EASISchool 3

SUPERCONDUCTING DIPOLES AND QUADRUPOLES FOR ACCELERATORS 1

October 5th, 2020

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Superconducting magnet design

- Superconducting magnet design is a true multiphysics problem involving several activities
  - Electromagnetic optimization (field quality, peak field on conductor, margin)
  - Choice of the conductor (transport properties)
  - Choice of the operating temperature and cryogenic design
  - Design of the mechanical support structure
  - Analysis of stability and quench protection
  - Manufacturing techniques
  - Cost analysis

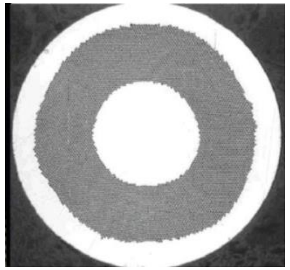
Outline

- Overview of superconducting magnets for particle accelerators (dipoles and quadrupoles)
- Conductor
- Magnetic design
- Mechanical design
  - Force, stress and pre-load
  - Support structures
- Quench protection

Conductor, Magnetic design: Stefania Farinon

Mechanical design, Quench protection: Susana Izquierdo Bermudez

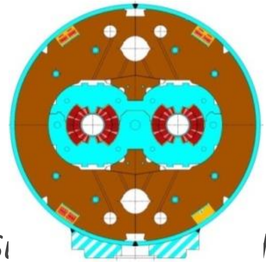
Superconducting strand



Superconducting cable

Superconducting coil

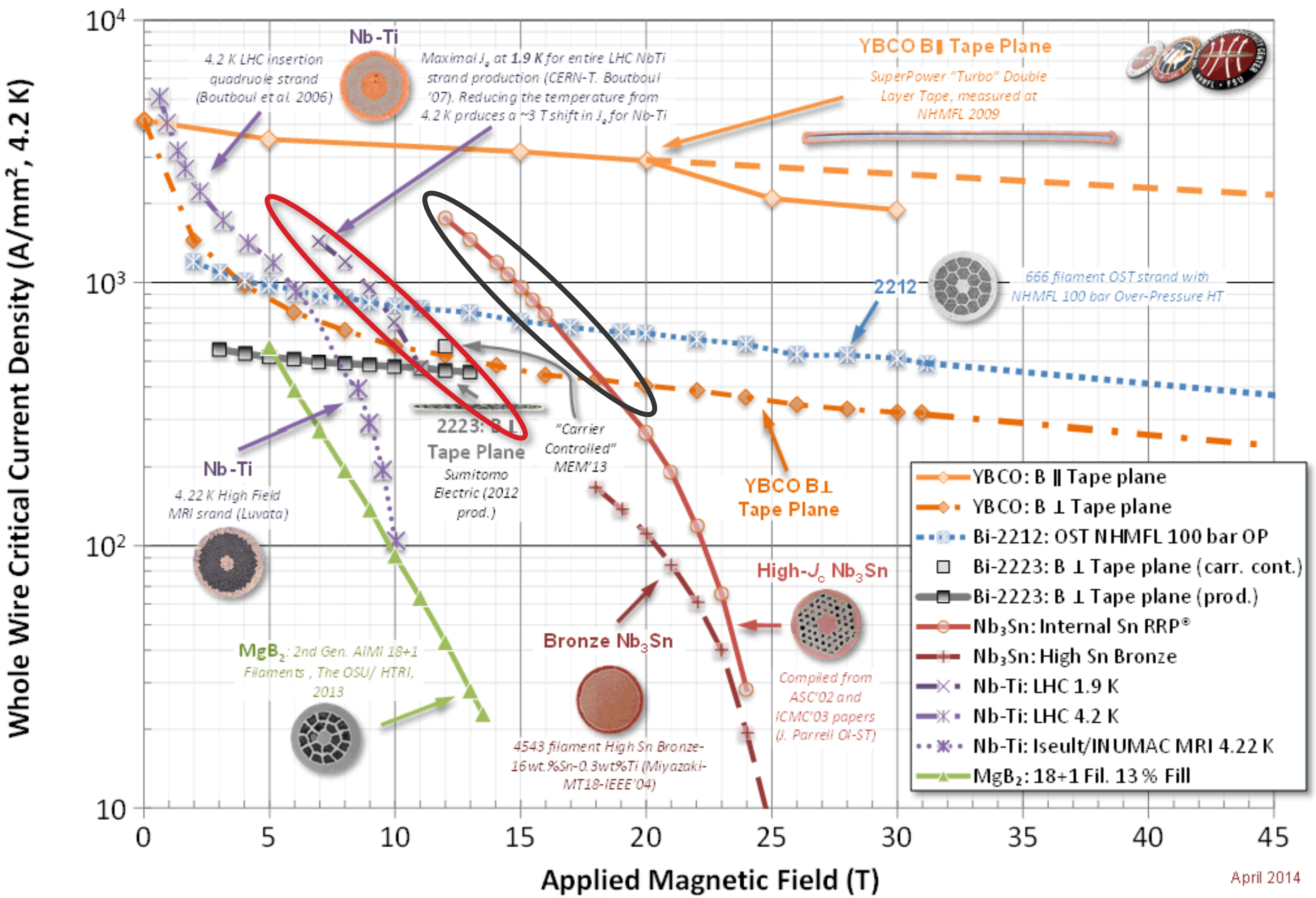
Superconducting magnet



Practical conductors for accelerator magnets

Comparison of critical current densities @ 4.2 K

https://fs.magnet.fsu.edu/~lee/plot/plot.htm



NbTi

- NbTi is the most widely used superconductor
- In High Energy Physics, NbTi has been used for Tevatron (Fermilab), HERA (DESY), RHIC (BNL), LHC (CERN)
- Critical surface parametrization (L. Bottura, IEEE TAS 10 (2000) 1054) :

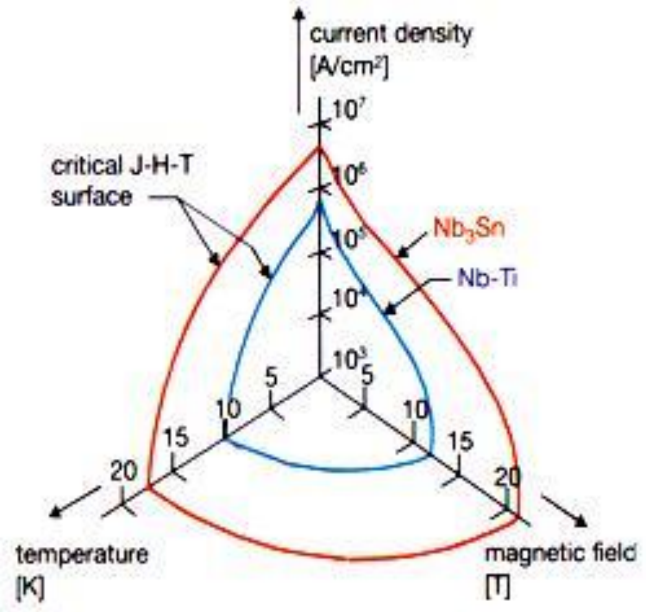
$$J_C(B,T) = \frac{C}{B}\left[\frac{B}{B_{C2}(T)}\right]^{\alpha}\left[1-\frac{B}{B_{C2}(T)}\right]^{\beta}\left[1-\left(\frac{T}{T_{C0}}\right)^{1.7}\right]^{\gamma}$$

with $T_{C0} = 9.2\ \mathrm{K}$ $\quad B_{C20} = 14.5\ \mathrm{T}$ $\quad B_{C2}(T) = B_{C20}\left[1-\left(\frac{T}{T_{C0}}\right)^{1.7}\right]$

- Fitting parameters for LHC wires ($J_C(5\,T, 4.2\,K) = 3000\ \mathrm{A/mm^2}$):

$$C = 92.1\ \mathrm{T\cdot kA/mm^2}, \alpha = 0.63,\ \beta = 1.0,\ \gamma = 2.3$$

- Practical limit for accelerator magnets:
  - $B_{C2}(1.9\,K) = 13.5\,T$, but $J_C(13.5\,T, 1.9\,K) = 0$
  - To have reasonable current density $B \lesssim 10\,T$, $J_C(10\,T, 1.9\,K) = 1680\ \mathrm{A/mm^2}$
  - Taking some margins (see next slides) $B \lesssim 8\,T$



Nb_3Sn

- Nb3Sn is the choice to go beyond the NbTi limits in accelerator magnets
  - 11 T dipoles and triplet quadrupoles in High Luminosity LHC
- Critical surface parametrization (L. Bottura et al., IEEE TAS 19 (2009) 1521) :

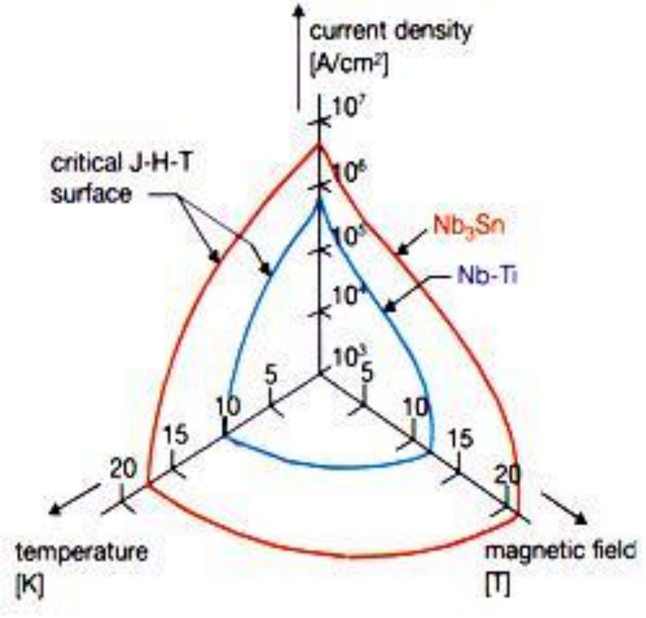
$$J_C(B,T) = \frac{C}{B}\left[\frac{B}{B_{C2}(T)}\right]^{\alpha}\left[1-\frac{B}{B_{C2}(T)}\right]^{\beta}\left[1-\left(\frac{T}{T_{C0}}\right)^{1.52}\right]^{\gamma}\left[1-\left(\frac{T}{T_{C0}}\right)^{2}\right]^{\gamma}$$

with $T_{C0} = 16\ \mathrm{K}$ $\quad B_{C20} = 29\ \mathrm{T}$ $\quad B_{C2}(T) = B_{C20}\left[1-\left(\frac{T}{T_{C0}}\right)^{1.7}\right]$

- Fitting parameters for target FCC wires ($J_C(16\ T, 4.2\ K) = 1500\ \mathrm{A/mm^2}$):

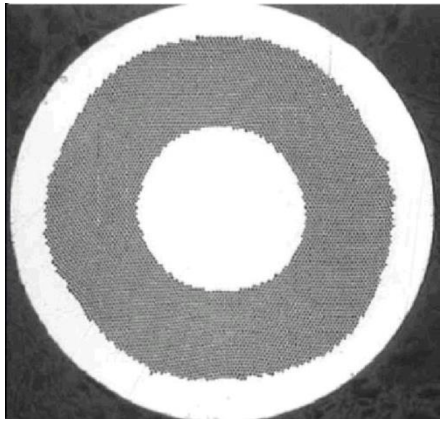
$$C = 267.845\ \mathrm{T \cdot kA/mm^2}, \alpha = 0.5,\ \beta = 2,\ \gamma = 0.96$$

- Practical limit for accelerator magnets:
  - $B_{C2}(1.9\ K) = 28\ T$
  - To have reasonable current density $B \lesssim 18\ T, J_C(18\ T, 1.9\ K) = 1480\ \mathrm{A/mm^2}$
  - Taking some margins (see next slides) $B \lesssim 16\ T$, that double the performance WRT NbTi

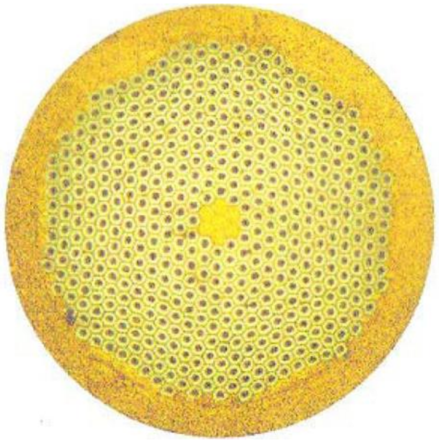


Multifilament wires

- For practical applications, superconducting materials are produced in small filaments and surrounded by a stabilizer (typically copper) to form a multifilament wire or strand
  - Typical filament diameter is in the range 3-10 μm for NbTi and $\lesssim$50 μm for Nb_3Sn; typical strand diameter is $\lesssim$ 1 mm
  - Fine filaments to
    - Reduce effects due to magnetization
    - Limit flux jumps
  - Copper matrix
    - For protection and stability (see Susana presentation)
  - Twisting
    - to reduce interfilament coupling and AC losses



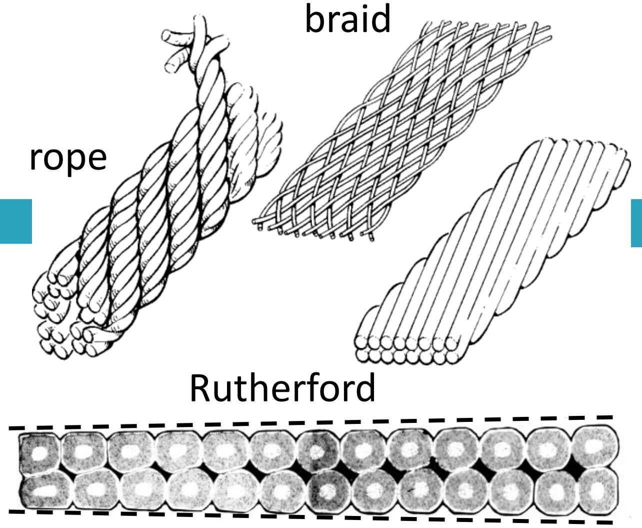
NbTi LHC wire

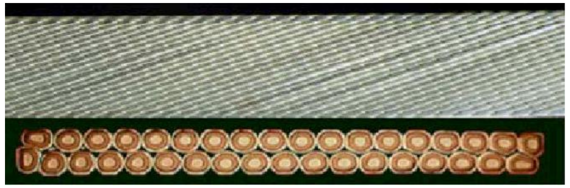


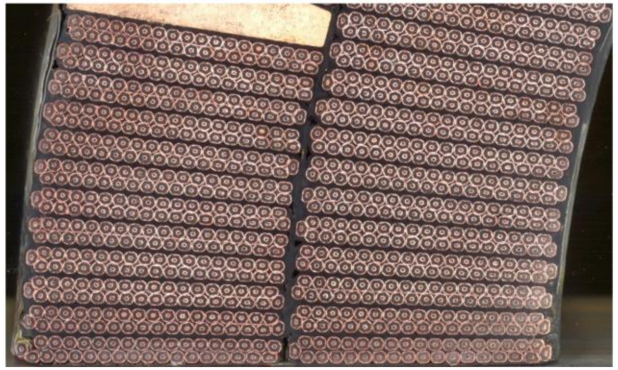
Nb_3Sn PIT process wire

Multistrand cables

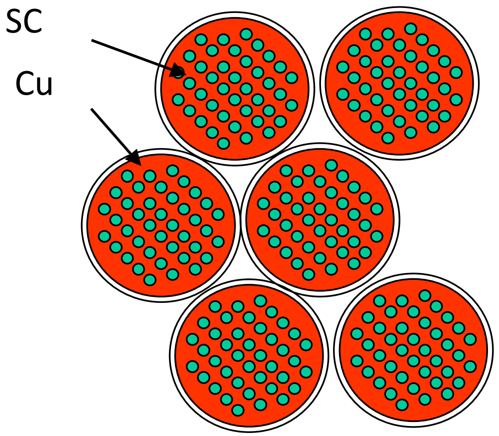
- Most of the superconducting coils for particle accelerators are wound from a multi-strand Rutherford cable
- Main advantages:
  - Rutherford cables
    - high density compaction of strands
    - rectangular or trapezoidal shape (to stack arc-shaped coils)
  - multi-strand cables
    - large current density with small number of turns
    - smaller coil inductance
    - current redistribution in case of problem in a portion of a strand
  - strand twisting
    - to reduce inter-strand coupling and AC losses
  - strand transposition
    - to eliminate the flux enclosed
    - to increase the mechanical stability



