tr>
<th>( B_1 ) (T)</th>
<th>( b_3 )</th>
<th>( b_5 )</th>
<th>( b_7 )</th>
<th>( b_9 )</th>
<th>( b_{11} )</th>
<th>( b_{13} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4.9</td>
<td>0.5</td>
<td>0.3</td>
<td>-0.4</td>
<td>-29</td>
<td>12</td>
<td>1.5</td>
</tr>
</tbody>
</table>
Real dipoles

- Due to the geometrical constraints of the cables, more parameters are needed to set to zero more multipoles.

HiLumi D2 dipole

LHC dipole
Perfect dipole #4: canted $\cos \theta$ (CCT) dipoles

- The simplest CCT dipole consists of 2 inclined solenoids wound in the opposite direction: the solenoidal component cancels, and only the dipolar component remains.

- The parametric equations of the two helices ($a_2 > a_1 >> p$) are:

  \[ P_1(\theta) = \begin{cases} 
  a_1 \cos \theta \\
  a_1 \sin \theta \\
  \frac{p}{2\pi} + \frac{a_1}{\tan \beta} \sin \theta 
  \end{cases} \quad \cup \quad P_2(\theta) = \begin{cases} 
  a_2 \cos \theta \\
  a_2 \sin \theta \\
  \frac{p}{2\pi} - \frac{a_2}{\tan \beta} \sin \theta 
  \end{cases} \quad -\pi N < \theta < \pi N

- The resulting surface current densities, in polar coordinates, are given by:

  \[ j_1 = \begin{cases} 
  j_{1r} = \frac{I}{p} \\
  j_{1\theta} = \frac{1}{p} \left( \frac{p}{2\pi a_1} + \frac{\cos \theta}{\tan \beta} \right) \\
  j_{1z} = 0 
  \end{cases} \quad \cup \quad j_2 = \begin{cases} 
  j_{2r} = -\frac{I}{p} \\
  j_{2\theta} = -\frac{1}{p} \left( \frac{p}{2\pi a_2} - \frac{\cos \theta}{\tan \beta} \right) \\
  j_{2z} = 0 
  \end{cases} \quad \text{(derivation at the end of the slides)}\]
Magnetic field from $j_\vartheta$

- Reminding that an infinitely long solenoid generates a magnetic field given by  
  \[ B_z = \mu_0 \frac{N I}{L} = \mu_0 \frac{I}{L} = \mu_0 \frac{I}{p}, \]  
  where  \( \frac{I}{p} = j_\vartheta \)

- The azimuthal components of the current density in the 2-layer CCT dipole generate a solenoidal magnetic field given by:
  \[ B_z = \mu_0 j_{1\vartheta} + \mu_0 j_{2\vartheta} = \mu_0 \frac{I}{p} + \mu_0 \frac{-I}{p} = 0 \]
Magnetic field from $j_z$

Let’s start from the harmonic components generated by a line current:

$$B_n(\rho, \theta) = -\frac{\mu_0 I}{2\pi R_{ref}} \left(\frac{R_{ref}}{r}\right)^n \cos n \theta$$

In our case $r = a_1$ for $P_1$ and $r = a_2$ for $P_2$

replacing $I \rightarrow j_z ad\theta$ and integrating we get that the harmonic components of a 2-layer CCT dipole are given by:

$$B_n = -\frac{\mu_0}{2\pi a_1} \left(\frac{R_{ref}}{a_1}\right)^{n-1} \frac{a_1 I}{p} \int_0^{2\pi} \cos n \theta \left( \frac{p}{2\pi a_1} + \frac{\cos \theta}{\tan \beta} \right) d\theta - \frac{\mu_0}{2\pi a_2} \left(\frac{R_{ref}}{a_2}\right)^{n-1} \frac{-a_2 I}{p} \int_0^{2\pi} \cos n \theta \left( \frac{p}{2\pi a_2} - \frac{\cos \theta}{\tan \beta} \right) d\theta$$

$$B_1 = B_y = -\frac{\mu_0 I}{p \tan \beta} \quad \text{and} \quad B_n = 0 \quad n \neq 1$$
Quadrupoles
Quadrupoles

