Quantum Chromodynamics

V. Ravindran

The Institute of Mathematical Science,
Chennai, India

AEPSHEP 2022 05-18 OCTOBER 2022, Pyeongchang, SOUTH KOREA

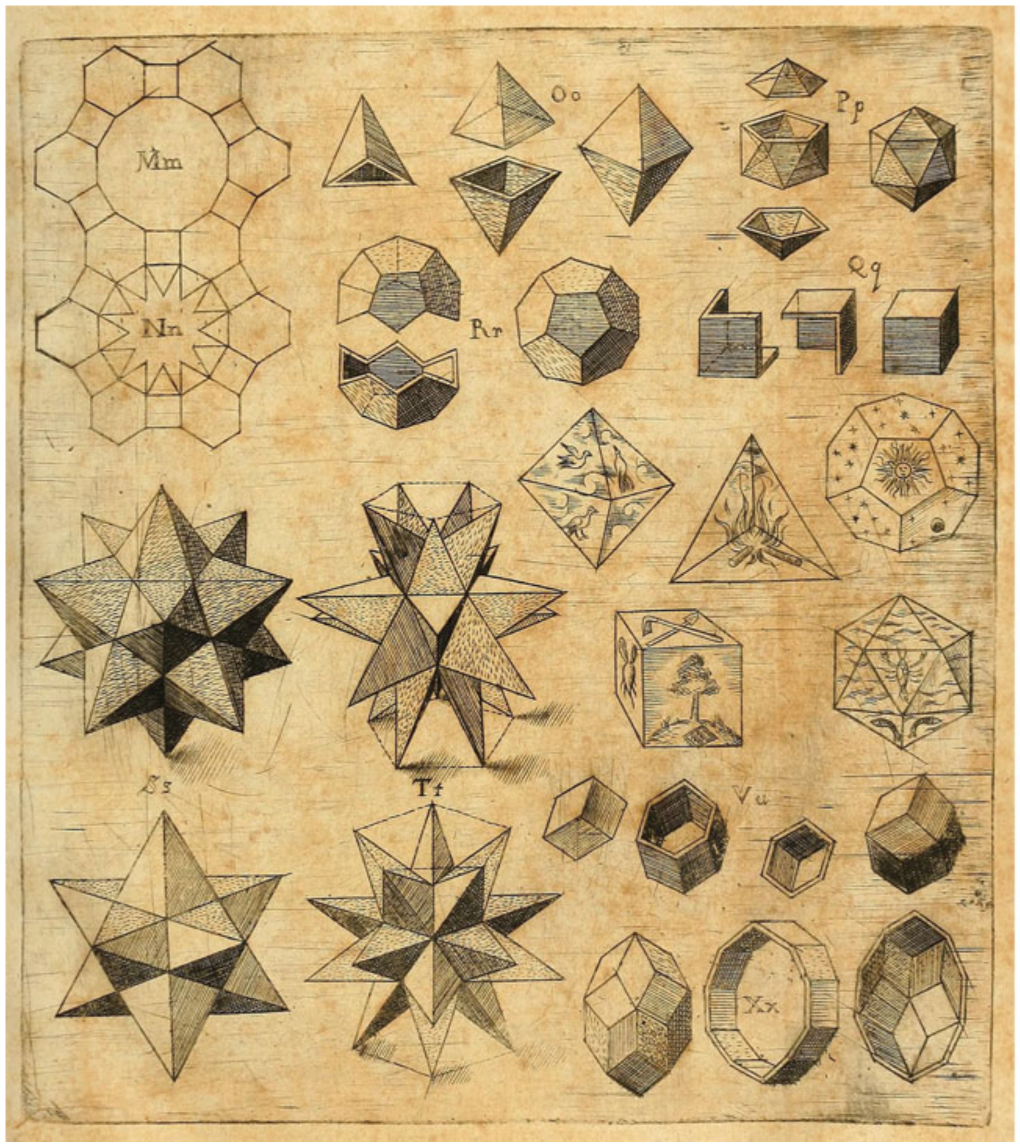
QCD-1

Lecture-1

- Quark Model, Form Factor etc
- Deep Inelastic Scattering
  - Bjorken Scaling
  - Naive Parton Model
- Quantum Chromodynamics (QCD)
- QCD improved Parton Model
  - NLO Coefficient
  - DGLAP evolution
- NNLO and Beyond
- Higher twist, Heavy flavours

Question:

What is everything made up ?
What are the fundamental building blocks?

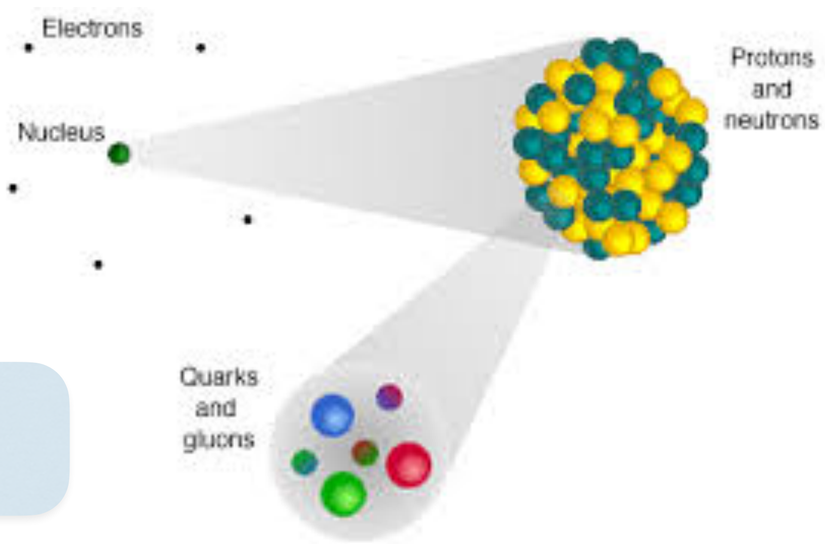


Earliest answer by Plato:

Earth, Air, Fire and Water

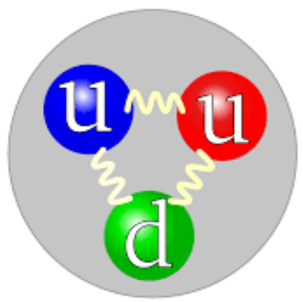
Structure of Matter

Atoms

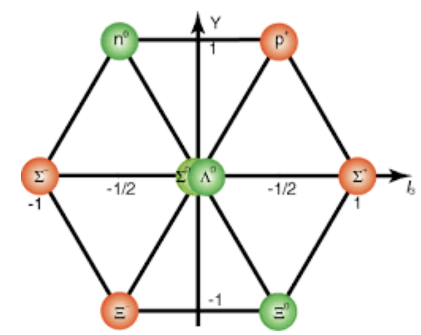


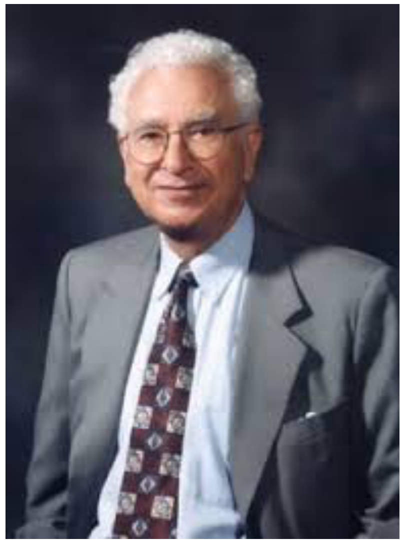
- Graviation
- Electromagnetic
- Weak
- Strong interaction

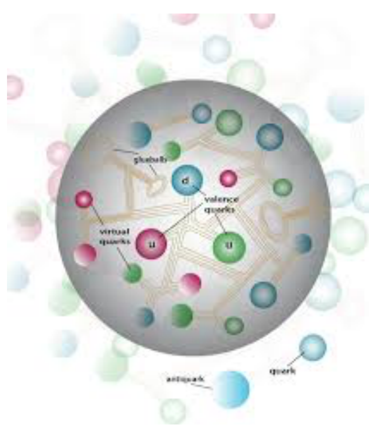
- Strong interaction



Quark Model







Parton Model

QCD

Journey Continues .....