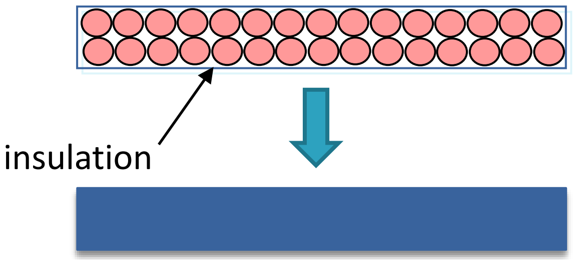




WARNING: engineering current density



$\lambda_{strand} = A_{SC}/A_{strand}$



$\lambda_{cable} = N_{strand}A_{strand}/A_{cable}$

- In SC magnets what really matters is the overall 'engineering' current density J_{eng}
- $J_{eng} = \frac{N_{strand}I}{A_{cable}} = J_C\lambda_{strand}\lambda_{cable}$
  - Typical $R_{\text{cu-non Cu}}$ ranges from 1 to 2, then λ_{strand} ranges from 0.33 to 0.5 ($\lambda_{strand} = \frac{1}{1+R_{\text{Cu}-\text{non Cu}}}$)
  - λ_{cable} takes into account the total space occupied by each turn, and is typically 0.7 to 0.8
- So typically J_{eng} is only 20% to 40% of J_C

Margin on the loadline

- The margin of a magnet is defined with respect to its weakest point, i.e. the peak field
- J_{SS} (Short Sample) corresponds to the intersection of the loadline for the peak field and the critical current density curve: ideally is the maximum performance of the magnet
- The loadline fraction is the ratio J_{op}/J_{SS}
- The margin on the loadline is $1 - J_{op}/J_{SS}$

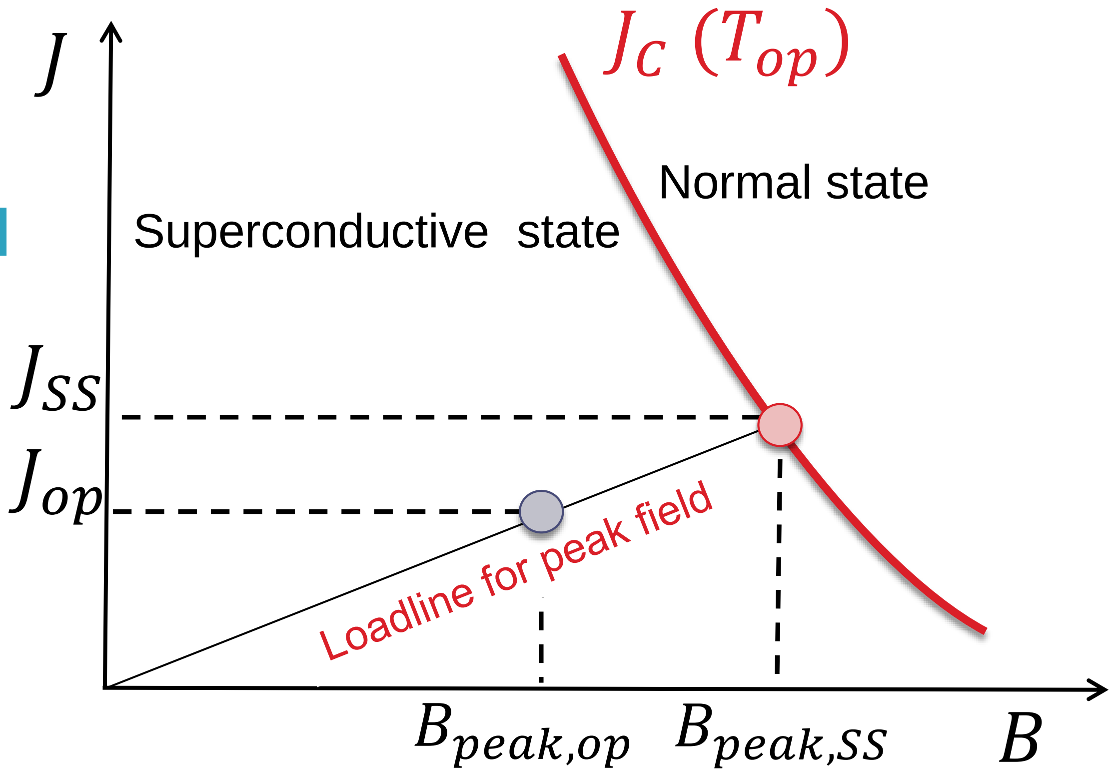
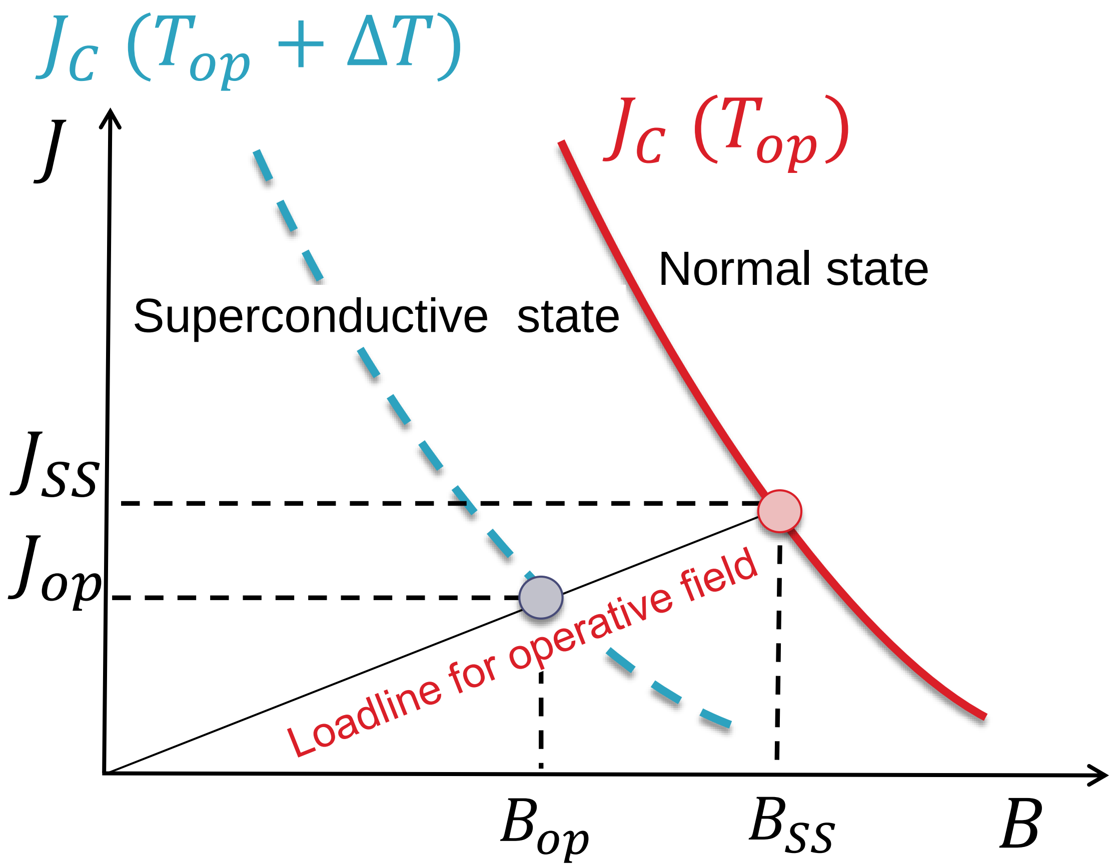


TABLE I. MARGIN FOR DIFFERENT ACCELERATORS

	Nominal			Actual		
	Temp. (K)	Field (T)	Margin	Temp. (K)	Field (T)	Margin
Tevatron	4.2	4.3	22%	4.2	4.2	24%
Hera	4.6	4.7	23%	3.9	5.3	23%
RHIC	4.5	3.5	30%	4.5	3.5	30%
LHC	1.9	8.3	14%	1.9	*7.8**	*19%*

Temperature margin ΔT

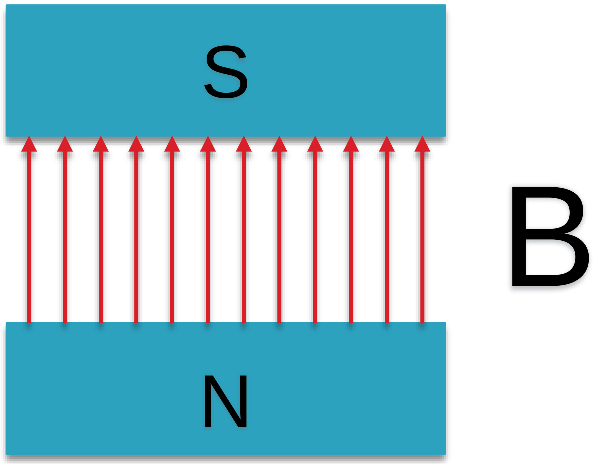
- The temperature margin is a physical quantity related to the energy which can be released before crossing the critical surface (order of few K)
- It is the temperature rise necessary for the J_C curve to intersect the loadline at operative field



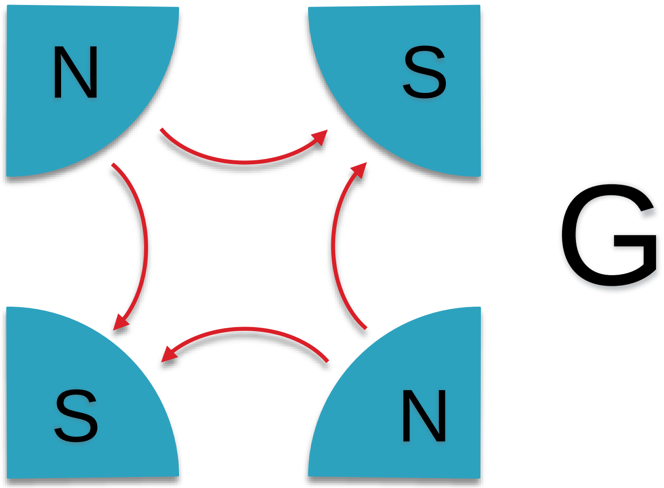
Dipole and quadrupole definition

Dipole and quadrupole definition

- Dipole magnets generate constant and uniform field B:



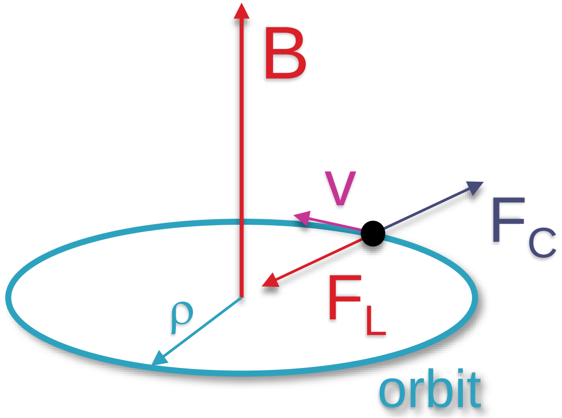
- Quadrupole magnets generate constant and uniform gradient G:



What is the effect of a dipole on a travelling particle?

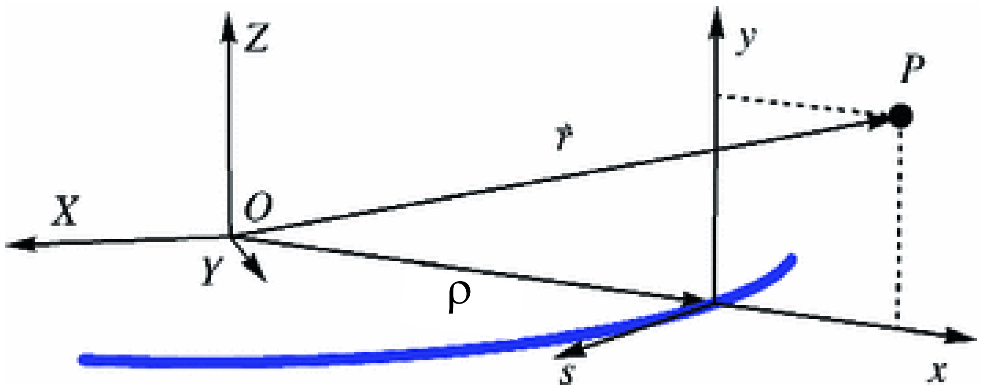
- A particle of charge q travelling in a uniform magnetic field B at speed v is subjected to the Lorentz force $F_L = q\, v \times B$
- The Lorentz force is balanced by the centrifugal force $F_C = mv^2/r$
- The results is that the Lorentz force keeps particles in a circular orbit:

$$mv^2/\rho = qvB \quad \Rightarrow \quad \rho = mv/qB = p/qB$$



What is the effect of a quadrupole on a travelling particle?

Frenet-Serret coordinate system



Time is replaced by s, which is the reference orbit given by the bending magnets and is moving with the beam

- Since the only force is magnetic:

$$\vec{F} = m\frac{d^2\vec{R}}{dt^2} = q\,\vec{v}\times\vec{B}, \qquad \vec{R} = (\rho + x)\hat{x} + y\hat{y}$$

- If $v \sim v_s \gg v_x, v_y$

$$\vec{v}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & v \\ B_x & B_y & 0 \end{vmatrix} = -vB_y\hat{x} + vB_x\hat{y}$$

$$m\frac{d^2\vec{R}}{dt^2} = m\left(\frac{d^2x}{dt^2}\hat{x} + \frac{d^2y}{dt^2}\hat{y}\right) = -qvB_y\hat{x} + qvB_x\hat{y}$$

- Along s direction, $s = vt$ then $\frac{d}{dt} = \frac{d}{ds}\frac{ds}{dt} = v\frac{d}{ds}$:

$$v^2\frac{d^2x}{ds^2} = -\frac{1}{m}qvB_y \qquad \Rightarrow \qquad \frac{d^2x}{ds^2} + \frac{q}{p}B_y = 0$$

$$v^2\frac{d^2y}{ds^2} = \frac{1}{m}qvB_x \qquad \Rightarrow \qquad \frac{d^2y}{ds^2} - \frac{q}{p}B_x = 0$$

where $p = mv$

What is the effect of a quadrupole on a travelling particle?

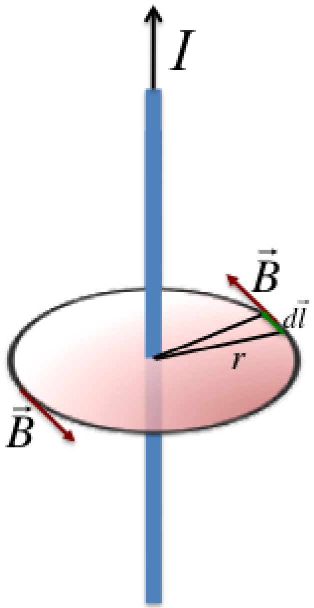
- Quadrupole field: $B_x = Gy, \quad B_y = Gx, \quad G$ is the field gradient
- $\frac{d^2x}{ds^2} + \frac{q}{p} B_y = 0 \Rightarrow \frac{d^2x}{ds^2} + \frac{qG}{p} x = 0$
- this is a (mass-spring) harmonic oscillator; the motion can be described by the function:

$$x(s) = A\cos(\psi(s))$$

- The global effect is focusing in the x direction and defocusing in the y direction (QF)
- If fields have the opposite sign, we get focusing in the y direction and defocusing in the x direction (QD)
- In an accelerator, quadrupoles give the force necessary to stabilize the linear motion

Multipolar expansion of magnetic field

Magnetic field of a current line



- From the Maxwell equation:

  $\nabla \times \vec{B} = \mu_0 \vec{J} \quad \oint \vec{B} d\ell = \mu_0 I$

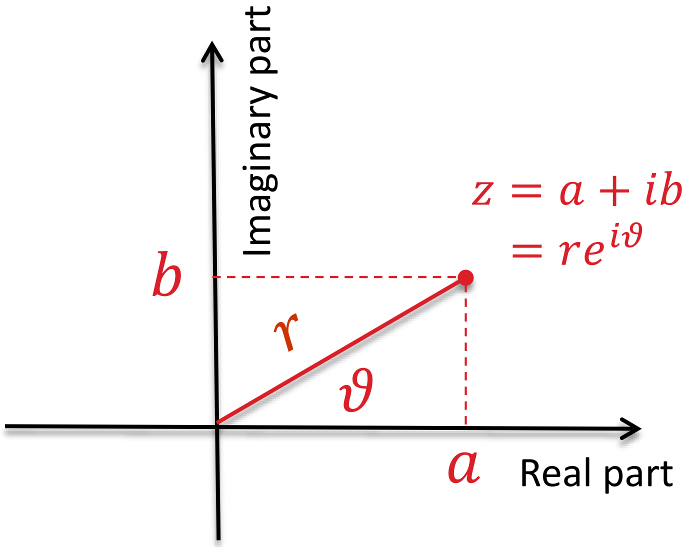
- It's easy to find that

  $$B(r) = \frac{\mu_0 I}{2\pi r}$$

  lying on a plane perpendicular to the current line and tangent to the circumference of radius r

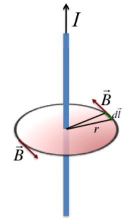
Basics of complex numbers

- By definition, the complex number i is the solution of the equation $i^2 = -1$
- A general complex number is identified by 2 components: $z = a + ib$
  - where a is the *real part*
  - and b is the *imaginary part*
- It can be also written in the exponential form $z = re^{i\vartheta} = r(\cos\vartheta + i\sin\vartheta)$