LHC

HiLumi-LHC

HTS model (Fermilab)

SIS300

LEP

Stefania Farinon
Perfect quadrupoles

\[
B_n = -8 \frac{\mu_0 J}{2\pi R_{ref}} \int_0^{\pi/4} \cos n \vartheta \cos n \vartheta d\vartheta
\]

\[
B_2 = -\frac{\mu_0 J R_{ref}}{2} \ln \left(1 + \frac{w}{R}\right)
\]

\[
G = \frac{B_2}{R_{ref}} = \frac{\mu_0 J}{2} \ln \left(1 + \frac{w}{R}\right)
\]
Symmetrical line currents

\[ B_n = -\frac{\mu_0 I}{2\pi R_{ref}} \left( \frac{R_{ref}}{r} \right)^n \cos n \vartheta - \frac{\mu_0 I}{2\pi R_{ref}} \left( \frac{R_{ref}}{r} \right)^n \cos n (\pi - \vartheta) \]

\[ B_n = -\frac{\mu_0 I}{2\pi R_{ref}} \left( \frac{R_{ref}}{r} \right)^n [\cos n \vartheta + \cos n (\pi - \vartheta)] \]

\[ B_n = -\frac{\mu_0 I}{2\pi R_{ref}} \left( \frac{R_{ref}}{r} \right)^n \cos n \vartheta [1 + \cos n \pi] \]

\[ = \begin{cases} 
-2 \frac{\mu_0 I}{2\pi R_{ref}} \left( \frac{R_{ref}}{r} \right)^n \cos n \vartheta & \text{if } n \text{ is even} \\
0 & \text{if } n \text{ is odd}
\end{cases} \]
Line currents symmetrical with respect to the bisector

\[ B_n = -\frac{\mu_0 I}{2\pi R_{\text{ref}}} \left(\frac{R_{\text{ref}}}{r}\right)^n \cos n\vartheta - \frac{\mu_0 (-I)}{2\pi R_{\text{ref}}} \left(\frac{R_{\text{ref}}}{r}\right)^n \cos n\left(\frac{\pi}{2} - \vartheta\right) \]

\[ B_n = -\frac{\mu_0 I}{2\pi R_{\text{ref}}} \left(\frac{R_{\text{ref}}}{r}\right)^n \left[ \cos n\vartheta - \cos n\left(\frac{\pi}{2} - \vartheta\right) \right] \]

\[ B_n = -\frac{\mu_0 I}{2\pi R_{\text{ref}}} \left(\frac{R_{\text{ref}}}{r}\right)^n \cos n\vartheta \left[ 1 - \cos \frac{n\pi}{2} \right] \]

\[ = \begin{cases} 
-2\frac{\mu_0 I}{2\pi R_{\text{ref}}} \left(\frac{R_{\text{ref}}}{r}\right)^n \cos n\vartheta & \text{if } \frac{n}{2} \text{ is odd} \\
0 & \text{if } \frac{n}{2} \text{ is even}
\end{cases} \]
Only harmonic components with even $n$ and odd $n/2$ survive ($B_2$, $B_6$, $B_{10}$, ..)

Integrating as usual the harmonics of a line current:

$$B_n = -8 \frac{\mu_0 J}{2\pi R_{ref}} \int_R^{R+w} \left( \frac{R_{ref}}{r} \right)^n r dr \int_0^\alpha \cos n \vartheta d\vartheta$$

$$B_n = \left\{ \begin{array}{ll}
-2\frac{\mu_0 J R_{ref}}{n} \sin 2\alpha \ln \left( 1 + \frac{w}{R} \right) & n = 2 \\
-\frac{4}{n(n-2)} \frac{\mu_0 J R_{ref}^{n-1}}{\pi} \sin n\alpha \left( \frac{1}{R^{n-2}} - \frac{1}{(R+w)^{n-2}} \right) & n = 6,10,14,\ldots
\end{array} \right.$$
Sector quadrupole