Quark Model

Quark Model - Gell-Mann and Zweig

Strongly interacting particles: Hadrons

Mesons and Baryons

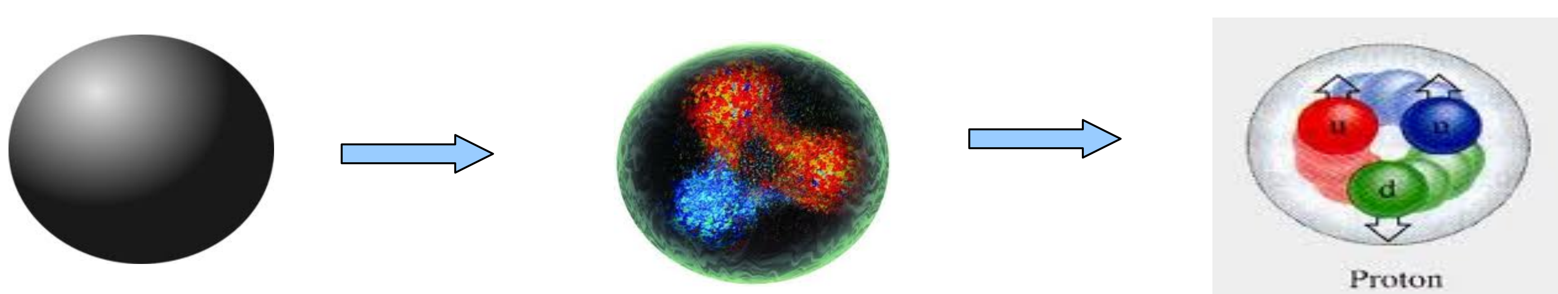
Classification:

Assume that they are composite objects Made up of point-like spin - 1/2 partilces called

Quarks and Anti-quarks

Baryons: Three quarks

Mesons: Quark and anti-quark



Proton

Quark Model

Gell-Mann and Zweig

Quarks and antiquarks are constituents of hadrons and are spin-1/2 particles carrying fractional charges

Realisation of the model through Symmetry group: Flavour-SUf(3)

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

-group of special unitary matrices

$$U(\vec{\alpha}) \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} u' \\ d' \\ s' \end{pmatrix}, \quad U(\vec{\alpha}) = \exp\left(i\vec{\alpha}\cdot\vec{t}\right) \in SU_f(3)$$

$$U^\dagger U = 1, \quad detU = 1$$

$(t^1, t^2, \cdots t^8)$ - generators

$[t^a, t^b] = if^{abc}t^c, \quad f^{abc}$ - structure constants

- Mesons: bound states of a quark and an antiquark
- Baryons: bound states of three quarks

$$\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{8} + \mathbf{1}$$

$J^P = 0^-$: $(\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta)$

$J^P = 1^-$: $(\rho^\pm, \rho^0, K^{*\pm}, K^{*0}, \bar{K}^{*0}, \omega)$,

$$\mathbf{3} \times \mathbf{3} \times \mathbf{3} = \mathbf{10} + \mathbf{8} + \mathbf{8} + \mathbf{1}$$

Baryon octet, $J^P = 1/2^+$: $(p, n, \Sigma^\pm, \Sigma^0, \Xi^-, \Xi^0, \Lambda)$.

Baryon decouplet, $J^P = 3/2^+$: $(\Delta^{++}, \Delta^+, \Delta^0, \Delta^-, \Sigma^\pm, \Sigma^0, \Xi^-, \Xi^0, \Omega^-)$

Color Quantum Number

Problem with Fermi-Dirac Statisics

Δ^{++} $u\uparrow u\uparrow u\uparrow$

Quark Model predicts fully symmetric wave function

Spacial part is symmetric

Flavor and Spin part is symmetric (no anti-symmetric combination)

Need for New Quantum number: Color Quantum number

Han and Nambu, Greenberg and Gell-Mann

$q_a \to q_{a,i} \quad i = 1, 2, 3$ a = u,d and s each has (red,green,blue)

Use this color quantum number to construct anti-symmetric combination to resolve problem with Fermi-Dirac statistics

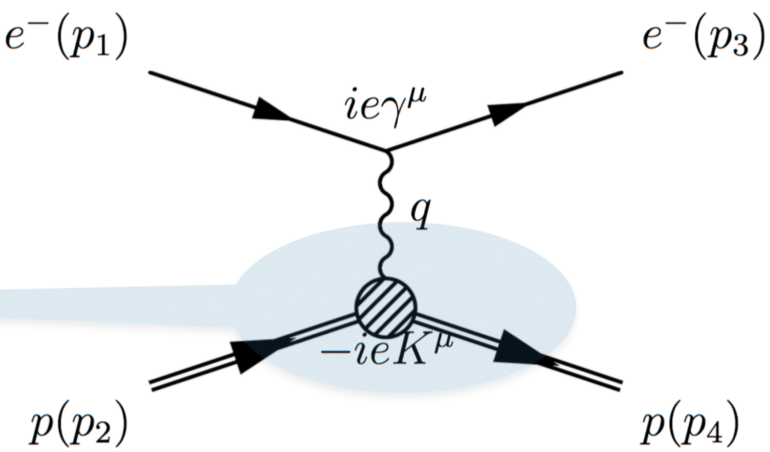
$$\epsilon_{ijk} u^i u^j u^k$$

Birth of QUANTUM CHROMODYNAMICS

Form Factors of hadrons

Elastic Scattering

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}\bigg|_{\text{point}} |F(q^2)|^2$$



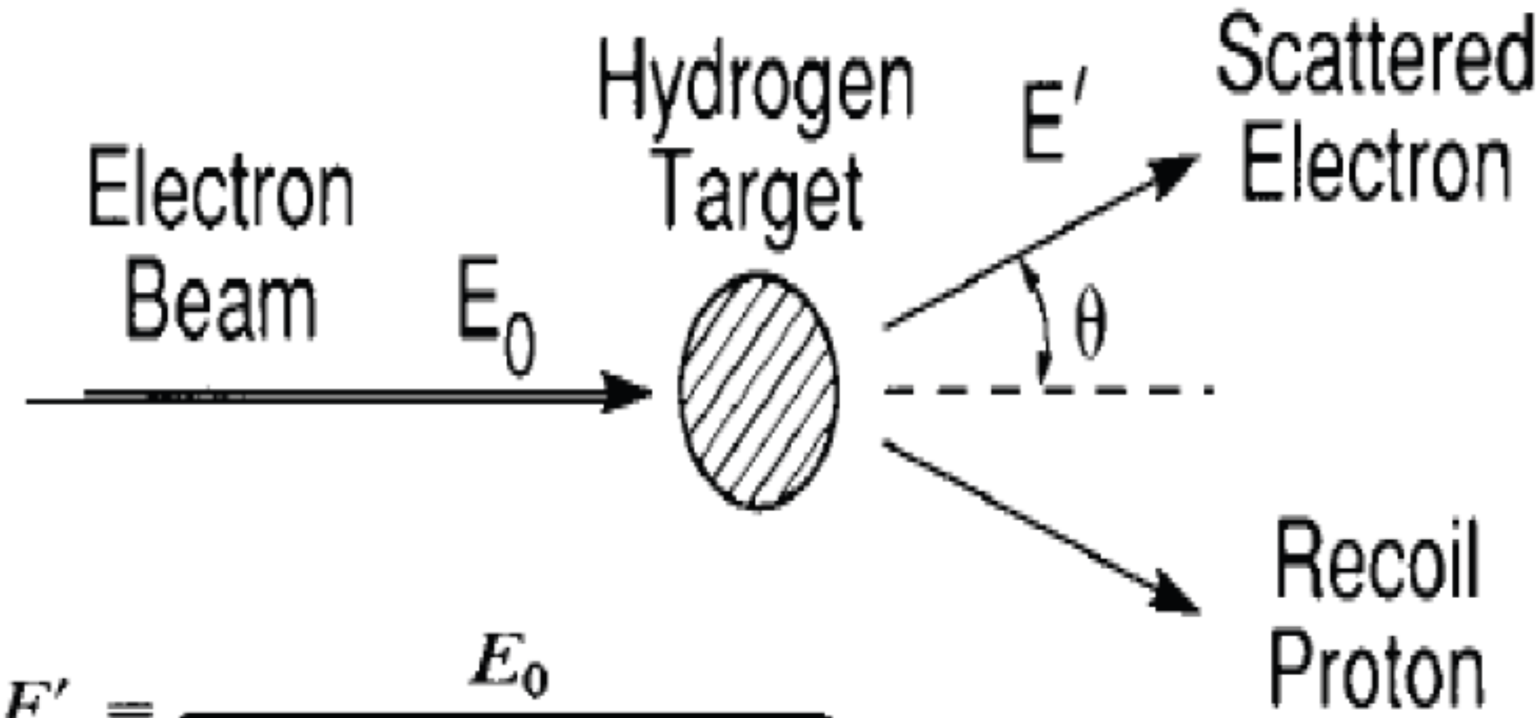
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \frac{\theta}{2}} \frac{E_3}{E_1} \left\{ \left(F_1^2 - \frac{\kappa_p^2 q^2}{4m_p^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2m_p^2} (F_1 + \kappa_p F_2)^2 \sin^2 \frac{\theta}{2} \right\}$$