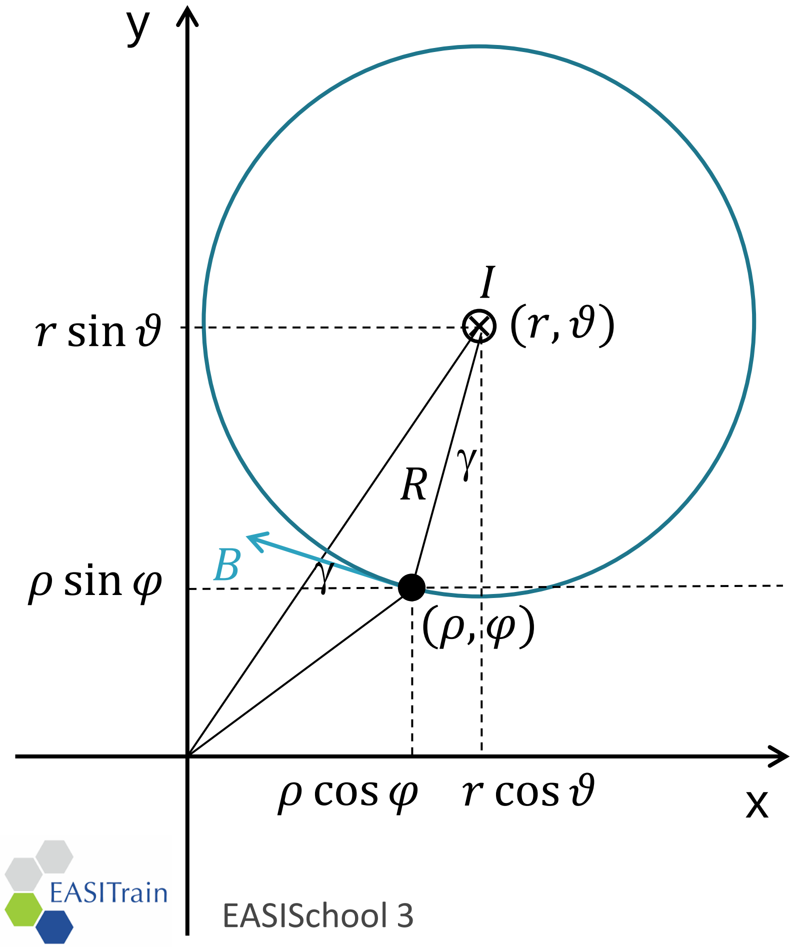
$$r = \sqrt{a^2 + b^2}$$
$$\vartheta = \operatorname{atan}\frac{b}{a}$$

Magnetic field of a current line



$$B(r) = \frac{\mu_0 I}{2\pi r}$$

- In a more general coordinate system, using complex notation:



- $\boldsymbol{B}(\boldsymbol{z}) = \frac{\mu_0 I}{2\pi(\boldsymbol{z}-\boldsymbol{a})}$, con $\boldsymbol{z} = \rho e^{i\varphi}$ e $\boldsymbol{a} = re^{i\vartheta}$
- In facts:

$$\boldsymbol{B}(\boldsymbol{z}) = \frac{\mu_0 I}{2\pi(\rho e^{i\varphi} - re^{i\vartheta})} = \frac{\mu_0 I}{2\pi[(\rho\cos\varphi - r\cos\vartheta) + i(\rho\sin\varphi - r\sin\vartheta)]}$$

$$= \frac{\mu_0 I[(\rho\cos\varphi - r\cos\vartheta) - i(\rho\sin\varphi - r\sin\vartheta)]}{2\pi[r^2 + \rho^2 - 2r\rho\cos(\vartheta - \varphi)]}$$

$$= \frac{\mu_0 I}{2\pi R}\frac{(r\cos\vartheta - \rho\cos\varphi) + i(\rho\sin\varphi - r\sin\vartheta)}{R}$$

$$= \frac{\mu_0 I}{2\pi R}(\sin\gamma + i\cos\gamma) = B_y + iB_x$$

Magnetic field in $\boldsymbol{z} = \rho e^{i\varphi}$ of a current line in $\boldsymbol{a} = re^{i\vartheta}$ if $\rho < r$

- $\boldsymbol{B}(\boldsymbol{z}) = \frac{\mu_0 I}{2\pi(\boldsymbol{z}-\boldsymbol{a})} = \frac{\mu_0 I}{2\pi(\rho e^{i\varphi} - re^{i\vartheta})} = -\frac{\mu_0 I}{2\pi re^{i\vartheta}} \frac{1}{1-\frac{\rho}{r}e^{i(\varphi-\vartheta)}}$
- Reminding that if $\epsilon < 1$: $\frac{1}{1-\epsilon} = \sum_{n=1}^{\infty} \epsilon^{n-1}$
- $\boldsymbol{B}(\boldsymbol{z}) = -\frac{\mu_0 I}{2\pi r} e^{-i\vartheta} \sum_{n=1}^{\infty} \left[\frac{\rho}{r} e^{i(\varphi-\vartheta)}\right]^{n-1} = -\frac{\mu_0 I}{2\pi r} \sum_{n=1}^{\infty} e^{-in\vartheta} \left(\frac{\rho e^{i\varphi}}{r}\right)^{n-1} = -\frac{\mu_0 I}{2\pi r} \sum_{n=1}^{\infty} e^{-in\vartheta} \left(\frac{R_{ref}}{r}\right)^{n-1} \left(\frac{\rho e^{i\varphi}}{R_{ref}}\right)^{n-1}$
- $= \sum_{n=1}^{\infty} (B_n + iA_n)(\cos(n-1)\varphi + i\sin(n-1)\varphi) \left(\frac{\rho}{R_{ref}}\right)^{n-1}$

with $B_n = -\frac{\mu_0 I}{2\pi r} \left(\frac{R_{ref}}{r}\right)^{n-1} \cos n\vartheta = -\frac{\mu_0 I}{2\pi R_{ref}} \left(\frac{R_{ref}}{r}\right)^{n} \cos n\vartheta$ and $A_n = \frac{\mu_0 I}{2\pi r} \left(\frac{R_{ref}}{r}\right)^{n-1} \sin n\vartheta = \frac{\mu_0 I}{2\pi R_{ref}} \left(\frac{R_{ref}}{r}\right)^{n} \sin n\vartheta$

A_n and B_n are in T

Harmonic components of magnetic field

- The magnetic field can be expandend in series as

$$B_x + iB_y = \sum_{n=1}^{\infty} (B_n + iA_n)(\cos(n-1)\varphi + i\sin(n-1)\varphi)\left(\frac{\rho}{R_{ref}}\right)^{n-1}$$

- where B_n coefficients are the **normal** multipoles and A_n coefficients are the **skew** multipoles: dipole (n=1), quadrupole (n=2), sextupole (n=3), octupole (n=4), ….
- To get them in practical adimensioned units, harmonics are often normalized:

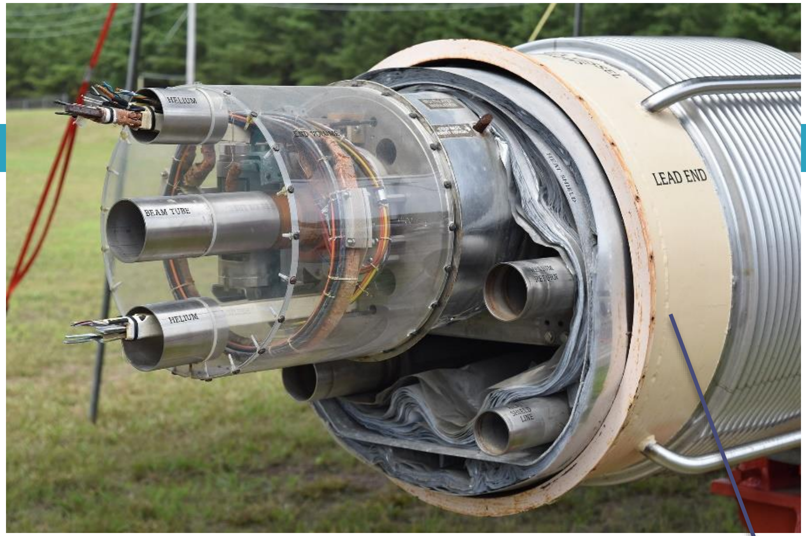
$$B_x + iB_y = 10^{-4} B_{norm} \sum_{n=1}^{\infty} (b_n + ia_n)(\cos(n-1)\varphi + i\sin(n-1)\varphi)\left(\frac{r}{R_{ref}}\right)^{n-1}$$

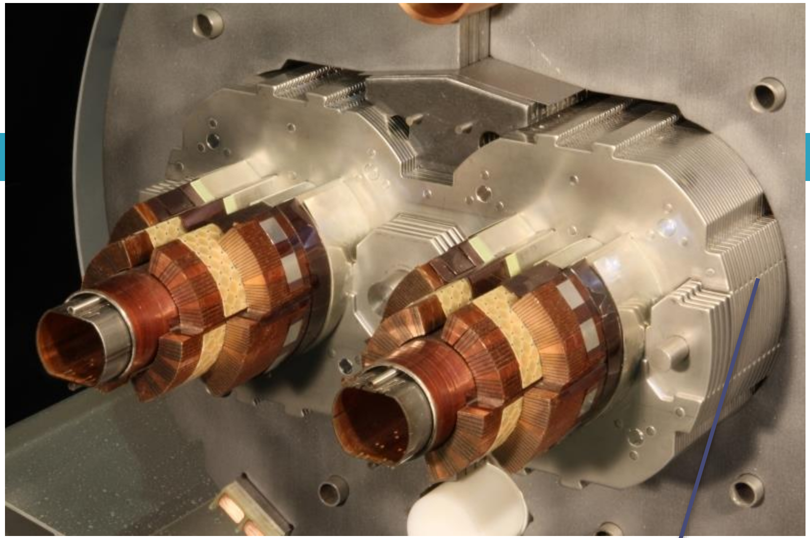
with $b_n = -\frac{10^4}{B_{norm}} \frac{\mu_0 I}{2\pi R_{ref}} \left(\frac{R_{ref}}{r}\right)^n \cos n\vartheta \qquad a_n = \frac{10^4}{B_{norm}} \frac{\mu_0 I}{2\pi R_{ref}} \left(\frac{R_{ref}}{r}\right)^n \sin n\vartheta$

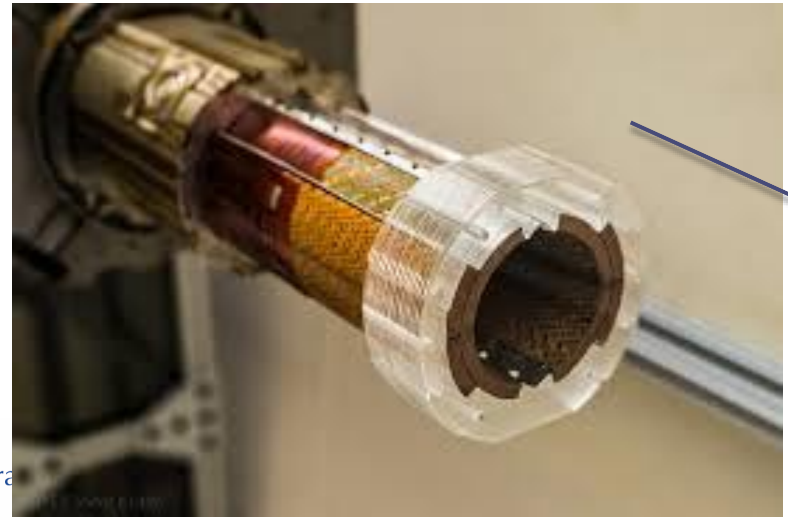
- B_{norm} [T] is the normalization field, $B_{norm} = B_1$ for dipoles, $B_{norm} = B_2 = GR_{ref}$ for quadrupoles, etc.

Dipols

Dipoles







4.5T

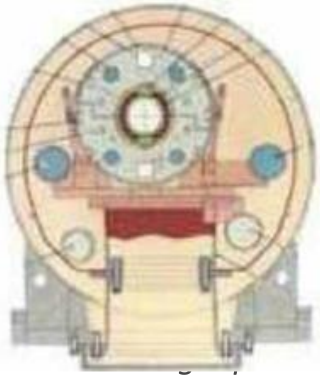
Tevatron,
6 m, 76 mm
774 dipoles

5.3T

HERA,
9 m, 75 mm
416 dipoles

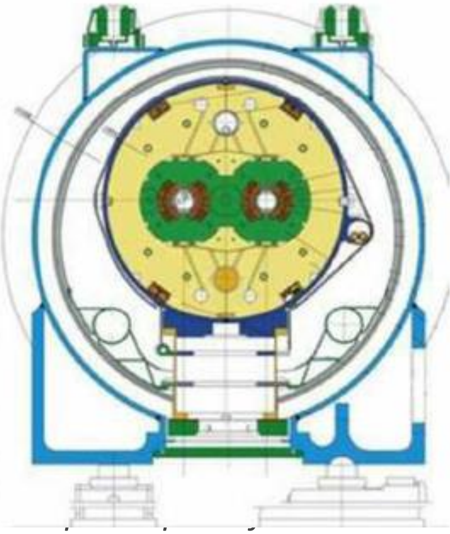
3.5T

RHIC,
9 m, 80 mm
264 dipoles



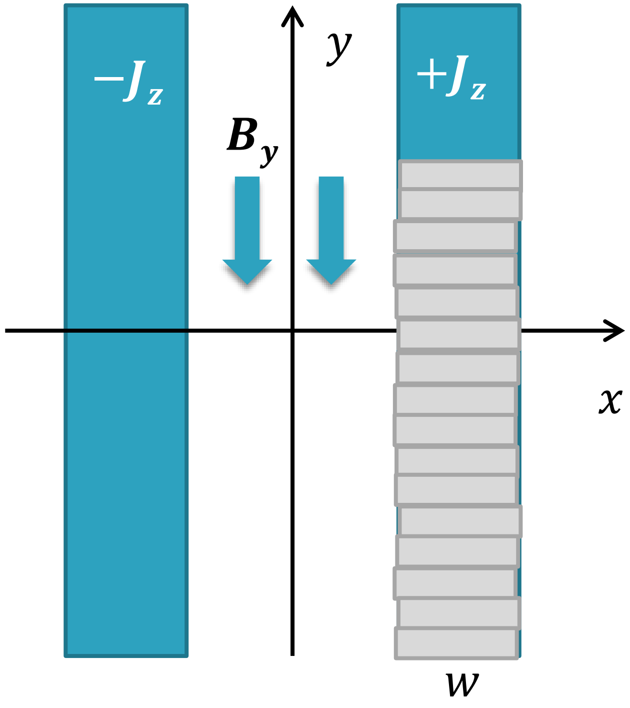
8.3T

LHC,
15 m, 56 mm
1276 dipoles

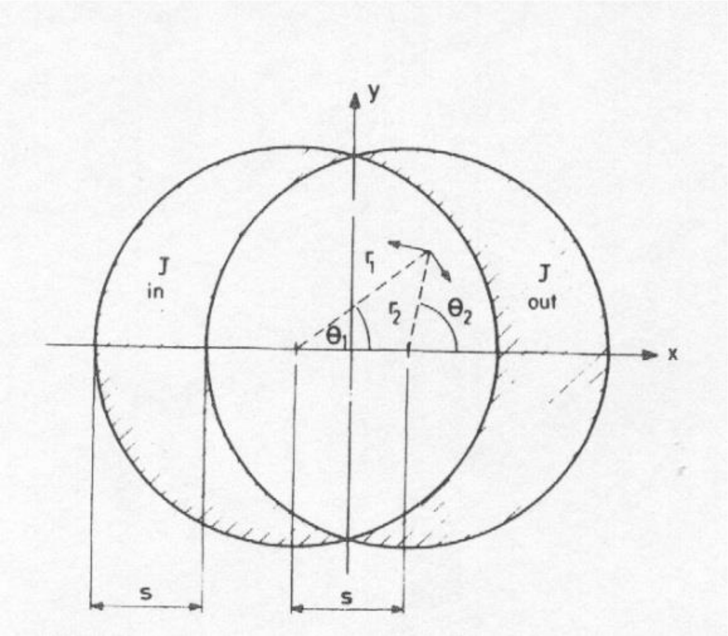


Perfect dipole #1: «wall-dipole»

- A uniform current density flowing in two parallel walls of infinite height generates a pure dipolar field
  - winding and mechanical structure are not particularly complicated
  - the coil is theoretically infinite
  - coil truncation results in an acceptable field quality only for large dimensions
  - simply applying the Biot-Savart law $B_y = -\frac{\mu_0 J w}{2}$



Perfect dipole #2: intersecting circles



From "Superconducting Magnets", M.N.Wilson

- Within a cylinder carrying uniform J, the field is $B(r) = \frac{\mu_0 J r}{2}$, directed tangentially
- Combining the effect of the two cylinders:

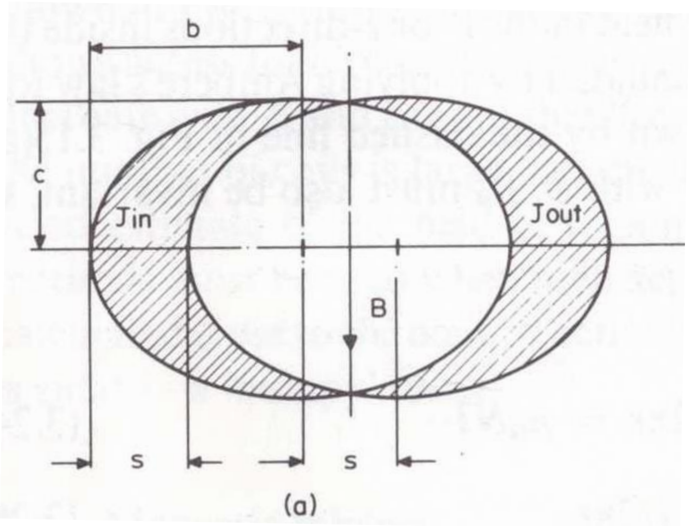
$$B_y = \frac{\mu_0 J}{2}(-r_1 \cos\theta_1 + r_2 \cos\theta_2) = -\frac{\mu_0 J s}{2}$$

$$B_x = \frac{\mu_0 J}{2}(+r_1 \sin\theta_1 - r_2 \sin\theta_2) = 0$$

Perfect dipole #2: intersecting ellipses

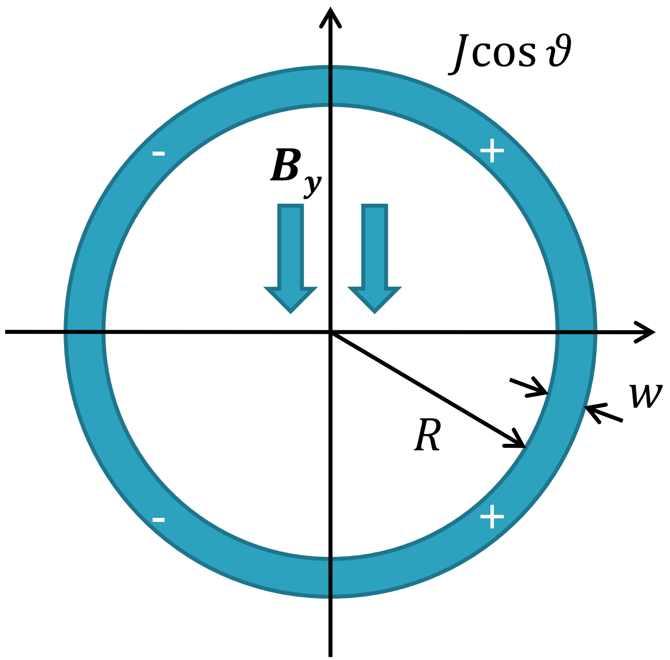
- Analogously, two intersecting ellipses of semi-axes b and c generate a pure dipolar field given by:

$$B_y = -\mu_0 J s \frac{c}{(b+c)}$$



- The shape of intersecting circles and ellipses is not particularly favourable to winding:
  - central aperture is not circular
  - an inner mechanical support could be needed (further reducing available aperture)

Perfect dipole #3: $J\cos\vartheta$ distribution



- Let us consider a current density distribution $J\cos\vartheta$ in a shell of inner radius R and thickness w

- I remind that the normal harmonic component of a line current in (r,ϑ) is given by:

$$B_n(\rho,\vartheta) = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\vartheta$$

- To get the total cotribution we replace I with $JdS = J\cos\vartheta \cdot rdrd\vartheta$ and integrate from 0 to 2π

Perfect dipole #3: $J\cos\vartheta$ distribution

$$B_n = -\frac{\mu_0 J}{2\pi R_{ref}} \int_R^{R+w} \left(\frac{R_{ref}}{r}\right)^n r dr \int_0^{2\pi} \cos\vartheta \cos n\,\vartheta d\vartheta$$

$$\int_0^{2\pi} \cos\vartheta \cos n\,\vartheta d\vartheta = \begin{cases} \pi & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

- The only surviving term is B_1, i.e. the dipole field:

$$B_1 = -\frac{\mu_0 J_0}{2\pi R_{ref}} \int_R^{R+w} \left(\frac{R_{ref}}{r}\right) r dr \cdot \pi = -\frac{\mu_0 J w}{2}$$

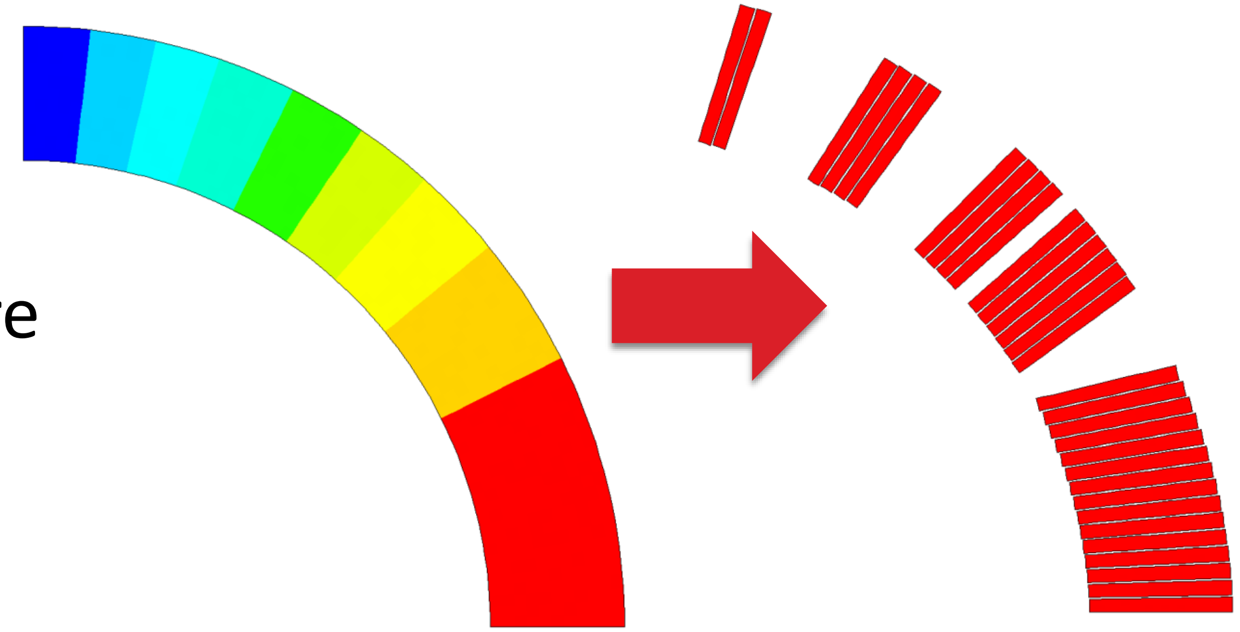
- $B_1 \propto$ current density (obvious)
- $B_1 \propto$ coil width w (less obvious)
- B_1 is independent of the aperture R (surprising)

- How can we approach this distribution using real conductors?

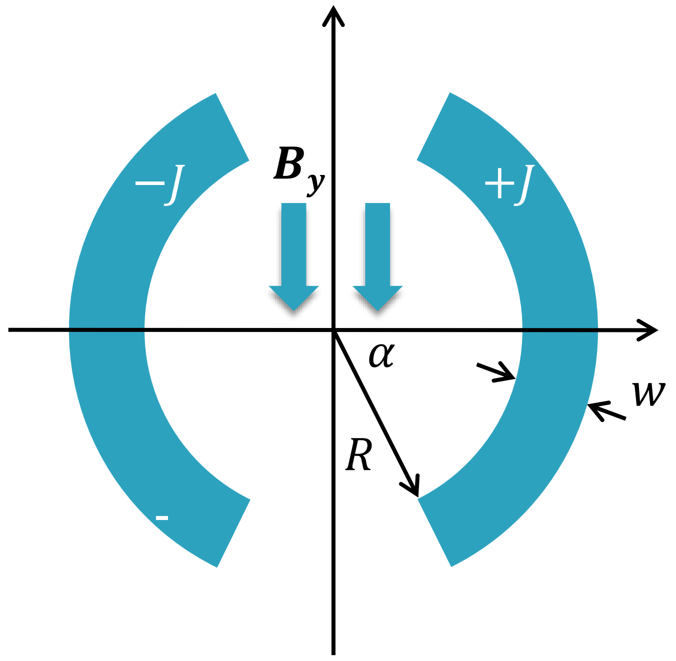
Perfect dipole vs real dipole

- Using real conductors, current density need to be uniform

- The perfect $J\cos\vartheta$ distribution is approached accumulating turn close to the midplane (where $\cos\vartheta \sim 1$) and reducing them at 90° (where $\cos\vartheta \rightarrow 0$)
  - the aperture is circular
  - the winding is self-supporting (roman arc)



Sector dipoles



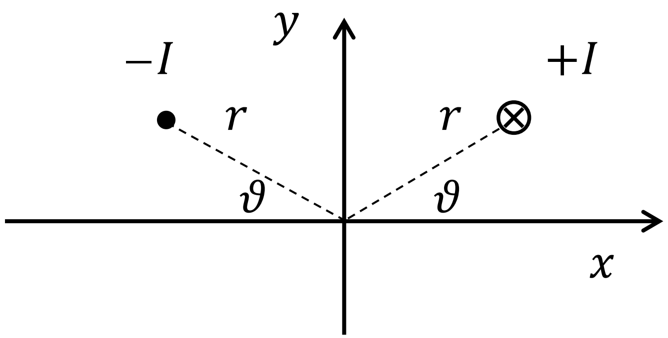
- The simplest approach is the sector dipole
- To get the dipole field B_1 we start again from the general expression for a current line

$$B_n(r,\vartheta) = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\,\vartheta$$

- Replacing I $\rightarrow JdS = J \cdot rdrd\vartheta$ and integrating for $n = 1$ we find:

$$B_1 = -2\frac{\mu_0 J}{2\pi R_{ref}} \int_R^{R+w} \left(\frac{R_{ref}}{r}\right) rdr \int_{-\alpha}^{\alpha} \cos\theta\, d\theta$$
$$= -\frac{2\mu_0 J w \sin\alpha}{\pi}$$

Symmetrical line currents



$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\,\vartheta - \frac{\mu_0(-I)}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\,(\pi-\vartheta)$$

$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n [\cos n\,\vartheta - \cos n\,(\pi-\vartheta)]$$

$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\,\vartheta[1-\cos n\,\pi]$$

$$= \begin{cases} -2\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\,\vartheta & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

Multipoles of a sector dipole

- Following the result for symmetrical line currents, $B_n = 0$ for even n
- For odd n:

$$B_n = -2\frac{\mu_0 J}{2\pi R_{ref}}\int_{R}^{R+w}\left(\frac{R_{ref}}{r}\right)^n r dr\int_{-\alpha}^{\alpha}\cos n\,\vartheta d\vartheta$$

$$= -\frac{2}{n(n-2)}\frac{\mu_0 J R_{ref}^{n-1}}{\pi}\sin n\,\alpha\left(\frac{1}{R^{n-2}}-\frac{1}{(R+w)^{n-2}}\right)$$

- Normalizing to the dipole field $B_1 = -\dfrac{2\mu_0 J w\sin\alpha}{\pi}$

$$b_n = \frac{1}{n(n-2)}\frac{R_{ref}^{n-1}\sin n\,\alpha}{w\sin\alpha}\left(\frac{1}{R^{n-2}}-\frac{1}{(R+w)^{n-2}}\right)\cdot 10^4$$

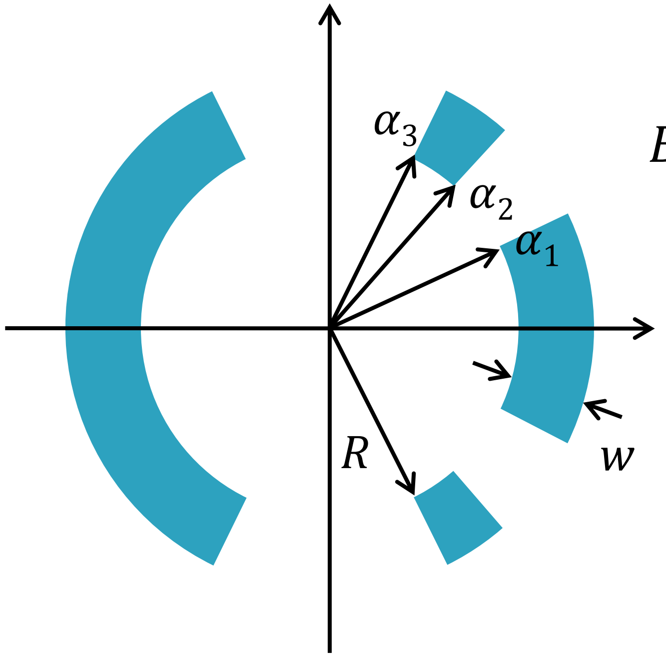
Multipoles of a sector dipole

- Multipoles are proportional to $\sin n\,\alpha$
- The solution of the equation $\sin n\,\alpha = 0$ is $\alpha = k\frac{\pi}{n}$ with k integer such that $0 < \alpha < \frac{\pi}{2}$
- With 1 sector we can set to zero only one multipole:
  - $b_3 = 0$ if $\alpha = \frac{\pi}{3}$
  - $b_5 = 0$ if $\alpha = \frac{\pi}{5}, \frac{2}{5}\pi$
  - $b_7 = 0$ if $\alpha = \frac{\pi}{7}, \frac{2}{7}\pi, \frac{3}{7}\pi$

R=50 mm, w=15 mm, J_0=5·10[8] A/m[2]

α	B_1 (T)	b_3	b_5	b_7	b_9
$\frac{3}{7}\pi$	-5.9	-914	106	0	-8
$\frac{\pi}{3}$	-5.2	0	-239	61	0
$\frac{2}{7}\pi$	-4.7	632	-298	0	22
$\frac{\pi}{5}$	-3.5	1844	0	-99	-17
$\frac{\pi}{7}$	-2.6	2560	431	0	-31

2-sector dipole



$$B_n = -4\frac{\mu_0 J}{2\pi R_{ref}}\int_R^{R+w}\left(\frac{R_{ref}}{r}\right)^n r dr\left(\int_0^{\alpha_1}\cos n\,\theta d\theta + \int_{\alpha_2}^{\alpha_3}\cos n\,\theta d\theta\right) \quad \text{for odd } n$$
$$= -\frac{2}{n(n-2)}\frac{\mu_0 J R_{ref}^{n-1}}{\pi}(\sin n\,\alpha_1 - \sin n\,\alpha_2 + \sin n\,\alpha_3)\left(\frac{1}{R^{n-2}} - \frac{1}{(R+w)^{n-2}}\right)$$

- 3 free parameters, means that we ca set to zero 3 multipoles at a time: $\begin{cases}(\sin 3\,\alpha_1 - \sin 3\,\alpha_2 + \sin 3\,\alpha_3) = 0 & B_3 = 0\\ (\sin 5\,\alpha_1 - \sin 5\,\alpha_2 + \sin 5\,\alpha_3) = 0 & B_5 = 0\\ (\sin 7\,\alpha_1 - \sin 7\,\alpha_2 + \sin 7\,\alpha_3) = 0 & B_7 = 0\end{cases}$

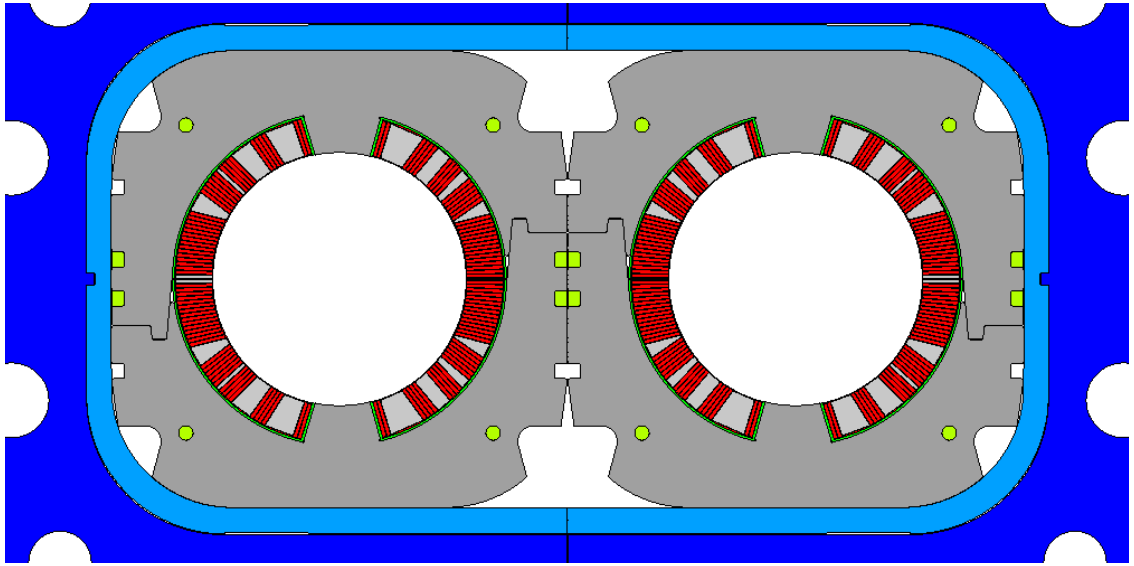
- A possible solution is nearly $\alpha_1 = 43.2°, \quad \alpha_2 = 52.2°, \quad \alpha_3 = 67.3°$