- The gradient [T/m] is given by:
  \[ G = \frac{B_2}{R_{\text{ref}}} = -\frac{2\mu J}{\pi} \sin 2\alpha \ln\left(1 + \frac{w}{R_{\text{ref}}}\right) \]

- With 1 sector we can set to zero only one multipole:
  \[ \sin n\alpha = 0 \rightarrow \alpha = k\frac{\pi}{n} \]
  with \( k \) integer such that \( 0 < \alpha < \frac{\pi}{4} \)
  - \( b_6 = 0 \) if \( \alpha = 30^\circ \)
  - \( b_{10} = 0 \) if \( \alpha = 18^\circ, 36^\circ \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( G ) (T/m)</th>
<th>( b_4 ) (units)</th>
<th>( b_{10} ) (units)</th>
<th>( b_{14} ) (units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30(^\circ)</td>
<td>-91</td>
<td>0</td>
<td>-32</td>
<td>3</td>
</tr>
<tr>
<td>18(^\circ)</td>
<td>-62</td>
<td>660</td>
<td>0</td>
<td>-5</td>
</tr>
<tr>
<td>36(^\circ)</td>
<td>-100</td>
<td>-252</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>
Perfect CCT quadrupoles

- In the same notation used for dipoles, the simplest CCT quadrupole consists of 2 inclined helices wound in the opposite direction.

- The parametric equations of the two helices ($a_2 > a_1 >> p$) are

  \[
  P_1(\varphi) = \begin{cases} 
  a_1 \cos \varphi \\
  a_1 \sin \varphi \\
  \frac{p \varphi}{2\pi} + \frac{a_1}{2 \tan \beta} \sin 2\varphi
  \end{cases} \quad \cup \quad
  P_2(\varphi) = \begin{cases} 
  a_2 \cos \varphi \\
  a_2 \sin \varphi \\
  \frac{p \varphi}{2\pi} - \frac{a_2}{2 \tan \beta} \sin 2\varphi
  \end{cases}
  \quad \text{for} \quad -\pi N < \varphi < \pi N
  \]

- The resulting surface current densities, in polar coordinates, are given by

  \[
  j_1 = \begin{cases} 
  j_{1r} = \frac{I}{p} \left\{ \begin{array}{c} 0 \\
  1 \\
  \frac{1}{2\pi a_1} + \frac{\cos 2\varphi}{\tan \beta}
  \end{array} \right. \\
  j_{1\theta} = \frac{I}{p} \left\{ \begin{array}{c} 1 \\
  \frac{\cos 2\varphi}{\tan \beta}
  \end{array} \right.
  \end{cases}
  \quad \cup \quad
  j_2 = \begin{cases} 
  j_{2r} = -\frac{I}{p} \left\{ \begin{array}{c} 0 \\
  1 \\
  \frac{1}{2\pi a_2} - \frac{\cos 2\varphi}{\tan \beta}
  \end{array} \right.
  \end{cases}
  \]
Magnetic field from $j_\vartheta$

- Reminding that an infinitely long solenoid generates a magnetic field given by
  \[ B_z = \mu_0 \frac{NI}{L} = \mu_0 \frac{I}{L} = \mu_0 \frac{I}{p}, \]
  where $\frac{I}{p} = j_\vartheta$.

- The azimuthal components of the current density in the 2-layer CCT quadrupole generate a solenoidal magnetic field given by:
  \[ B_z = \mu_0 j_1 \vartheta + \mu_0 j_2 \vartheta = \mu_0 \frac{I}{p} + \mu_0 \frac{-I}{p} = 0 \]
Magnetic field from $j_z$

- Let’s start from the harmonic components generated by a line current:

$$B_n(\rho, \theta) = -\frac{\mu_0 I}{2\pi R_{ref}} \left( \frac{R_{ref}}{r} \right)^n \cos n \theta$$