Normalisation $\quad F_1^p(0) = 1, F_2^p(0) = 1$

$$\mu_p = \frac{(1+\kappa_p)e}{2m_p} \qquad \kappa_p = 1.79$$

κ_P is the anomalous magnetic moment

Electron-proton elastic scattering:



$$E' = \frac{E_0}{1 + \frac{2E_0}{M}\sin^2\theta/2}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \cdot \frac{E'}{E}\left[\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right]$$

The functions G_E and G_M take account of the size of the proton; they are called ***form factors***

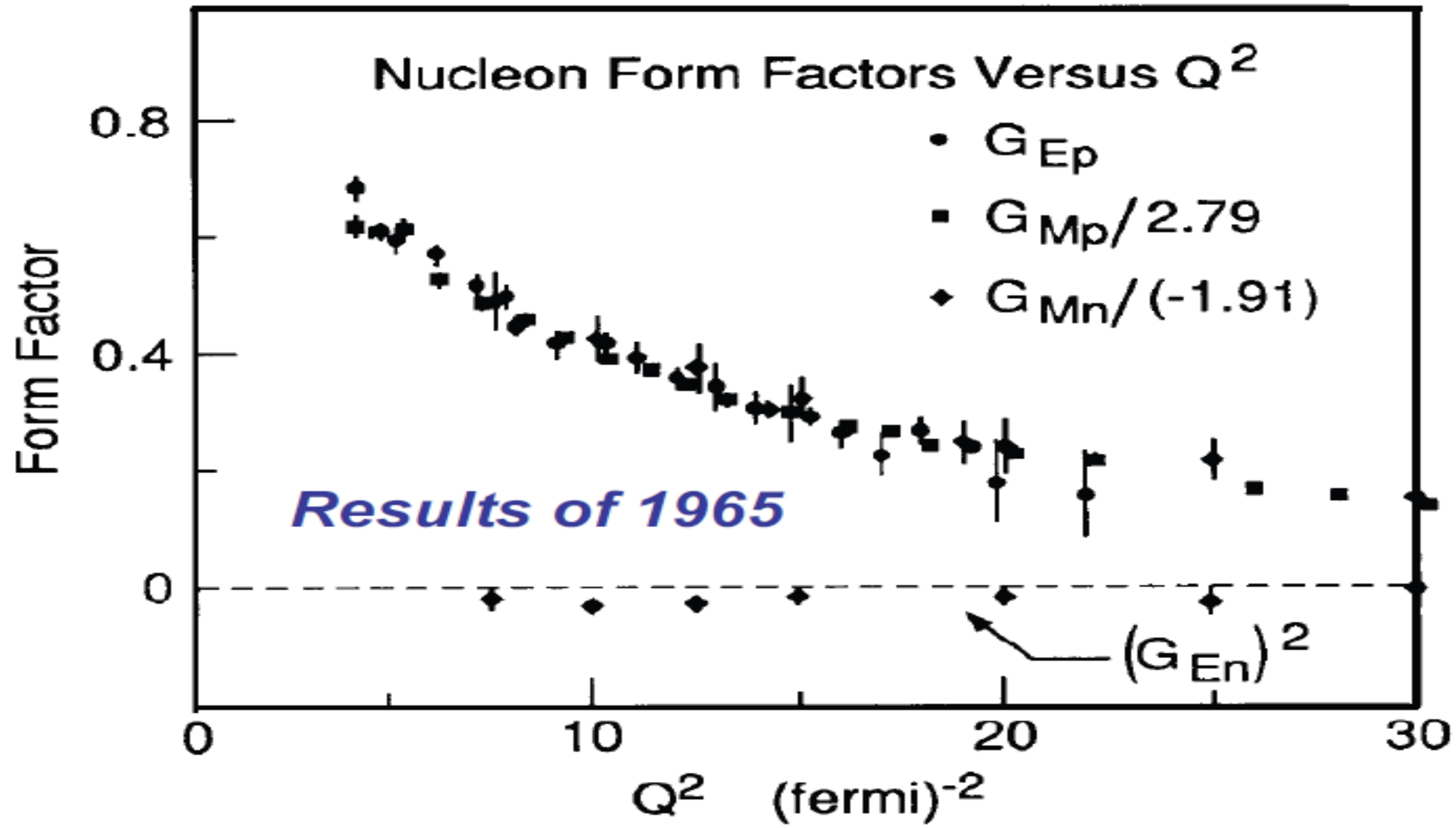
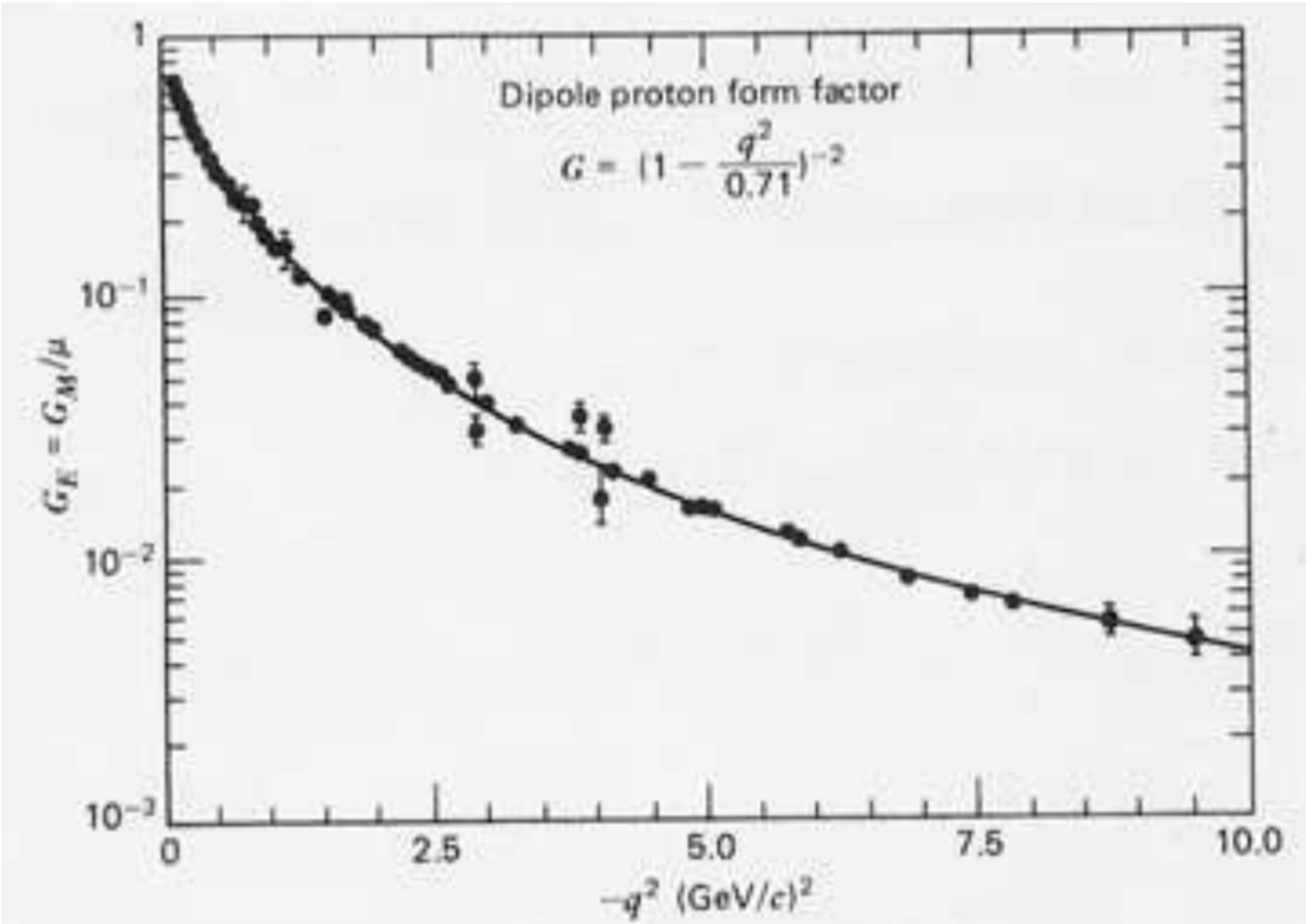