B_1 (T)	b_3	b_5	b_7	b_9	b_{11}	b_{13}
-4.9	0.5	0.3	-0.4	-29	12	1.5

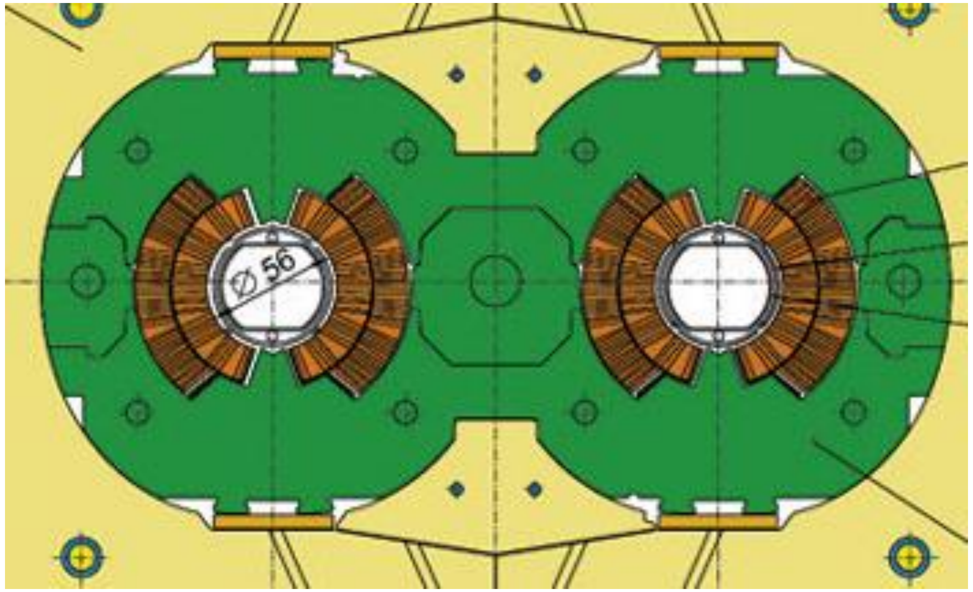
Real dipoles

- Due to the geometrical constraints of the cables, more parameters are needed to se to zero more multipoles

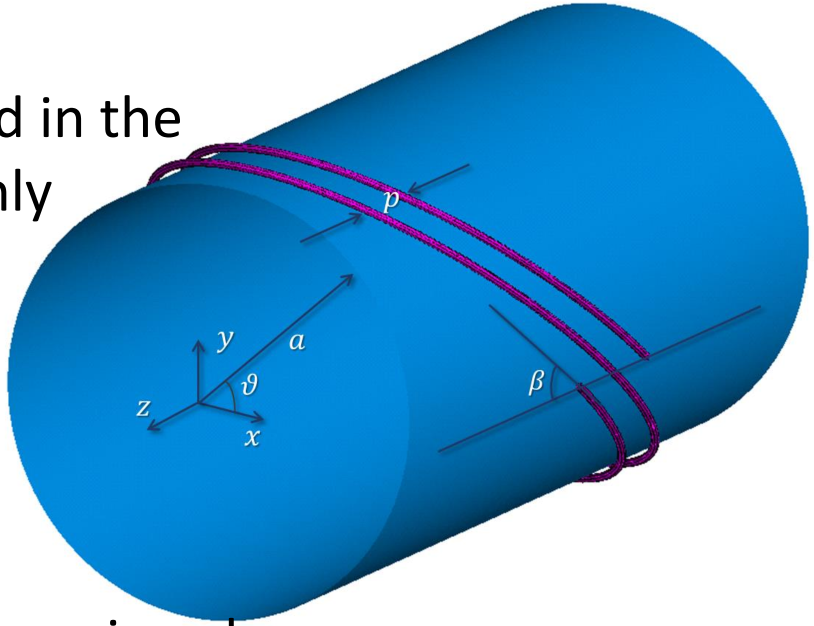
HiLumi D2 dipole



LHC dipole



Perfect dipole #4: canted $\cos\vartheta$ (CCT) dipoles



- the simplest CCT dipole consists of 2 inclined solenoids wound in the opposite direction: the solenoidal component cancels, and only the dipolar component remains
- The parametric equations of the two helices ($a_2 > a_1 >> p$) are

$$\boldsymbol{P_1}(\vartheta) = \begin{cases} a_1 \cos\vartheta \\ a_1 \sin\vartheta \\ \frac{p\vartheta}{2\pi} + \frac{a_1}{\tan\beta}\sin\vartheta \end{cases} \cup \quad \boldsymbol{P_2}(\vartheta) = \begin{cases} a_2 \cos\vartheta \\ a_2 \sin\vartheta \\ \frac{p\vartheta}{2\pi} - \frac{a_2}{\tan\beta}\sin\vartheta \end{cases} \qquad -\pi N < \vartheta < \pi N$$

- The resulting surface current densities, in polar coordinates, are given by

$$\boldsymbol{j_1} = \begin{cases} j_{1r} \\ j_{1\vartheta} \\ j_{1z} \end{cases} = \frac{I}{p}\begin{cases} 0 \\ 1 \\ \frac{p}{2\pi a_1} + \frac{\cos\vartheta}{\tan\beta} \end{cases} \cup \quad \boldsymbol{j_2} = \begin{cases} j_{2r} \\ j_{2\vartheta} \\ j_{2z} \end{cases} = \frac{-I}{p}\begin{cases} 0 \\ 1 \\ \frac{p}{2\pi a_2} - \frac{\cos\vartheta}{\tan\beta} \end{cases}$$

(derivation at the end of the slides)

Magnetic field from j_ϑ

- Reminding that an infinitely long solenoid generates a magnetic field given by $B_z = \mu_0 \frac{NI}{L} = \mu_0 \frac{I}{\frac{L}{N}} = \mu_0 \frac{I}{p}$, where $\frac{I}{p} = j_\vartheta$
- The azimuthal components of the current density in the 2-layer CCT dipole generate a solenoidal magnetic field given by:

$$B_z = \mu_0 j_{1\vartheta} + \mu_0 j_{2\vartheta} = \mu_0 \frac{I}{p} + \mu_0 \frac{-I}{p} = 0$$

Magnetic field from j_z

- Let's start from the harmonic components generated by a line current:

$$B_n(\rho,\theta) = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\,\vartheta$$

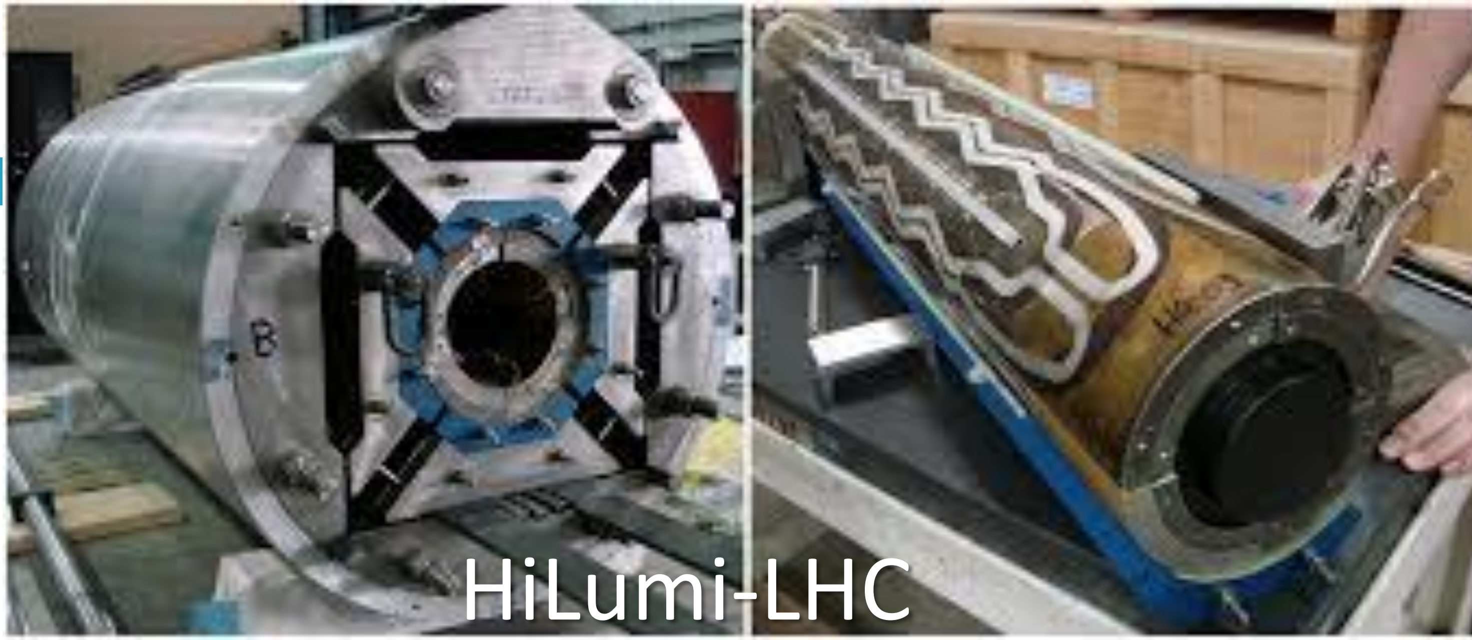
- In our case $r = a_1$ for $\boldsymbol{P_1}$ and $r = a_2$ for $\boldsymbol{P_2}$
- replacing $I \rightarrow j_z a d\vartheta$ and integrating we get that the harmonic components of a 2-layer CCT dipole are given by:

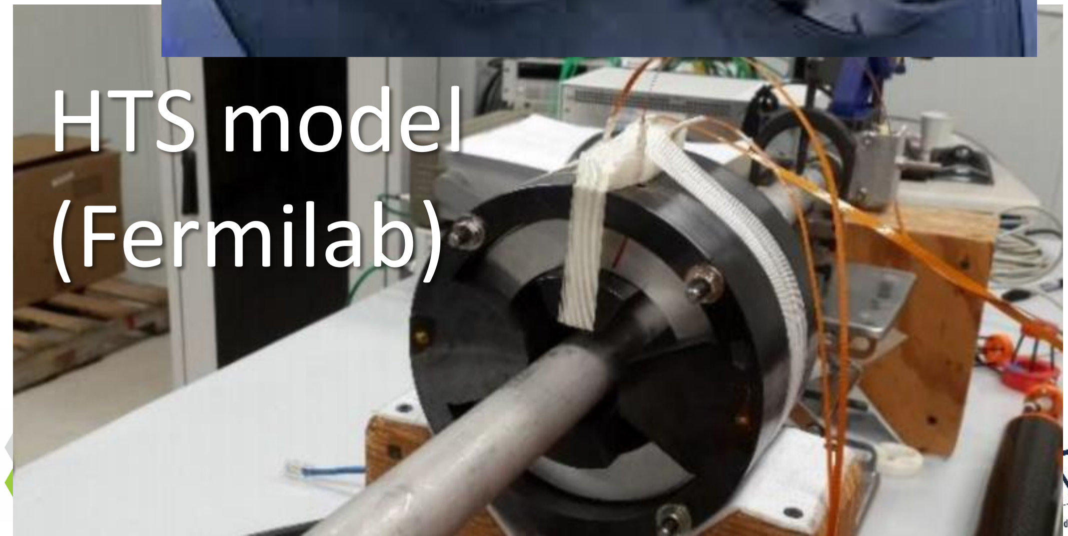
$$B_n = -\frac{\mu_0}{2\pi a_1}\left(\frac{R_{ref}}{a_1}\right)^{n-1}\frac{a_1 I}{p}\int_0^{2\pi}\cos n\,\vartheta\left(\frac{p}{2\pi a_1}+\frac{\cos\vartheta}{\tan\beta}\right)d\vartheta - \frac{\mu_0}{2\pi a_2}\left(\frac{R_{ref}}{a_2}\right)^{n-1}\frac{-a_2 I}{p}\int_0^{2\pi}\cos n\,\vartheta\left(\frac{p}{2\pi a_2}-\frac{\cos\vartheta}{\tan\beta}\right)d\vartheta$$

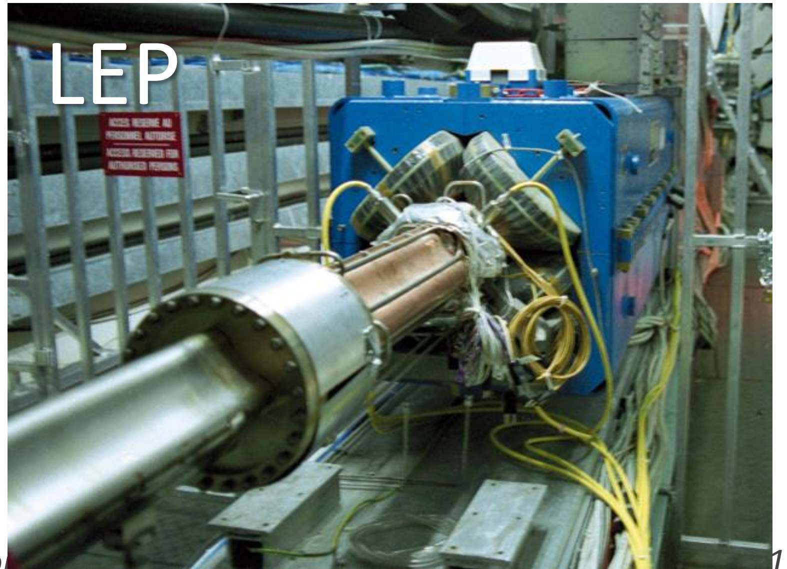
$$B_1 = B_y = -\frac{\mu_0 I}{p\tan\beta} \quad \text{and} \quad B_n = 0 \;\; n \neq 1$$

Quadrupoles

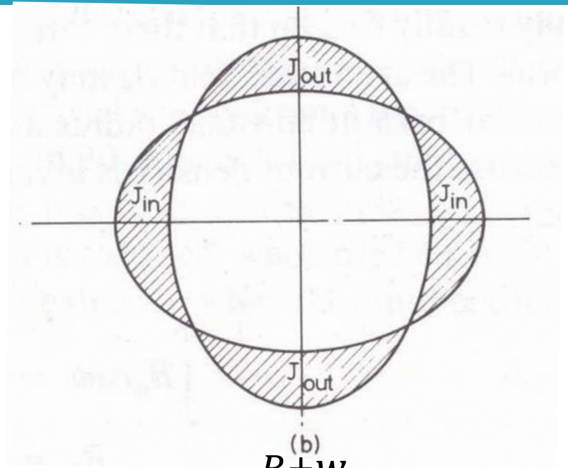
Quadrupoles

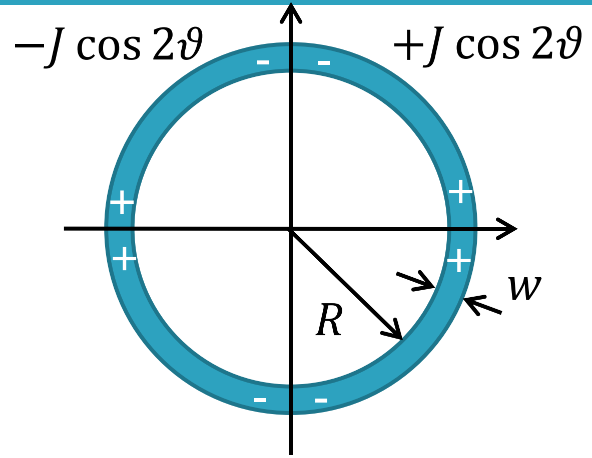






Perfect quadrupoles





$$B_n = -8\frac{\mu_0 J}{2\pi R_{ref}}\int_R^{R+w}\left(\frac{R_{ref}}{r}\right)^n r dr\int_0^{\pi/4}\cos n\vartheta\cos n\vartheta d\vartheta,\quad \text{if } \frac{n}{2} \text{ is odd}$$