- In our case $r = a_1$ for $P_1$ and $r = a_2$ for $P_2$

- replacing $I \rightarrow j_z a_1 d\theta$ and integrating we get that the harmonic components of a 2-layer CCT quadrupole are given by:

$$B_n = -\frac{\mu_0}{2\pi a_1} \left( \frac{R_{ref}}{a_1} \right)^{n-1} \frac{a_1 I}{p} \int_0^{2\pi} \cos n \theta \left( \frac{p}{2\pi a_1} + \frac{\cos 2\theta}{\tan \beta} \right) d\theta - \frac{\mu_0}{2\pi a_2} \left( \frac{R_{ref}}{a_2} \right)^{n-1} \frac{a_2 I}{p} \int_0^{2\pi} \cos n \theta \left( \frac{p}{2\pi a_2} - \frac{\cos 2\theta}{\tan \beta} \right) d\theta$$

$$B_2 = G R_{ref} = -\frac{\mu_0 I R_{ref}}{2p \tan \beta} \left( \frac{1}{a_1} + \frac{1}{a_2} \right) \text{ and } B_n = 0 \quad n \neq 2$$
Iron yoke
Why magnets are surrounded by iron yoke?

- Accelerator magnets are usually surrounded by iron yoke:
  - It considerably enhances the bore field for a given current density
  - It modifies the loadline (increasing $B_{SS}$)
  - It considerably reduces the fringe field
  - It can contribute to mechanical structure (see Susana presentation)
The harmonic components of a line current inside a cylindrical iron shell of radii $R_{in}$ and $R_{out}$ is given by

$$B_n(r, \vartheta) = -\frac{\mu_0 I}{2\pi R_{ref}} \left(\frac{R_{ref}}{r}\right)^n \cos n \vartheta \left[1 + k \left(\frac{r}{R_{in}}\right)^{2n}\right]$$

$$k = \frac{\mu_r - 1}{\mu_r + 1} \frac{1 - \left(\frac{R_{in}}{R_{out}}\right)^{2n}}{1 - \left(\frac{\mu_r - 1}{\mu_r + 1}\right)^2 \left(\frac{R_{in}}{R_{out}}\right)^{2n}} \approx 1 \quad \text{se} \quad \mu_r >> 1$$
Sector dipole inside a cylindrical shell

- Integrating the line current harmonics we get the resulting dipole field:

\[ B_1 = -4 \frac{\mu_0 J}{2\pi R_{ref}} \int_{R}^{R+w} \left( \frac{R_{ref}}{r} \right) \left[ 1 + k \left( \frac{r}{R_{in}} \right)^2 \right] r dr \int_{0}^{\alpha} \cos \theta d\theta \]

\[ = - \frac{2\mu_0 J \sin \alpha}{\pi} \left[ w + k \frac{(R + w)^3 - R^3}{3R_{in}^2 \cos \theta} \right] \]

\[ = - \frac{2\mu_0 J w \sin \alpha}{\pi} \left[ 1 + k \frac{R^2 + w R + \frac{w^2}{3}}{R_{in}^2} \right] \]

- The contribution is relevant (15-50%) when iron is not far from the winding \((R_{in} \gtrsim R + w)\), i.e. for small collar widths, and it affect the main components (dependence on \(\left( \frac{r}{R_{in}} \right)^{2n}\))
Impact on short sample field

- $B_{1\text{ iron}} = -\frac{2\mu_0 J_{op} w \sin \alpha}{\pi} \left[ 1 + k \frac{R^2 + wR + \frac{w^2}{3}}{R_{in}^2} \right]$

- $B_{1\text{ iron}} = B_{1\text{ no iron}} \left[ 1 + k \frac{R^2 + wR + \frac{w^2}{3}}{R_{in}^2} \right]$

- To get the same bore field, we need less current density:
  - $J_{op\text{–no iron}} = \frac{\pi B_1}{2\mu_0 w \sin \alpha}$
  - $J_{op\text{–iron}} = \frac{\pi B_1}{2\mu_0 w \sin \alpha} \left[ 1 + k \frac{R^2 + wR + \frac{w^2}{3}}{R_{in}^2} \right]$

$J$ vs $B$ diagram:

- $J_{op\text{–no iron}}$
- $J_{op\text{–iron}}$
- $J_{SS\text{–no iron}}$
- $J_{SS\text{–iron}}$

Loadline without iron
Loadline with iron
Sector quadrupole inside a cylindrical shell

- Integrating the line current harmonics we get the resulting quadrupole field field:

\[
B_2 = -8 \frac{\mu_0 J}{2\pi R_{ref}} \int_R^{R+w} \left( \frac{R_{ref}}{r} \right)^2 \left[ 1 + k \left( \frac{r}{R_{in}} \right)^4 \right] r dr \int_0^\alpha \cos 2\theta d\theta
\]

\[
= - \frac{2\mu_0 J R_{ref} \sin 2\alpha}{\pi} \left[ \ln \left( 1 + \frac{w}{R} \right) + k \frac{(R+w)^4 - R^4}{4R_{in}^4} \right]
\]

- The contribution is less relevant than dipole
Iron saturation

- Previous considerations are valid when iron yoke works in its linear range, i.e. below saturation

- Typical iron saturates for $B \sim 2\ T$
  - If $B < 2\ T$ BH curve is roughly linear with a pendency of $\mu_r \sim 10^3 - 10^4$
  - If $B > 2\ T$ $\mu_r \sim 1$ and iron gives no further contribution
  - The correct iron yoke contribution to magnetic field, including saturation, can only be determined via finite element analysis

\[
B = \mu_0 \mu_r H
\]
Grading techniques
Grading techniques

- The field map inside a coil is highly nonuniform (inner layers have larger peak fields than outer layers)
- In the low field outer layers it is possible to:
  - use larger current density and narrower conductor
  - Use a less performant (and cheaper) material
An example: 16 T cos dipole for the FCC

- Both inner and outer layers are dimensioned so that the margin on the loadline is 14%
Winding shapes
Dipole winding shapes – EuroCirCol project

- I will show the results of the optimization of a double aperture 16 T dipole for the FCC in 4 different options as part of WP5 of Eurocircol project (www.eurocircol.eu)
- All optimizations share common assumption: same magnet aperture (50 mm), conductor performance \( J_c (16 \, T, 4.2 \, K) = 1500 \, A/mm^2 \), margin on the loadline (>14%), allowed mechanical constraints (\( \sigma <150 \, \text{MPa at warm and} <200 \, \text{MPa at cold} \))
Cos-theta coil

- **Pros**
  - Natural choice (LHC dipoles)
  - Circular aperture fully available for beam
  - Self-supporting winding (roman arc)

- **Cons**
  - Hardway bending in coil ends
Block coil

- **Pros**
  - Particularly indicated for thick coils (turns are stacked vertically)
  - No wedges (saddle shape ends)
  - Peak stress during powering in the low field region

- **Cons**
  - Need of internal support, reducing available aperture
  - Very complicated coil ends (hardway bending)
Common coil

- **Pros**
  - Very simple coils (flat racetrack shape)

- **Cons**
  - Complicated stress management (huge radial Lorentz force)
  - Needs more superconductors

F. Toral, FCC week 2019
CCT – Canted Cos Theta coil

- **Pros**
  - Each turn is individually supported
  - 360° continuity of the winding: no azimuthal pre-load
  - No field distortion in coil ends
  - Small number of mechanical components

- **Cons**
  - Part of the current density lost in generating solenoidal field
  - Need more superconductors
  - Complicated winding if large Rutherford cables (bonding of cable inside channels, reliable insulation against former)

_B. Auchmann, FCC week 2019_
Results of the comparison

- The \( \cos \theta \) configuration has been selected as baseline for the Conceptual Design Report of the EuroCirCol project (http://cds.cern.ch/record/2651300/files/CERN-ACC-2018-0058.pdf?version=6)

- “Each of these alternatives features some interesting characteristics which may have a potential to become competitive to the baseline cosine-theta design in terms of performance, in particular if they would allow operation at a lower margin on the load-line, thus reducing the required amount of conductor”

- Short model magnets (~1.5 m lengths) of all the options will be built from 2018–2022
THANKS FOR THE ATTENTION