Fig. 23. Summary of results on nuclear form factors presented by the Stanford group at the 1965 "International Symposium on Electron and Photon Interactions at High Energies". (A momentum transfer of 1 GeV² is equivalent to 26 Fermis⁻².)

$$G_{Ep}(Q^2) \cong \left(\frac{1}{1 + \frac{Q^2}{0.71\ \mathrm{GeV}^2}}\right)^2 \text{ up to } Q^2 \backsim 10\ \mathrm{GeV}^2$$

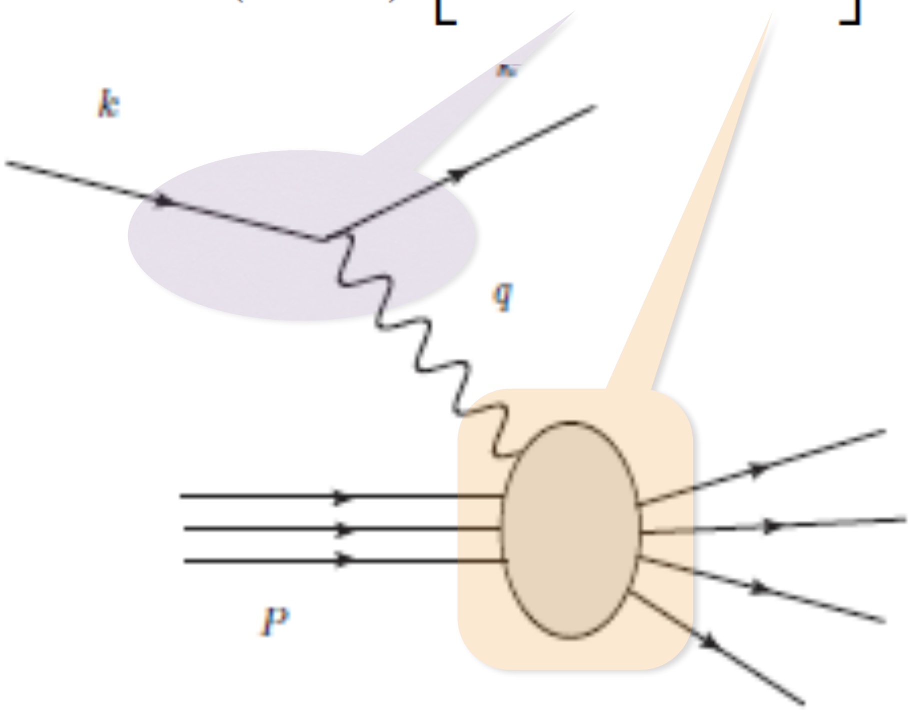


Dipole proton form factor
$G = (1 - \frac{q^2}{0.71})^{-2}$
1
10^{-1}
10^{-2}
10^{-3}
$G_E = G_M/\mu$
0
2.5
5.0
7.5
10.0
$-q^2$ (GeV/c)^2

Structure Functions from DIS

Inelastic Scattering Factorises

$$d\sigma = \frac{1}{4(k\cdot P)}\left[\frac{4\pi e^4}{q^4}L_{\mu\nu}W^{\mu\nu}\right]\frac{d^3k'}{2E'(2\pi)^3}$$



Leptonic Tensor

$$L_{\mu\nu} = 2\left[k_\mu k'_\nu + k'_\mu k_\nu - \frac{Q^2}{2}g_{\mu\nu}\right]$$

Hadronic Tensor

$$W^{\mu\nu} = \left(-g^{\nu\mu} + \frac{q^\nu q^\mu}{q^2}\right)W_1 + \left(P^\nu - \frac{P\cdot q}{q^2}q^\nu\right)\left(P^\mu - \frac{P\cdot q}{q^2}q^\mu\right)W_2$$

$W_i(\nu, Q^2)\quad i = 1,2$ Structure Function

$$x = \frac{Q^2}{2p.q};\quad y = \frac{p.q}{p.k};\quad Q^2 = xys$$

Not calculable

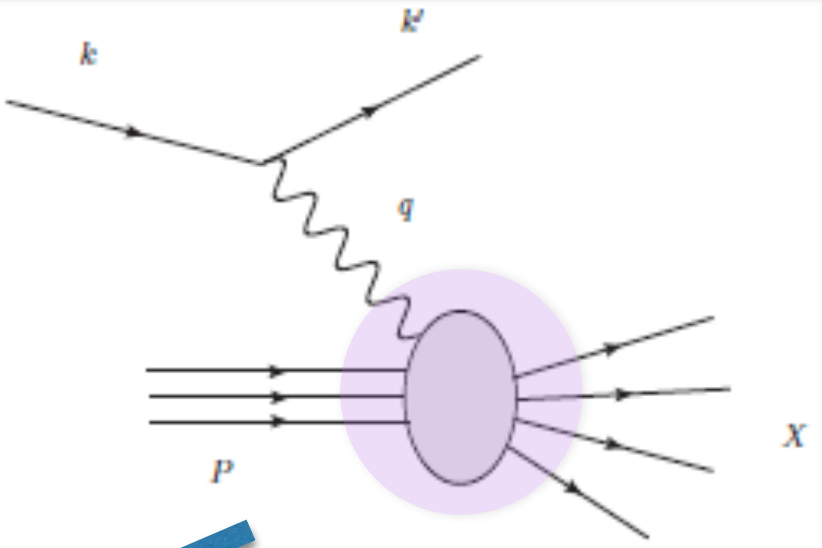
Inclusive Cross section

$$\frac{d\sigma}{dE_3 d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\frac{\theta}{2}}\left(W_2(\nu,Q^2)\cos^2\frac{\theta}{2} + 2W_1(\nu,Q^2)\sin^2\frac{\theta}{2}\right)$$

$$m_p W_1(\nu,Q^2)\to F_1(x)\qquad \nu W_2(\nu,Q^2)\to F_2(x)$$

Bjorken Scaling

Deep Inelastic Scattering



Hadronic Tensor

$$W^{\mu\nu}(P,q) = \int d^4\xi \; e^{iq\cdot\xi} < P|J^\mu(\xi)J^\nu(0)|P >$$
$$= \left(-g^{\nu\mu} + \frac{q^\nu q^\mu}{q^2}\right)W_1 + \left(P^\nu - \frac{P\cdot q}{q^2}q^\nu\right)\left(P^\mu - \frac{P\cdot q}{q^2}q^\mu\right)W_2$$

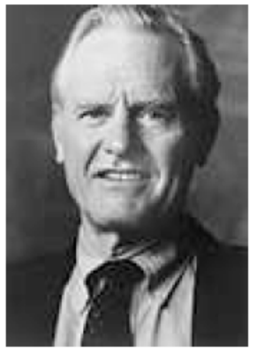
Bjorken Limit: $-q^2 \to \infty, P\cdot q \to \infty$ *with* $x = \frac{-q^2}{2P.q}$ *fixed*