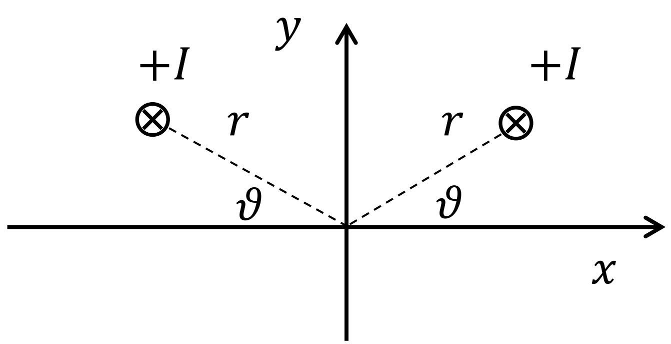
$$\int_0^{\pi/4}\cos 2\vartheta\cos n\vartheta d\vartheta = \begin{cases}\pi/8 & \text{se } n = 2\\ 0 & \text{se } n \neq 2\end{cases}$$

$$B_2 = -\frac{\mu_0 J R_{ref}}{2}\ln\left(1+\frac{w}{R}\right)$$

$$G = \frac{B_2}{R_{ref}} = \frac{\mu_0 J}{2}\ln\left(1+\frac{w}{R}\right)$$

Symmetrical line currents

$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\,\vartheta - \frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\,(\pi - \vartheta)$$



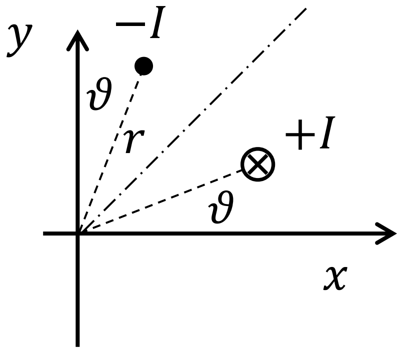
$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n [\cos n\,\vartheta + \cos n\,(\pi - \vartheta)]$$

$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\,\vartheta[1 + \cos n\,\pi]$$

$$= \begin{cases} -2\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\,\vartheta & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Line currents symmetrical with respect to the bisector

$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\vartheta - \frac{\mu_0(-I)}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\left(\frac{\pi}{2}-\vartheta\right)$$



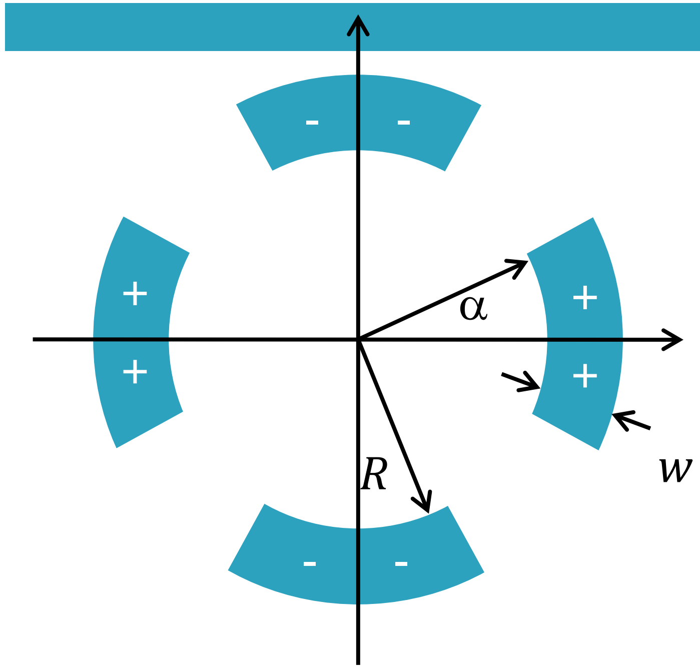
$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \left[\cos n\vartheta - \cos n\left(\frac{\pi}{2}-\vartheta\right)\right]$$

↓ if n is even

$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\vartheta\left[1-\cos\frac{n\pi}{2}\right]$$

$$= \begin{cases} -2\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\vartheta & \text{if } \frac{n}{2} \text{ is odd} \\ 0 & \text{if } \frac{n}{2} \text{ is even} \end{cases}$$

Sector quadrupole



- Only harmonic components with even n and odd $n/2$ survive (B_2, B_6, B_{10}, ..)
- Integrating as usual the harmonics of a line current:

$$B_n = -8\frac{\mu_0 J}{2\pi R_{ref}}\int_R^{R+w}\left(\frac{R_{ref}}{r}\right)^n r dr \int_0^\alpha \cos n\vartheta d\vartheta$$

$$B_n = \begin{cases} -\dfrac{2\mu_0 J R_{ref}}{\pi}\sin 2\alpha \ln\left(1+\dfrac{w}{R}\right) & n=2 \\ -\dfrac{4}{n(n-2)}\dfrac{\mu_0 J R_{ref}^{n-1}}{\pi}\sin n\alpha\left(\dfrac{1}{R^{n-2}}-\dfrac{1}{(R+w)^{n-2}}\right) & n=6,10,14,\dots \end{cases}$$

Sector quadrupole

- The gradient [T/m] is given by:

$$G = \frac{B_2}{R_{ref}} = -\frac{2\mu J}{\pi} \sin 2\,\alpha \ln\left(1 + \frac{w}{R_{ref}}\right)$$

- With 1 sector we can set to zero only one multipole :

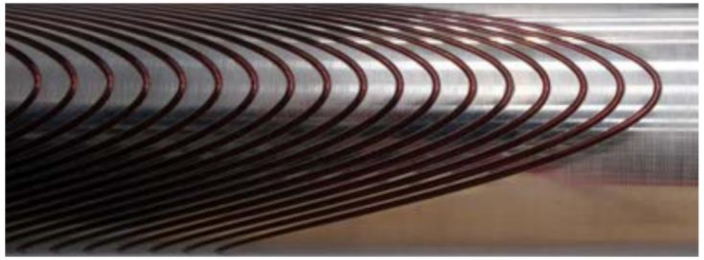
$\sin n\,\alpha = 0 \;\rightarrow \alpha = k\frac{\pi}{n}$ with k integer such that $0 < \alpha < \frac{\pi}{4}$

  - $b_6 = 0$ if $\alpha = 30°$
  - $b_{10} = 0$ if $\alpha = 18°, 36°$

R=50 mm, w=15 mm, J_0=5·10[8] A/m[2]

α	G (T/m)	b_4 (units)	b_{10} (units)	b_{14} (units)
30°	-91	0	-32	3
18°	-62	660	0	-5
36°	-100	-252	0	2

Perfect CCT quadrupoles



- In the same notation used for dipoles, the simplest CCT quadrupole consists of 2 inclined helices wound in the opposite direction
- The parametric equations of the two helices ($a_2 > a_1 >> p$) are

$$\boldsymbol{P_1}(\vartheta) = \begin{cases} a_1 \cos\vartheta \\ a_1 \sin\vartheta \\ \dfrac{p\vartheta}{2\pi} + \dfrac{a_1}{2\tan\beta}\sin 2\vartheta \end{cases} \cup \quad \boldsymbol{P_2}(\vartheta) = \begin{cases} a_2 \cos\vartheta \\ a_2 \sin\vartheta \\ \dfrac{p\vartheta}{2\pi} - \dfrac{a_2}{2\tan\beta}\sin 2\vartheta \end{cases} \quad -\pi N < \vartheta < \pi N$$

- The resulting surface current densities, in polar coordinates, are given by

$$\boldsymbol{j_1} = \begin{cases} j_{1r} \\ j_{1\vartheta} \\ j_{1z} \end{cases} = \frac{I}{p}\begin{cases} 0 \\ 1 \\ \dfrac{p}{2\pi a_1} + \dfrac{\cos 2\vartheta}{\tan\beta} \end{cases} \cup \quad \boldsymbol{j_2} = \begin{cases} j_{2r} \\ j_{2\vartheta} \\ j_{2z} \end{cases} = \frac{-I}{p}\begin{cases} 0 \\ 1 \\ \dfrac{p}{2\pi a_2} - \dfrac{\cos 2\vartheta}{\tan\beta} \end{cases}$$

Magnetic field from j_ϑ

- Reminding that an infinitely long solenoid generates a magnetic field given by $B_z = \mu_0 \frac{NI}{L} = \mu_0 \frac{I}{\frac{L}{N}} = \mu_0 \frac{I}{p}$, where $\frac{I}{p} = j_\vartheta$
- The azimuthal components of the current density in the 2-layer CCT quadrupole generate a solenoidal magnetic field given by:

$$B_z = \mu_0 j_{1\vartheta} + \mu_0 j_{2\vartheta} = \mu_0 \frac{I}{p} + \mu_0 \frac{-I}{p} = 0$$

Magnetic field from j_z

- Let's start from the harmonic components generated by a line current:

$$B_n(\rho,\theta) = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\,\theta$$

- In our case $r = a_1$ for $\boldsymbol{P_1}$ and $r = a_2$ for $\boldsymbol{P_2}$
- replacing $I \rightarrow j_z a d\vartheta$ and integrating we get that the harmonic components of a 2-layer CCT quadrupole are given by:

$$B_n = -\frac{\mu_0}{2\pi a_1}\left(\frac{R_{ref}}{a_1}\right)^{n-1}\frac{a_1 I}{p}\int_0^{2\pi}\cos n\,\vartheta\left(\frac{p}{2\pi a_1}+\frac{\cos 2\vartheta}{\tan\beta}\right)d\vartheta - \frac{\mu_0}{2\pi a_2}\left(\frac{R_{ref}}{a_2}\right)^{n-1}\frac{-a_2 I}{p}\int_0^{2\pi}\cos n\,\vartheta\left(\frac{p}{2\pi a_2}-\frac{\cos 2\vartheta}{\tan\beta}\right)d\vartheta$$

$$B_2 = GR_{ref} = -\frac{\mu_0 I R_{ref}}{2p\tan\beta}\left(\frac{1}{a_1}+\frac{1}{a_2}\right) \quad \text{and} \quad B_n = 0 \;\; n \neq 2$$

Iron yoke

Why magnets are surrounded by iron yoke?

- Accelerator magnets are usually surrounded by iron yoke:
  - It considerably enhances the bore field for a given current density
  - It modifies the loadline (increasing B_{SS})
  - It considerably reduces the fringe field
  - It can contribute to mechanical structure (see Susana presentation)

Line current in a cylindrical iron shell

- The harmonic components of a line current inside a cylindrical iron shell of radii R_{in} and R_{out} is given by

$$B_n(r,\vartheta) = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{r}\right)^n \cos n\vartheta \left[1 + k\left(\frac{r}{R_{in}}\right)^{2n}\right]$$

$$k = \frac{\mu_r - 1}{\mu_r + 1}\frac{1-\left(\frac{R_{in}}{R_{out}}\right)^{2n}}{1-\left(\frac{\mu_r-1}{\mu_r+1}\right)^2\left(\frac{R_{in}}{R_{out}}\right)^{2n}} \approx 1 \quad \text{se} \quad \mu_r >> 1$$

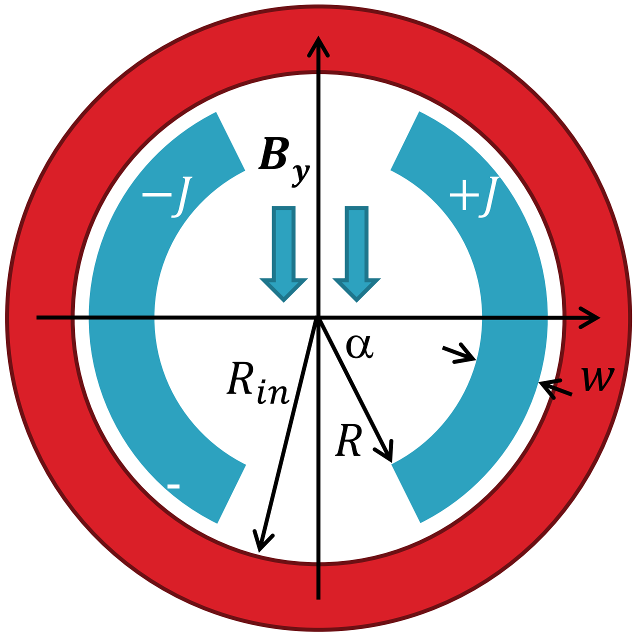
Sector dipole inside a cylindrical shell

- Integrating the line current harmonics we get the resulting dipole field:

$$B_1 = -4\frac{\mu_0 J}{2\pi R_{ref}}\int_{R}^{R+w}\left(\frac{R_{ref}}{r}\right)\left[1+k\left(\frac{r}{R_{in}}\right)^2\right]rdr\int_0^{\alpha}\cos\vartheta\, d\vartheta$$

$$= -\frac{2\mu_0 J\sin\alpha}{\pi}\left[w+k\frac{(R+w)^3-R^3}{3R_{in}^2}\right]$$

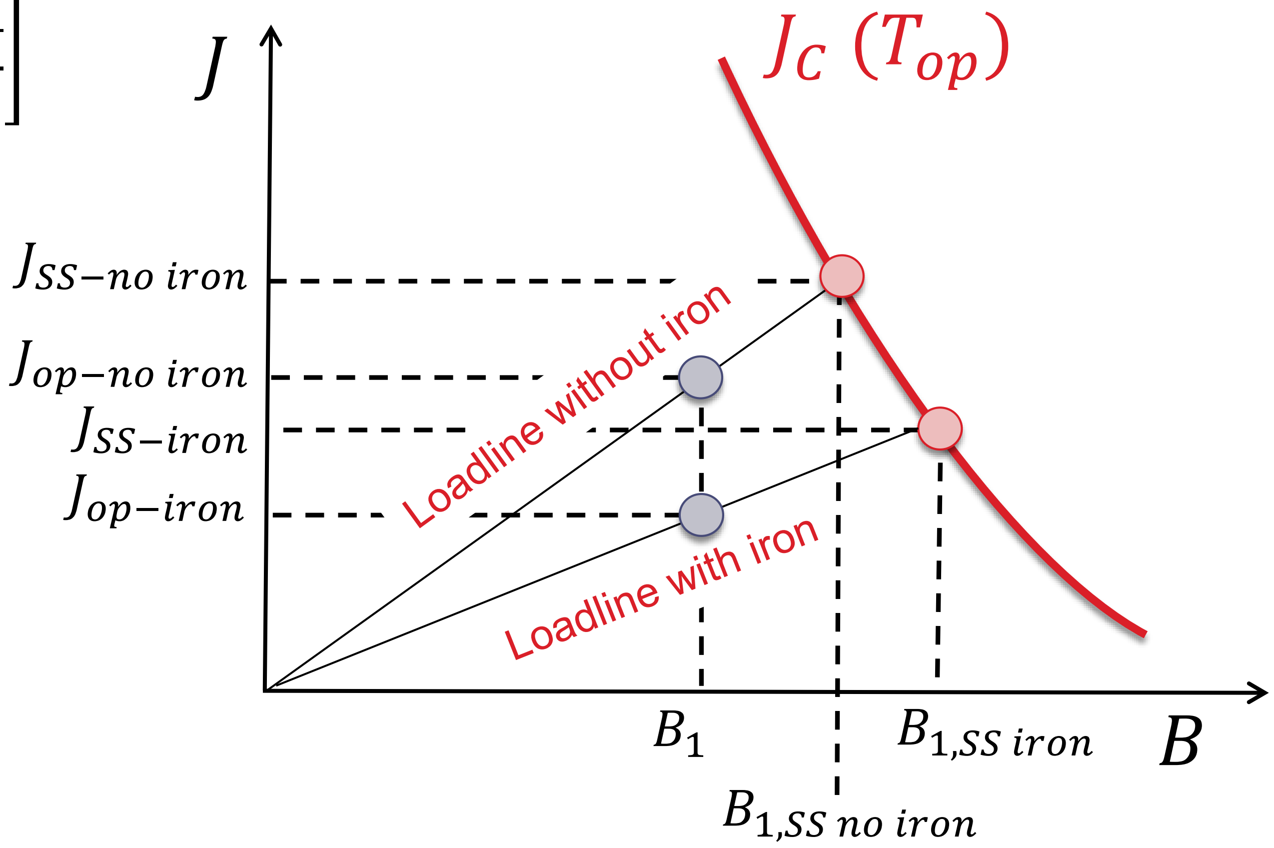
$$= -\frac{2\mu_0 Jw\sin\alpha}{\pi}\left[1+k\frac{R^2+wR+\frac{w^2}{3}}{R_{in}^2}\right]$$



- The contribution is relevant (15-50%) when iron is not far from the winding ($R_{in} \gtrsim R+w$), i.e. for small collar widths, and it affect the main components (dependence on $\left(\frac{r}{R_{in}}\right)^{2n}$)

Impact on short sample field

- $B_{1\,iron} = -\frac{2\mu_0 J_{op} w \sin\alpha}{\pi}\left[1 + k\frac{R^2+wR+\frac{w^2}{3}}{R_{in}^2}\right]$
- $B_{1\,iron} = B_{1\,no\,iron}\left[1 + k\frac{R^2+wR+\frac{w^2}{3}}{R_{in}^2}\right]$
- To get the same bore field, ,we need less current density:
  - $J_{op-no\,iron} = \frac{\pi B_1}{2\mu_0 w \sin\alpha}$
  - $J_{op-iron} = \frac{\pi B_1}{2\mu_0 w \sin\alpha\left[1+k\frac{R^2+wR+\frac{w^2}{3}}{R_{in}^2}\right]}$