A thorough Masterclass on superconducting magnets for particle accelerators by Ezio Todesco is available at https://indico.cern.ch/category/12408/
Derivation of current density in CCT magnet
Derivation of current density in CCT magnet

- Let's consider that a current I flows along the helix defined as
  \[ P(\vartheta) = \begin{cases} 
  a \cos \vartheta \\
  a \sin \vartheta \\
  h\vartheta \\
  \frac{h}{2\pi} + A \sin \vartheta 
  \end{cases} \]

- If the helix is infinitely long the current density will be given by:
  \[ j(\vartheta) = \frac{I}{\delta(\vartheta)} \hat{v}_r(\vartheta) \]
  where \( \delta(\vartheta) \) is the distance between two consecutive turns and \( \hat{v}_r(\vartheta) \) is the versor of the current direction.

- By definition \( |\hat{v}_r(\vartheta)| = 1 \) and the direction is the same of the derivative of \( P(\vartheta) \)
  \[ \hat{v}_r(\vartheta) = \frac{1}{\sqrt{a^2 + \left(\frac{h}{2\pi} + A \cos \vartheta\right)^2}} \begin{pmatrix} 
  -a \sin \vartheta \\
  a \cos \vartheta \\
  \frac{h}{2\pi} + A \cos \vartheta 
  \end{pmatrix} \]
Determination of $\delta(\vartheta)$

- $\delta(\vartheta)$ is the turn to turn distance, i.e. the distance between the two straight lines tangent to $P(\vartheta)$ in $\vartheta$ e $\vartheta + 2\pi$.

- The two straight lines being parallel, that distance can be calculated as

  - $\delta = |(P_1 - P_0) \times \hat{v}_r|$
  - where $P_1 = P(\vartheta + 2\pi)$ and $P_0 = P(\vartheta)$
  - $P_0 = \begin{pmatrix} a \cos\vartheta \\ a \sin\vartheta \\ \frac{h\vartheta}{2\pi} + A \sin\vartheta \end{pmatrix}$
  - $P_1 = \begin{pmatrix} a \cos\vartheta \\ a \sin\vartheta \\ \frac{h\vartheta}{2\pi} + h + A \sin\vartheta \end{pmatrix}$
  - $P_1 - P_0 = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$
Determination of $\delta(\vartheta)$ and $\mathbf{j}(\vartheta)$

$$\hat{\nu}_r = \frac{1}{\sqrt{a^2 + \left(\frac{\hbar}{2\pi} + A \cos \vartheta\right)^2}} \begin{pmatrix} -a \sin \vartheta \\ a \cos \vartheta \\ h/2\pi + A \cos \vartheta \end{pmatrix}$$

$$\mathbf{P}_1 - \mathbf{P}_0 = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$$

$$\delta(\vartheta) = |(\mathbf{P}_1 - \mathbf{P}_0) \times \hat{\nu}_r| = \frac{1}{\sqrt{a^2 + \left(\frac{\hbar}{2\pi} + A \cos \vartheta\right)^2}} \begin{vmatrix} -ah \cos \vartheta \\ -ah \sin \vartheta \\ 0 \end{vmatrix} = \frac{ah}{\sqrt{a^2 + \left(\frac{\hbar}{2\pi} + A \cos \vartheta\right)^2}}$$

$$\mathbf{j}(\vartheta) = \frac{I}{\delta(\vartheta)} \hat{\nu}_r(\vartheta) = \frac{I}{ah} \begin{pmatrix} -a \sin \vartheta \\ a \cos \vartheta \\ h/2\pi + A \cos \vartheta \end{pmatrix}$$

in cartesian coordinates

Since $j_x = j_x \cos \vartheta + j_y \sin \vartheta; j_\vartheta = -j_x \sin \vartheta + j_y \cos \vartheta$, we get in polar coordinates: $$\mathbf{j}(\vartheta) = \frac{I}{ah} \begin{pmatrix} 0 \\ \frac{\alpha h}{2\pi} + A \cos \vartheta \end{pmatrix}$$