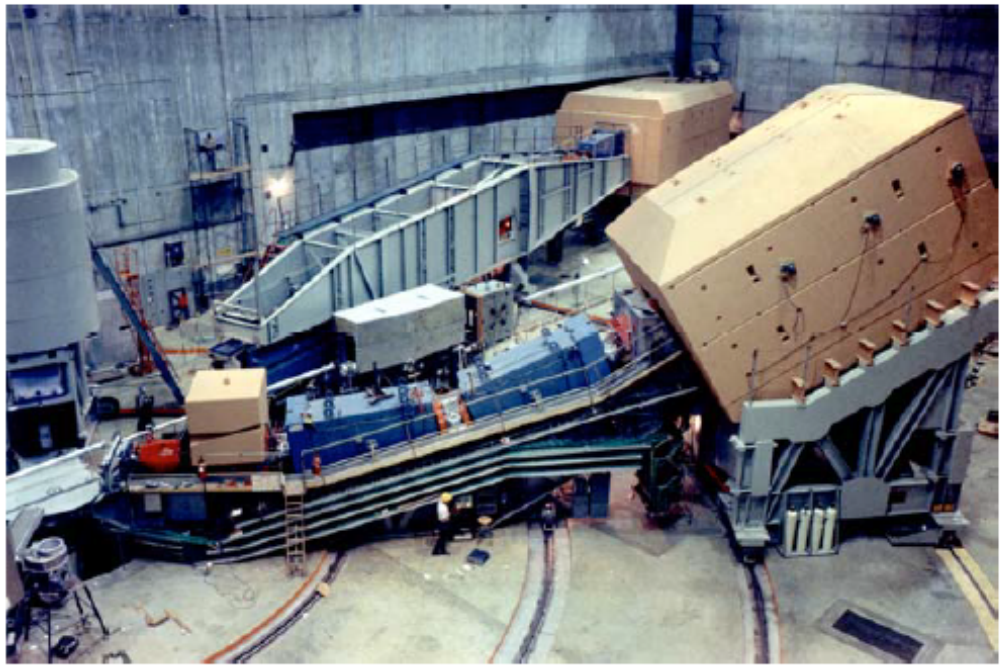
$$W_1(P,q) = F_1(x),$$
$$P.q\; W_2(P,q) = F_2(x)$$