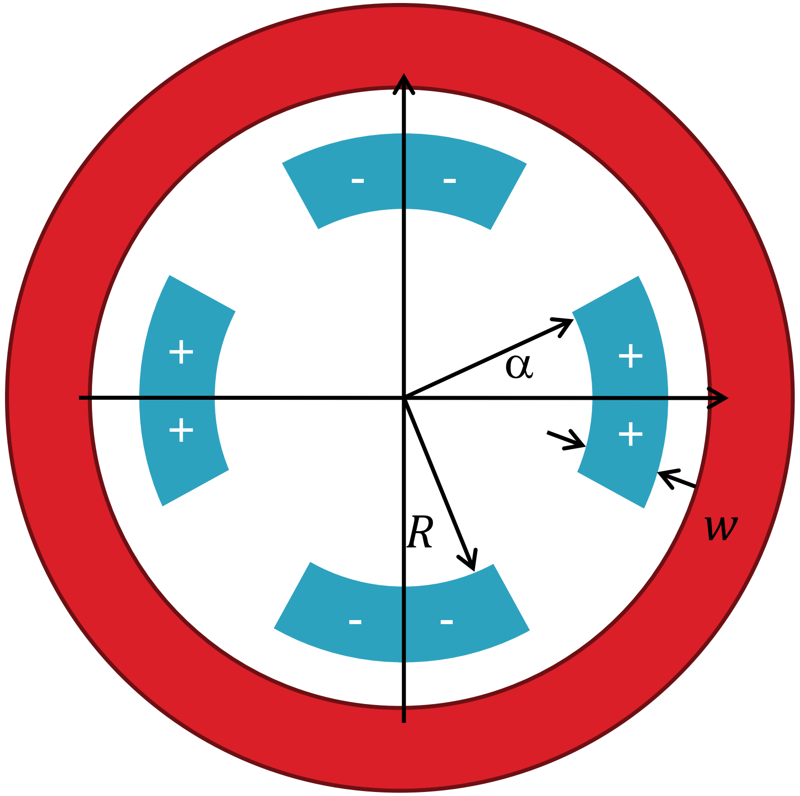
Sector quadrupole inside a cylindrical shell

- Integrating the line current harmonics we get the resulting quadrupole field field:

$$B_2 = -8\frac{\mu_0 J}{2\pi R_{ref}}\int_R^{R+w}\left(\frac{R_{ref}}{r}\right)^2\left[1+k\left(\frac{r}{R_{in}}\right)^4\right]rdr\int_0^{\alpha}\cos 2\vartheta\, d\vartheta$$

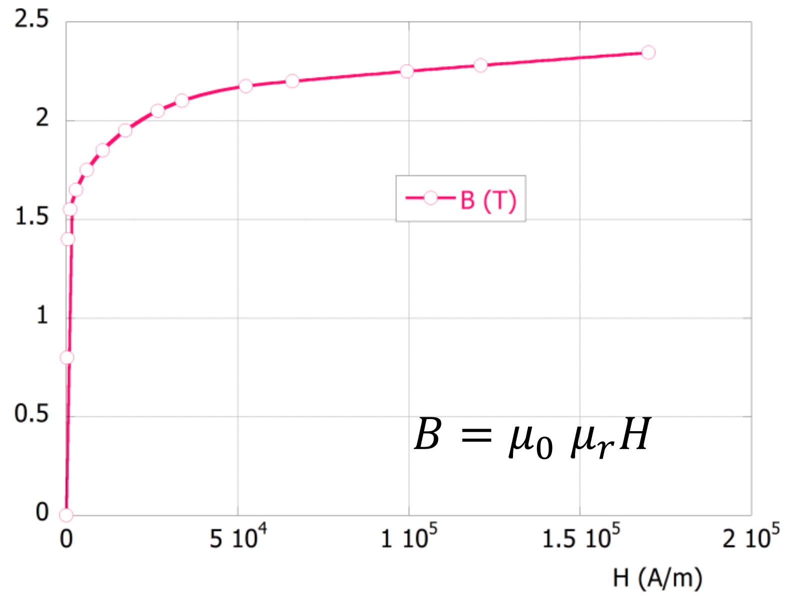
$$= -\frac{2\mu_0 J R_{ref}\sin 2\alpha}{\pi}\left[\ln\left(1+\frac{w}{R}\right)+k\frac{(R+w)^4-R^4}{4R_{in}^4}\right]$$

- The contribution is less relevant than dipole



Iron saturation

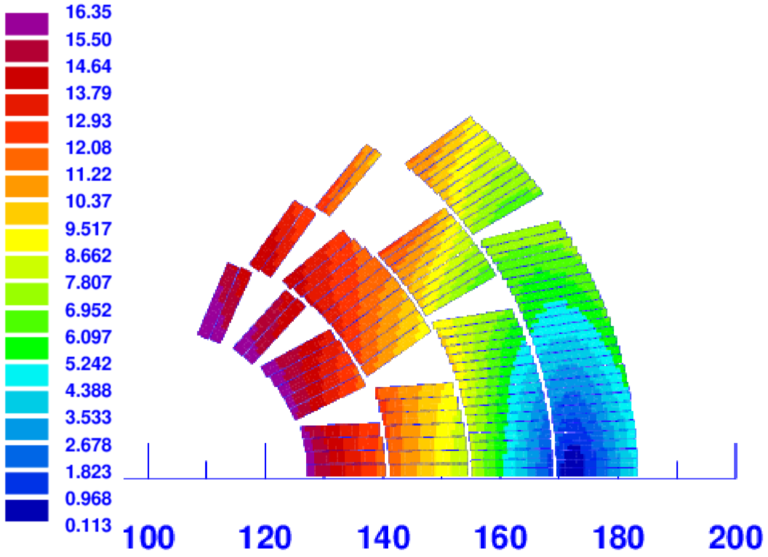
- Previous considerations are valid when iron yoke works in its linear range, i.e. below saturation
- Typical iron saturates for $B \sim 2\ T$
  - If $B < 2\ T$ BH curve is roughly linear with a pendency of $\mu_r \sim 10^3 - 10^4$
  - If $B > 2\ T$ $\mu_r \sim 1$ and iron gives no further contribution
  - The correct iron yoke contribution to magnetic field, including saturation, can only be determined via finite element analysis



Grading techniques

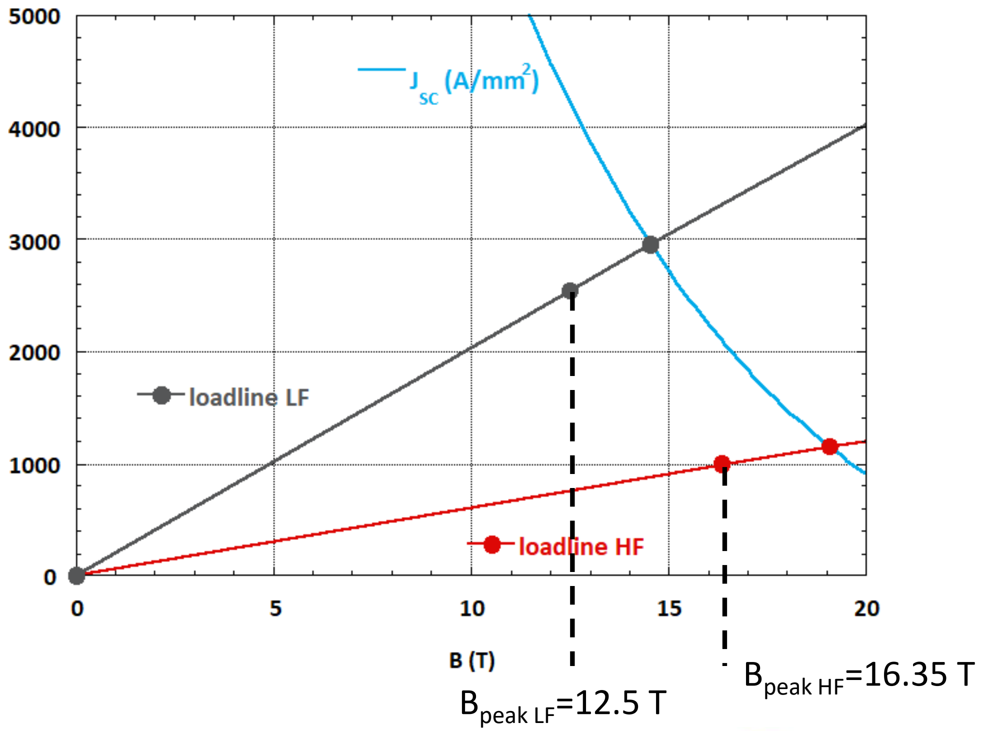
Grading techniques

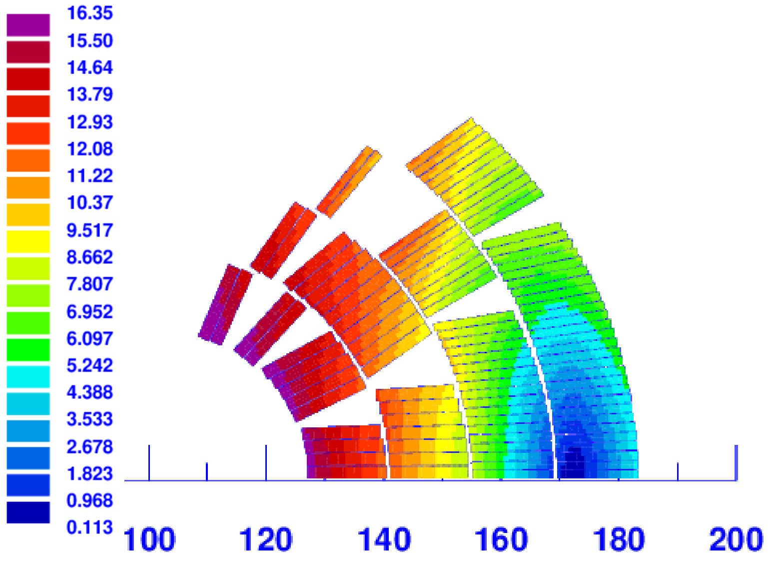
- The field map inside a coil is highly nonuniform (inner layers have larger peak fields than outer layers)
- In the low field outer layers it is possible to:
  - use larger current density and narrower conductor
  - Use a less performant (and cheaper) material



An example: 16 T cos dipole for the FCC

- Both inner and outer layers are dimensioned so that the margin on the loadline is 14%





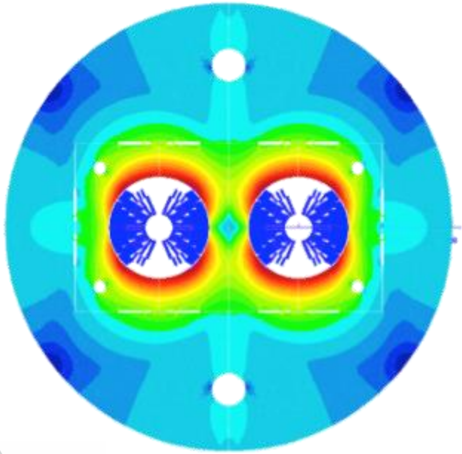
	HF Cable (inner)	LF Cable (outer)
Strand number	22	38
Strand diameter	1.1 mm	0.7 mm
Bare width	13.2 mm	14 mm
Bare inner thickness	1.892 mm	1.204 mm
Bare outer thickness	2.007 mm	1.3261 mm
Insulation	0.15 mm	0.15 mm
Keystone angle	0.5°	0.5°
Cu/Ncu	0.82	2.1
Operating current	11390 A	11390 A
Operating point on LL (1.9 K)	86 %	86 %

Winding shapes

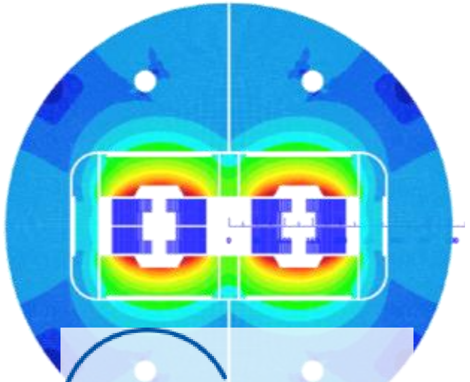
Dipole winding shapes – EuroCirCol project

- I will show the results of the optimization of a double aperture 16 T dipole for the FCC in 4 different options as part of WP5 of Eurocircol project (www.eurocircol.eu)
- All optimizations share common assumption: same magnet aperture (50 mm), conductor performance ($J_C(16\,T, 4.2\,K) = 1500\ \mathrm{A/mm^2}$), margin on the loadline (>14%), allowed mechanical constraints (σ <150 MPa at warm and <200 MPa at cold)

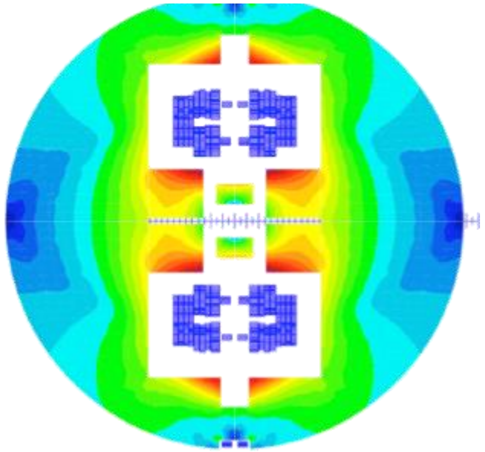
Cos-theta



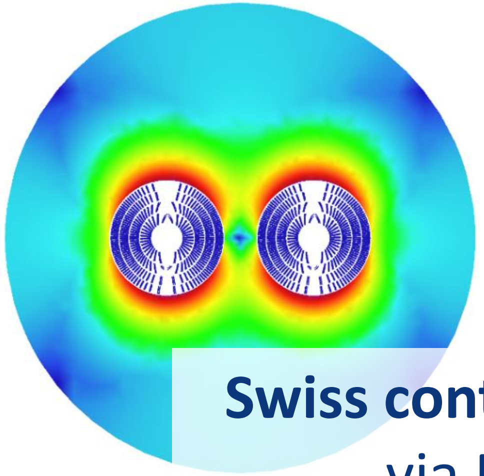
Blocks



Common coils



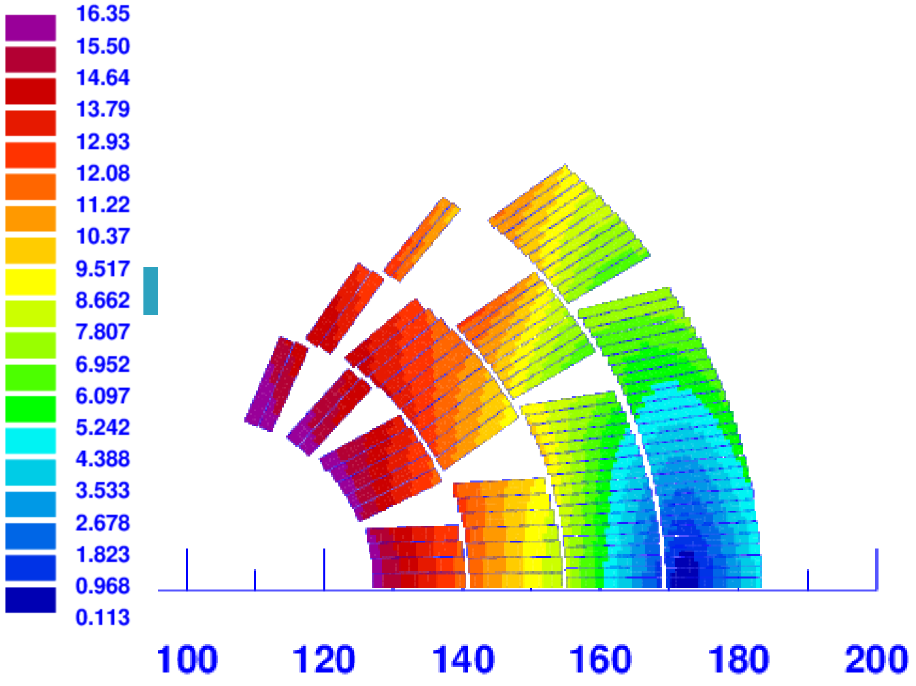
Canted Cos-theta

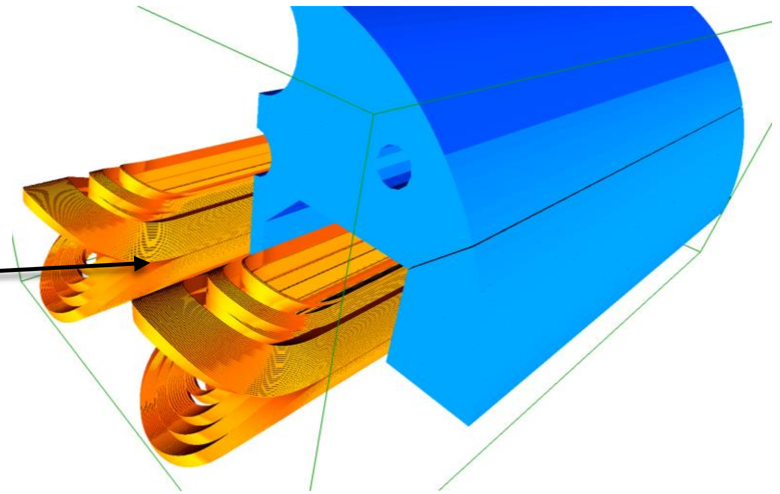


Swiss contribution via PSI

Cos-theta coil

- Pros
  - Natural choice (LHC dipoles)
  - Circular aperture fully available for beam
  - Self-supporting winding (roman arc)
- Cons
  - Hardway bending in coil ends

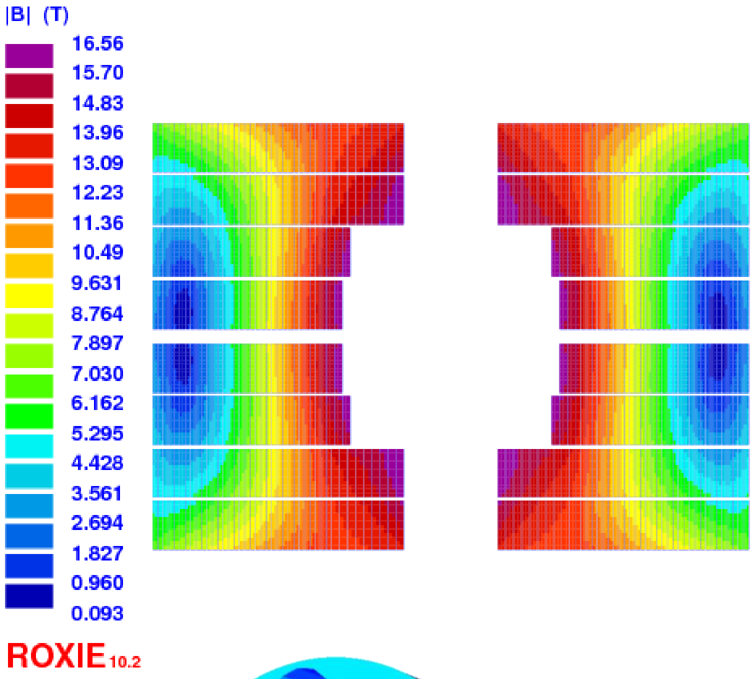


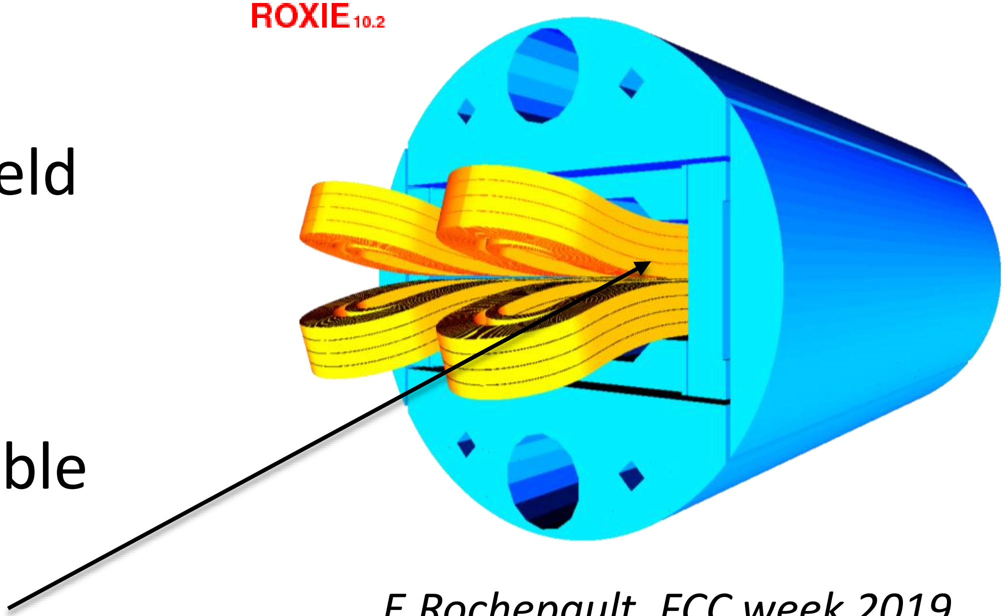


S.Farinon, FCC week 2019

Block coil

- Pros
  - Particularly indicated for thick coils (turn are stacked vertically)
  - No wedges (saddle shape ends)
  - Peak stress during powering in the low field region
- Cons
  - Need of internal support, reducing available aperture
  - Very complicated coil ends (hardway bending)

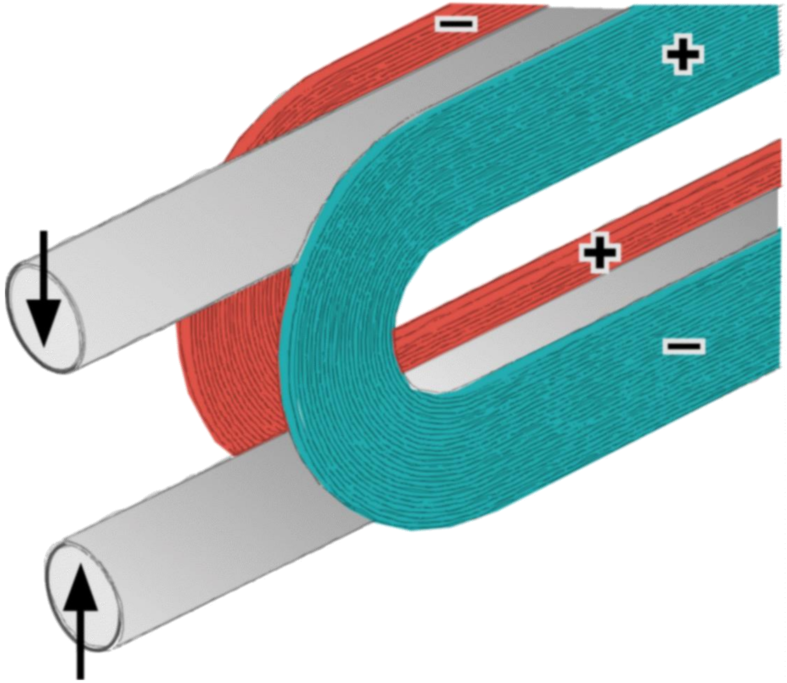


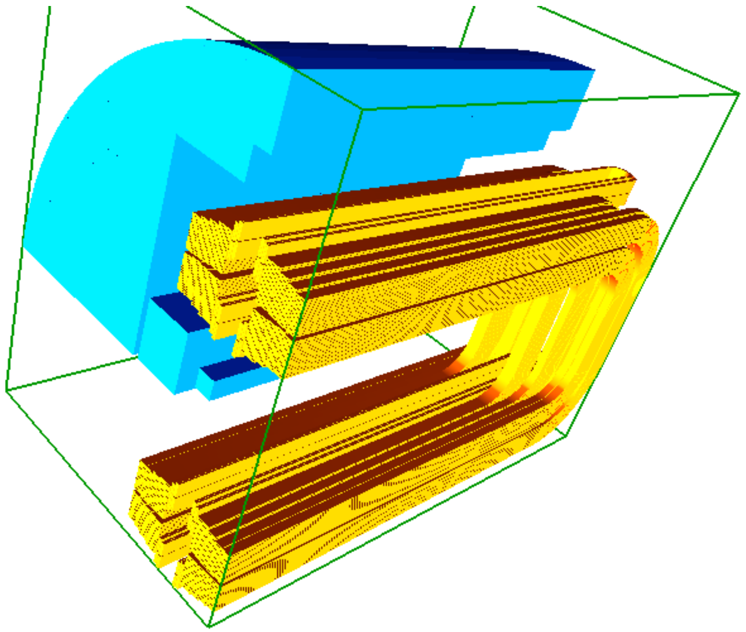


E.Rochepault, FCC week 2019

Common coil

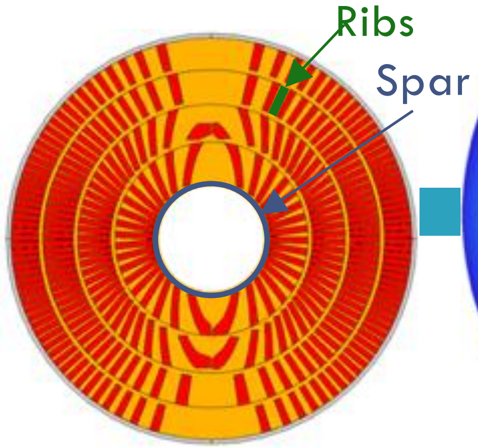
- Pros
  - Very simple coils (flat racetrack shape)
- Cons
  - Complicated stress management (huge radial Lorentz force)
  - Needs more superconductors

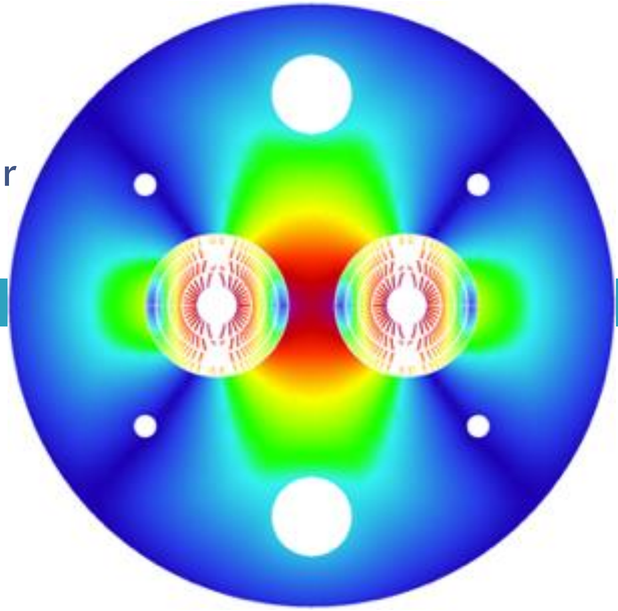




F. Toral, FCC week 2019

CCT – Canted Cos Theta coil





- Pros
  - Each turn is individually supported
  - 360° continuity of the winding: no azimuthal pre-load
  - No field distortion in coil ends
  - Small number of mechanical components
- Cons
  - Part of the current density lost in generating solenoidal field
  - Need more superconductors
  - Complicated winding if large Rutherford cables (bonding of cable inside channels, reliable insulation against former)

B. Auchmann, FCC week 2019

Results of the comparison

- The $\cos\vartheta$ configuration has been selected as baseline for the Conceptual Design Report of the EuroCirCol project (http://cds.cern.ch/record/2651300/files/CERN-ACC-2018-0058.pdf?version=6)
- *“Each of these alternatives features some interesting characteristics which may have a potential to become competitive to the baseline cosine-theta design in terms of performance, in particular if they would allow operation at a lower margin on the load-line, thus reducing the required amount of conductor”*

↓

- Short model magnets (~1.5 m lengths) of all the options will be built from 2018–2022

THANKS FOR THE ATTENTION

A thorough Masterclass on superconducting magnets for particle accelerators by Ezio Todesco is available at https://indico.cern.ch/category/12408/

Derivation of current density in CCT magnet

Derivation of current density in CCT magnet

- Let's consider that a current I flows along the helix defined as

$$\boldsymbol{P}(\vartheta) = \begin{cases} a\ cos\vartheta \\ a\ sin\vartheta \\ \dfrac{h\vartheta}{2\pi} + A\ sin\vartheta \end{cases}$$

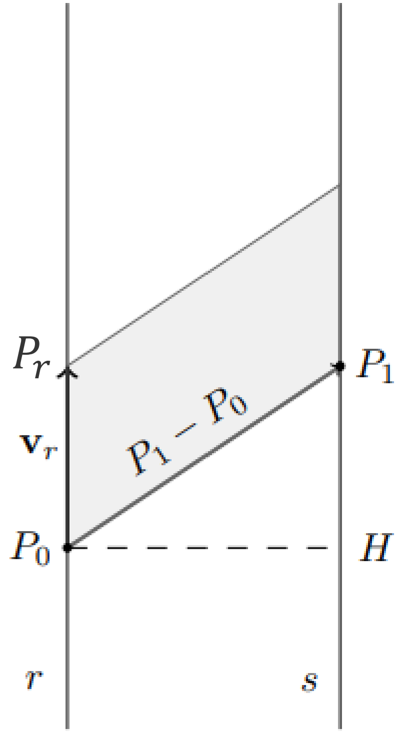
- If the helix is infintely long the current density will be given by:
  - $\boldsymbol{j}(\vartheta) = \frac{I}{\delta(\vartheta)} \widehat{\boldsymbol{v}}_{\boldsymbol{r}}(\vartheta)$

    where $\delta(\vartheta)$ is the distance bewteen two consecutive turns and ,$\widehat{\boldsymbol{v}}_{\boldsymbol{r}}(\vartheta)$ is the versor of the current direction
  - By definition $|\widehat{\boldsymbol{v}}_{\boldsymbol{r}}(\vartheta)| = 1$ and the direction is the same of the derivative of $\boldsymbol{P}(\vartheta)$

$$\widehat{\boldsymbol{v}}_{\boldsymbol{r}}(\vartheta) = \frac{1}{\sqrt{a^2 + \left(\dfrac{h}{2\pi} + A\ cos\vartheta\right)^2}} \begin{cases} -a\ sin\vartheta \\ \ \ a\ cos\vartheta \\ \dfrac{h}{2\pi} + A\ cos\vartheta \end{cases}$$

Determination of $\delta(\vartheta)$

- $\delta(\vartheta)$ is the turn to turn distance, i.e. the distance between the two straight lines tangent to $\boldsymbol{P}(\vartheta)$ in ϑ e $\vartheta + 2\pi$.
- The two straight lines being parallel, that distance can be calculated as
  - $\delta = |(\boldsymbol{P_1} - \boldsymbol{P_0}) \times \widehat{\boldsymbol{v}}_{\boldsymbol{r}}|$
    where $\boldsymbol{P_1} = \boldsymbol{P}(\vartheta + 2\pi)$ and $\boldsymbol{P_0} = \boldsymbol{P}(\vartheta)$
  - $\boldsymbol{P_0} = \begin{cases} a\,cos\vartheta \\ a\,sin\vartheta \\ \frac{h\vartheta}{2\pi} + A\,sin\vartheta \end{cases}$ $\quad \boldsymbol{P_1} = \begin{cases} a\,cos\vartheta \\ a\,sin\vartheta \\ \frac{h\vartheta}{2\pi} + h + A\,sin\vartheta \end{cases}$ $\quad \boldsymbol{P_1} - \boldsymbol{P_0} = \begin{cases} 0 \\ 0 \\ h \end{cases}$



Determination of $\delta(\vartheta)$ and $\mathbf{j}(\vartheta)$

- $\hat{\boldsymbol{v}}_{\boldsymbol{r}} = \frac{1}{\sqrt{a^2+\left(\frac{h}{2\pi}+A\cos\vartheta\right)^2}} \begin{cases} -a\sin\vartheta \\ a\cos\vartheta \\ \frac{h}{2\pi} + A\cos\vartheta \end{cases}$
- $\boldsymbol{P_1}-\boldsymbol{P_0} = \begin{cases} 0 \\ 0 \\ h \end{cases}$
- $\delta(\vartheta) = |(\boldsymbol{P_1}-\boldsymbol{P_0}) \times \hat{\boldsymbol{v}}_{\boldsymbol{r}}| = \frac{1}{\sqrt{a^2+\left(\frac{h}{2\pi}+A\cos\vartheta\right)^2}} \left| \begin{cases} -ah\cos\vartheta \\ -ah\sin\vartheta \\ 0 \end{cases} \right| = \frac{ah}{\sqrt{a^2+\left(\frac{h}{2\pi}+A\cos\vartheta\right)^2}}$
- $\boldsymbol{j}(\vartheta) = \frac{I}{\delta(\vartheta)} \hat{\boldsymbol{v}}_{\boldsymbol{r}}(\vartheta) = \frac{I}{ah} \begin{cases} -a\sin\vartheta \\ a\cos\vartheta \\ \frac{h}{2\pi} + A\cos\vartheta \end{cases}$ in cartesian coordinates
- Since $j_r = j_x\cos\vartheta + j_y\sin\vartheta$; $j_\vartheta = -j_x\sin\vartheta + j_y\cos\vartheta$, we get in polar coordinates: $\boldsymbol{j}(\vartheta) = \frac{I}{ah} \begin{cases} 0 \\ a \\ \frac{h}{2\pi} + A\cos\vartheta \end{cases}$