Bjorken Scaling

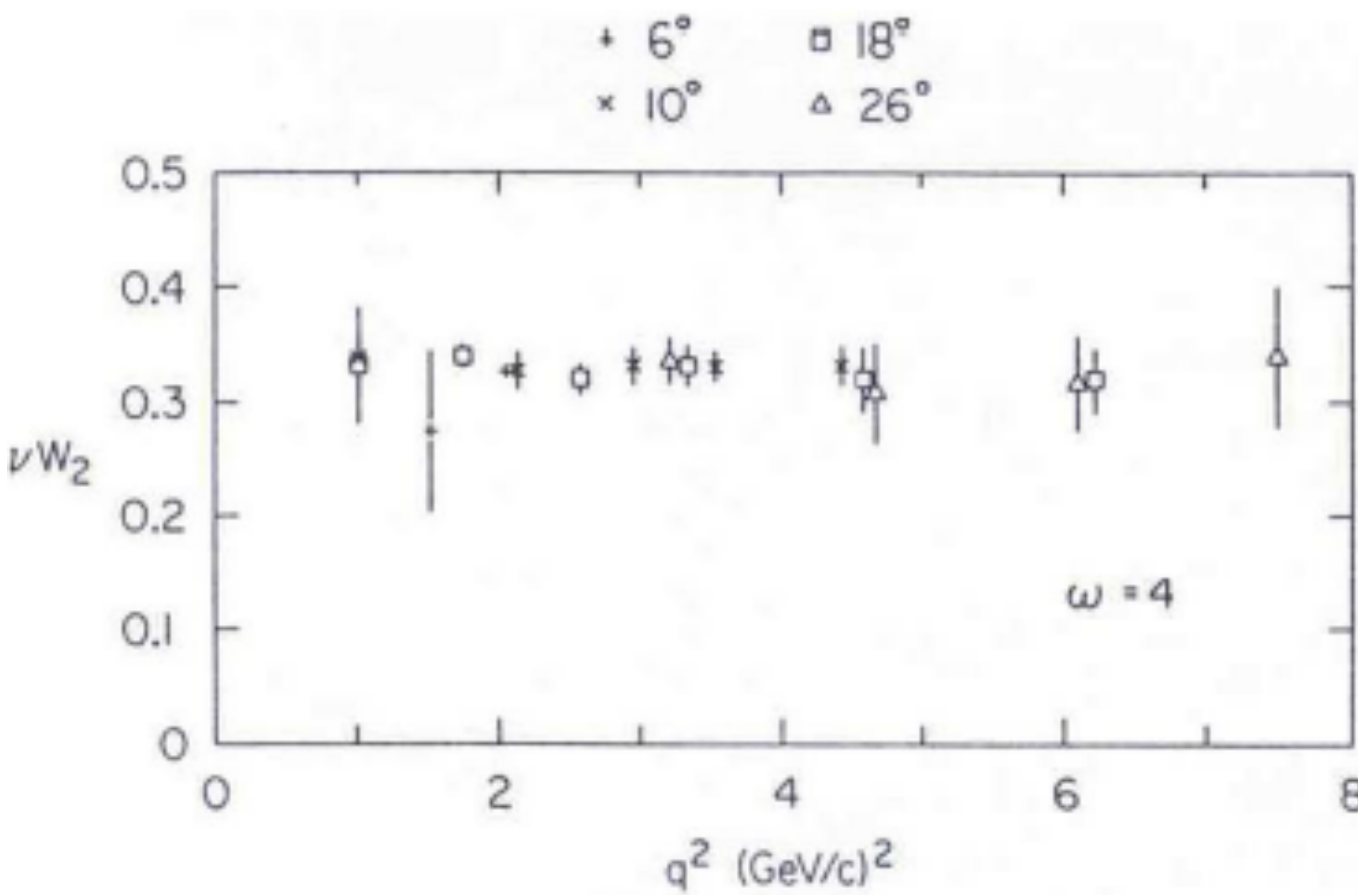
Birth of Parton Model

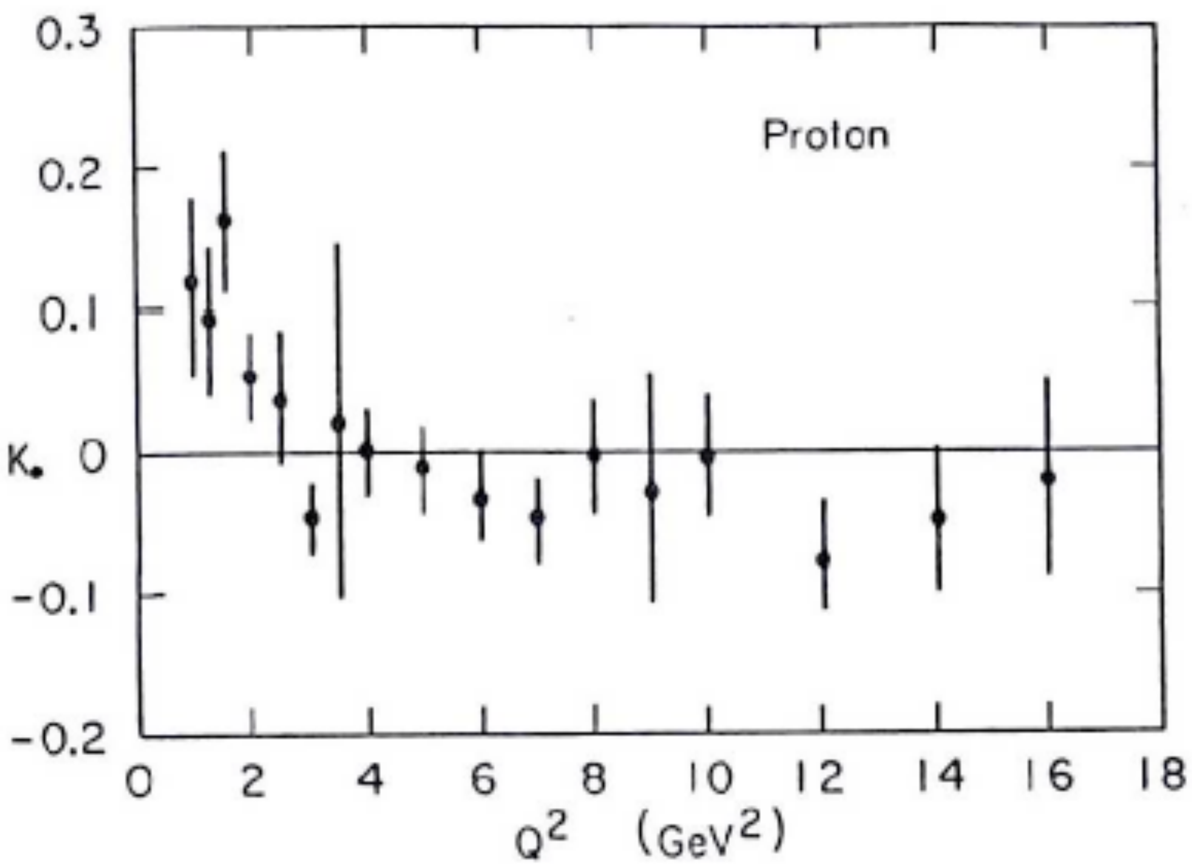
Deep Inelastic Scattering



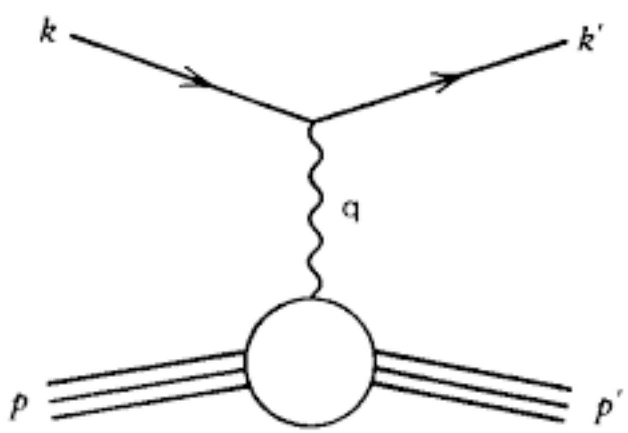
$Q^2 \approx 3M_N^2$ J. Friedman *1930 H. Kendall (1926-1999) R. Taylor *1929 (1968/69)

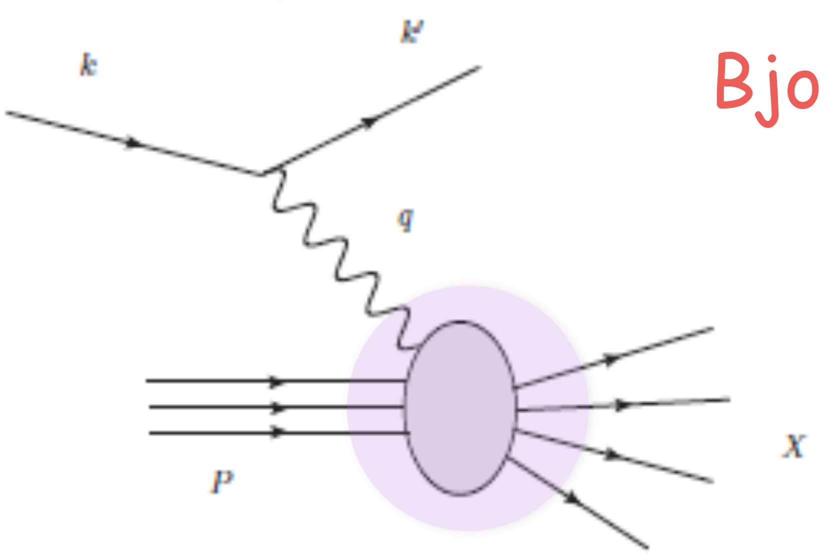




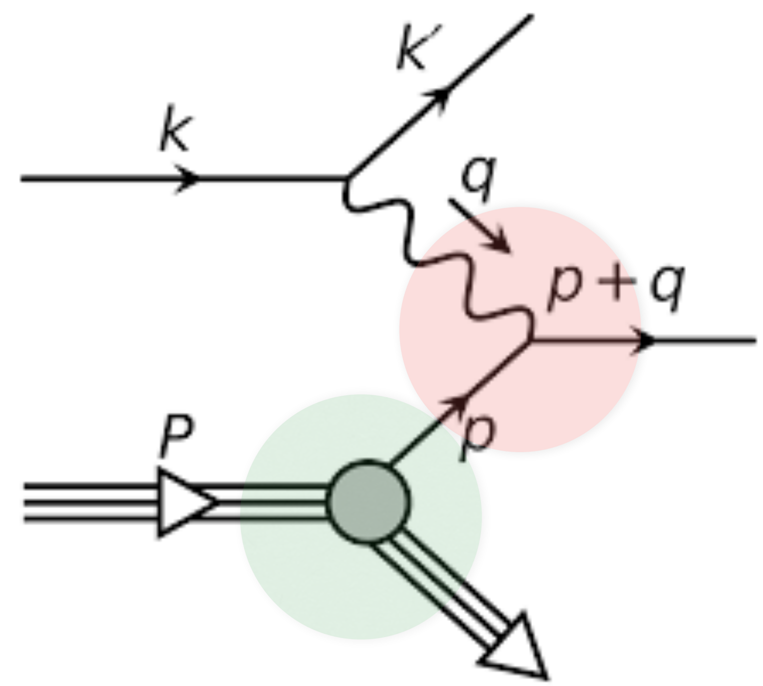
Parton Model

Elastic Scattering





Bjorken Scaling:



Deep Inelastic Scattering

Parton Model Picture

Parton Model

Note that:

- Hadrons are extended objects.

At high energies (infinite momentum frame):

- Hadron is Lorentz contracted in the direction of collision
- Interaction between constituents is time dilated

Model assumes:

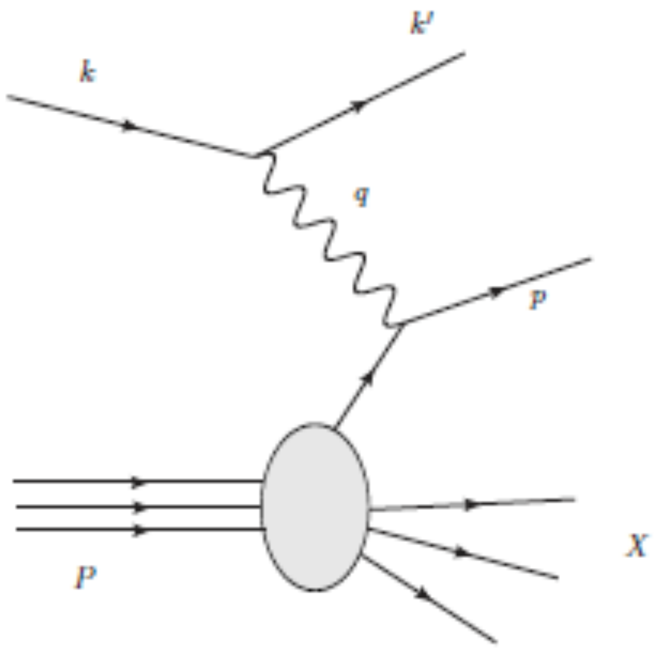
- At high energies they look like a collection of point like particles called partons held together by mutual interactions.
- Hadrons can be thought of one of the virtual states of these partons

Time dilation:

- Electron/photon interaction with partons takes place in a time scale shorter compared to time scales of virtual parton states.

Result: Inelastic scattering of electron proton can be thought of as a incoherent sum of elastic scattering of electron and a parton.

Naive Parton Model



$$d\sigma^{DIS}(P,q) = \sum_i \int_x^1 dz\, f_i(z)\, d\hat{\sigma}_i(zP,q)$$

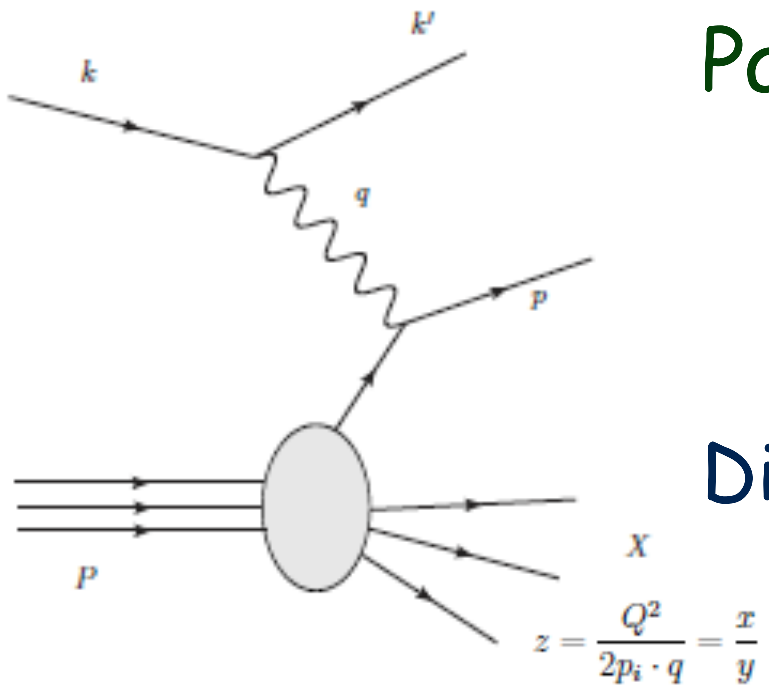
- Elastic scattering cross section with i-th parton
- Does not depend on the details of the target proton - Target Independent

$f_i(z)$ Parton Distribution Function (PDF)

- Probability of finding i-th parton with momentum fraction z of proton

- Does not depend on the future course of action of the i-th parton - Process Independent

Parton Model



$$x = \frac{Q^2}{2p.q}; \quad y = \frac{p.q}{p.k}; \quad Q^2 = xys$$

Parton Model - Master Formula

$$W_{\mu\nu}(P,q) = \sum_i \int_0^1 \frac{dy}{y} f_i(y) \hat{W}^{(i)}_{\mu\nu}(yP,q)\,,$$

Dimension-less

$$m_P W_1(\nu, Q^2) = F_1(x, Q^2),$$
$$\nu W_2(\nu, Q^2) = F_2(x, Q^2)$$

$$F_2(x,Q^2) = \sum_i \int_x^1 \frac{dy}{y} f_i(y) \hat{F}_2(x/y, Q^2)\,,$$

Parton level Cross sections

$$\hat{F}_1(x) = \frac{1}{2} e_q^2 \delta(x - \xi)\,,$$
$$\hat{F}_2(x) - 2x\hat{F}_1(x) = 0\,.$$

Bjorken scaling

$$2xF_1(x) = F_2(x) = \sum_i e_i^2 \; x \; f_i(x)$$

Momentum sum rule

$$2xF_1(x) = F_2(x) = \sum_i Q_i^2 \; x \; f_i(x)$$

Measurements for proton and neutron

$$\int_0^1 F_2^p(x)dx = \frac{4}{9}f_u + \frac{1}{9}f_d = 0.18$$

$$\int_0^1 F_2^n(x)dx = \frac{4}{9}f_d + \frac{1}{9}f_u = 0.12$$

SU(2) symmetry

where $f_q = \int_0^1 dx \; x \; f_q(x)$

Contribution to hadron momentum

$f_u = 0.36$ $\qquad$ $f_d = 0.18$

Only about 50% from quarks!

We now know that

Gluons also contribute significantly to momentum

what are these gluons?

Charged current DIS

Neutrino-Nucleon DIS can bring in parity violating SF

$$\frac{d\sigma^{CC}}{dxdy}(\nu N) = \frac{G_F^2 s}{2\pi}\left[(1-y)F_2^{\nu}(x) + y^2 x F_1^{\nu}(x) \pm y\left(1-\frac{y}{2}\right) x F_3^{\nu}\right]$$

Parton Model gives

$$F_2^{\nu p}(x) = 2x\left[d(x) + \bar{u}(x)\right]$$

$$F_2^{\nu n}(x) = 2x\left[u(x) + \bar{d}(x)\right]$$

Number of Valence quarks inside the Nucleon

$$\int_0^1 F_3^{\nu N}(x)dx = \int_0^1 \left[u(x) - \bar{u}(x) + d(x) - \bar{d}(x)\right] dx = 3$$

Quantum Chromodynamics

Han and Nambu, Greenberg and Gell-Mann

Solution to Δ^{++} $u\uparrow u\uparrow u\uparrow$ came from

Symmetry group: Color-SUc(3)

$$U(\vec{\beta}) = \exp\left(i\vec{\beta}\cdot\vec{T}\right) \in SU_c(3)$$

$$U(\vec{\beta})\begin{pmatrix} q_{i,1} \\ q_{i,2} \\ q_{i,3} \end{pmatrix} = \begin{pmatrix} q'_{i,1} \\ q'_{i,2} \\ q'_{i,3} \end{pmatrix}$$

$SU(N)$

$$[T^a, T^b] = i f^{abc} T^c \qquad i = 1, \cdots, N$$
$$a = 1, \cdots, N^2 - 1$$

Casimirs
$$(T^a T^a)_{ij} = C_F \delta_{ij}$$
$$f^{abc} f^{a'bc} = C_A \delta^{a'a}$$
$$C_F = \frac{N^2 - 1}{2N}, \quad C_A = N$$

Gauge the symmetry group SUc(3)

Gauge Theory of strong interaction

QUANTUM CHROMODYNAMICS

Matter Field:s ψ_i , $\overline{\psi}_i$ - Fundamental representation of SUc(3)

Gauge Fields A^a_μ - Adjoint representation

Gauge symmetry

QCD Lagrangian

$$\mathcal{L}_{QCD} = \overline{\psi}_j \left(i\not{D}_{jk} - m\delta_{jk} \right) \Psi_k - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \text{G.F}$$

where

$$D_\mu \psi(x) = \left(\partial_\mu - i g_s T^a A^a_\mu \right) \psi(x)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g_s [A_\mu, A_\nu]$$

$$A_\mu = T^a A^a_\mu$$

Invariant under

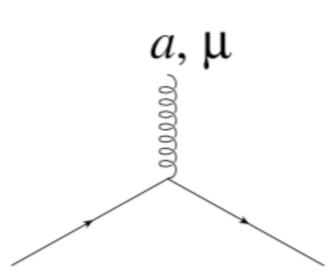
$$A_\mu \to U(\beta) \left(A_\mu - \frac{i}{g} \partial_\mu \right) U^\dagger(\beta)$$

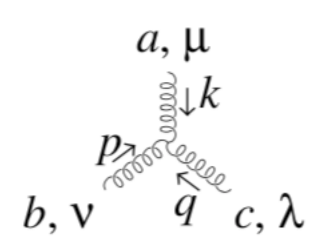
$$D_\mu \psi \to U(\beta) D_\mu \psi$$

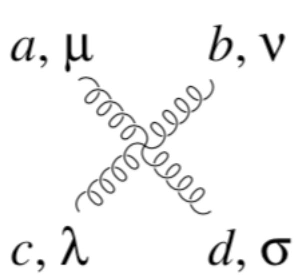
$$U(\vec{\beta}) \begin{pmatrix} q_{i,1} \\ q_{i,2} \\ q_{i,3} \end{pmatrix} = \begin{pmatrix} q'_{i,1} \\ q'_{i,2} \\ q'_{i,3} \end{pmatrix}$$

K.E + Interaction part

$$\mathcal{L}_{QCD} = \mathcal{L}_{K.E} + g_s A^a_\mu \overline{\psi} \gamma^\mu T^a \psi - g_s f^{abc} (\partial_\mu A^a_\nu) A^{b\mu} A^{c\nu} - g_s^2 f^{eab} f^{ecd} A^a_\mu A^b_\nu A^{\mu c} A^{\nu d}$$







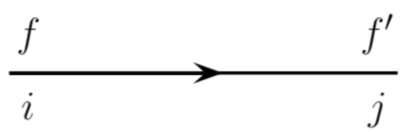
where

$$\mathcal{L}_{K.E} = -\frac{1}{2} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi} (\partial_\mu A^{a\mu})^2 + (\partial_\mu \bar{c}^a)(\partial^\mu c^a) + \sum_f \overline{\Psi}_{if} (i\not{\partial} - m_f) \Psi^{if}$$

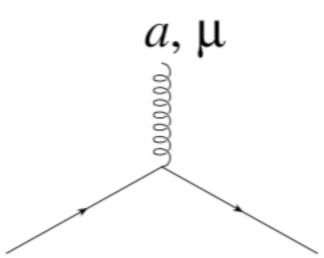
Feynman rules

QCD Feynman Rules

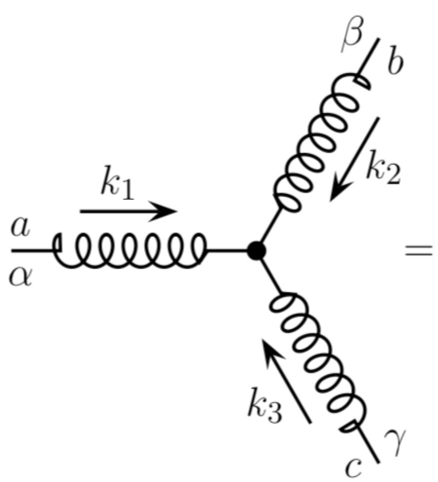
$$= \frac{-i\delta^{ab}}{k^2+i0}\left(g^{\mu\nu} + (\xi-1)\frac{k^\mu k^\nu}{k^2+i0}\right)$$



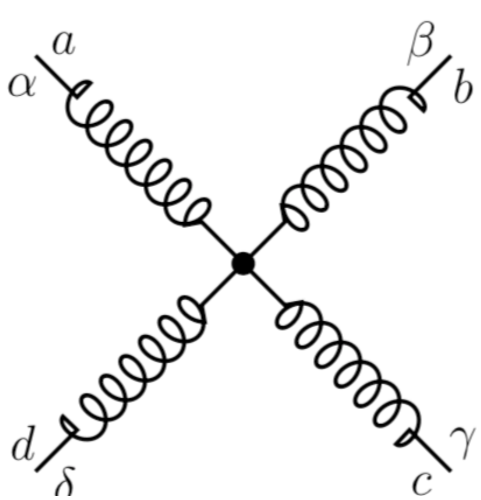
$$= \frac{i\delta^i_j\delta^f_{f'}}{\not{p} - m_f + i0} .$$



$$= \qquad ig\gamma^\mu T^a$$



$$= -gf^{abc}\left[g^{\alpha\beta}(k_1-k_2)^\gamma + g^{\beta\gamma}(k_2-k_3)^\alpha + g^{\gamma\alpha}(k_3-k_1)^\beta\right]$$



$$= -ig^2\left[\begin{array}{l} \quad f^{abe}f^{cde}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}) \\ + f^{ace}f^{bde}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\gamma\beta}) \\ + f^{ade}f^{bce}(g^{\alpha\beta}g^{\delta\gamma} - g^{\alpha\gamma}g^{\delta\beta}) \end{array}\right]$$

Ghost fields …

Gauge fixing:

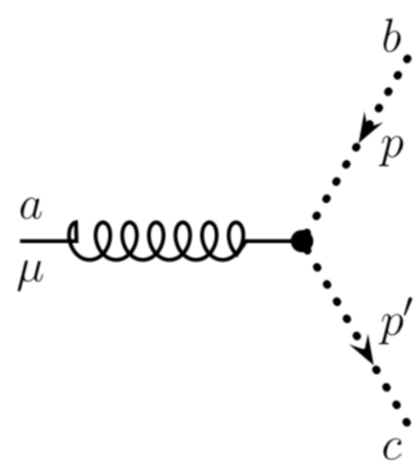
Gauge Fields contain two physical and two unphysical degrees of freedom

Ghost fields propagate as well as interact with gauge bosons but do not show up as physical particles

Ghost terms (Lorenz Gauge: $\partial_\mu A^\mu = 0$):

$$\mathcal{L}_{gh} = \partial_\mu \overline{c}^a \partial^\mu c^a - g_s f^{abc} (\partial_\mu \overline{c}^a) A^{\mu b} c^c$$

$$= \frac{i\delta^{ab}}{k^2 + i0} .$$



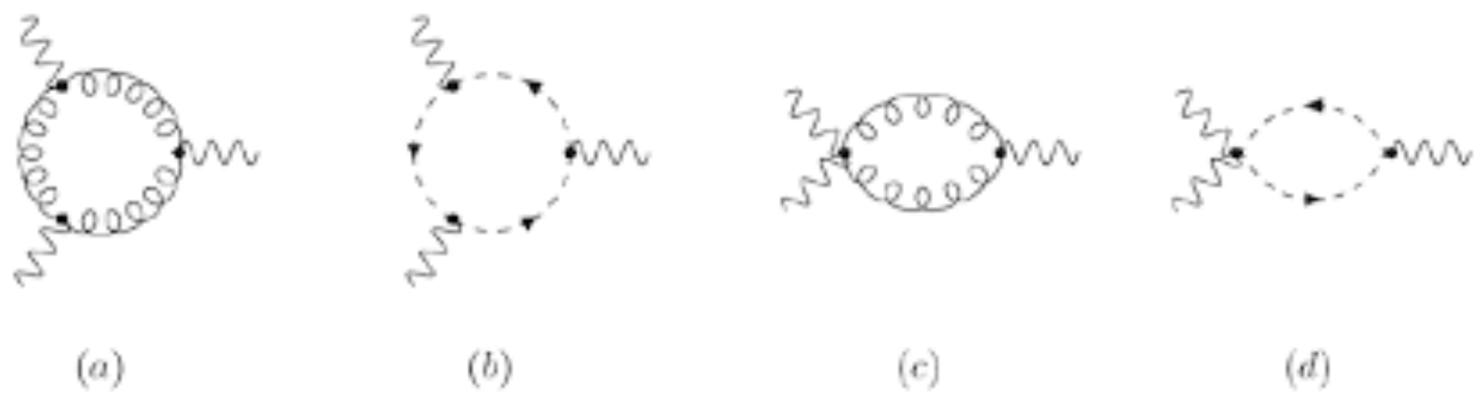
$$= +g f^{abc} p'^{\mu}$$

Quantum corrections

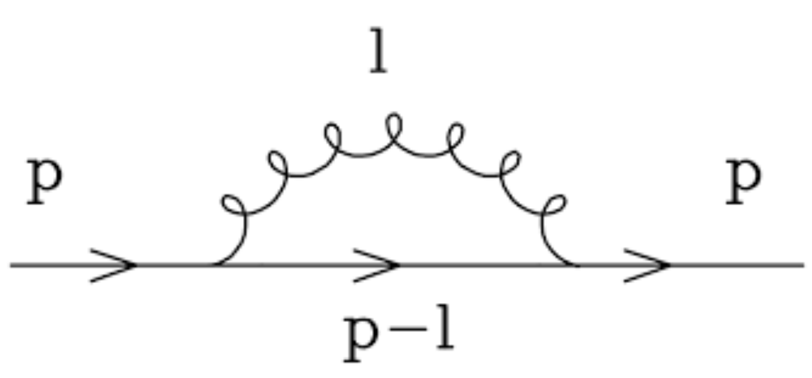
Perturbation theory to compute Green's functions

Vacuum polarization

Vertex Function



Selft energy



$$\int \frac{d^4 l}{(4\pi)^4} \frac{f(l,p)}{(l^2+i\epsilon)((p-l)^2+i\epsilon)}$$

diverge when $l^\mu \to \pm\infty$

Renormalisation

Perturbation theory to compute Green's functions

Coupling constant serves as an expansion parameter

Beyond leading order in coupling constant, they often diverge

Large momentum regions in Feynman Loop integrals

At short distances, local interactions give UV divergences

Practical resolution: Renormalisation

1. Regularise the theory so that we can proceed

2. Rescale the fields and parameters with infinite constants (Zi) in such a way that the Green's function computed in terms of recalled fields and parameters always give finite result

Counter Terms …

Regularisation: Dimension regularization : $d = 4 + \varepsilon$

d - space time dimension

$$\mathcal{L}_{QCD}\left(\psi, \overline{\psi}, A_\mu, c, \overline{c}, g_s, m\right) \rightarrow \mathcal{L}_{QCD}\left(\psi, \overline{\psi}, A_\mu, c, \overline{c}, g_s, m, \epsilon, \mu\right)$$

Rescaling of fields and parameters:

$$\psi = Z_\psi^{\frac{1}{2}}(\mu_R)\psi_R(\mu_R)$$

$$A_\mu = Z_A^{\frac{1}{2}}(\mu_R)A_{\mu R}(\mu_R)$$

• • • • •

$$g_s = Z_g(\mu_R)\left(\frac{\mu}{\mu_R}\right)^{\frac{\varepsilon}{2}} S_\varepsilon^{-\frac{1}{2}} g_{sR}(\mu_R)$$

μ Regularisation scale

μ_R Renormalisation scale

$$\mathcal{L}_{QCD}\left(\psi, \cdots, g_s, m, \epsilon, \mu\right) = \mathcal{L}_{QCD,R}\left(\psi_R, \cdots, g_{sR}, m_R, \epsilon, \mu_R\right)$$

$$+\mathcal{L}_{CT}\left(\{Z_i(\mu_R)\}, \psi_R, \cdots, g_{sR}, \epsilon, \mu_R\right)$$

Choose Zi in such a way all the Green's functions in terms of renormalized quantities give finite result to all orders in perturbation theory

Renormalisation Group:

Renoramliation group invariance:

$$\mathcal{L}_{QCD}\left(\psi,\cdots,g_s,m,\epsilon,\mu\right)=\mathcal{L}_{QCD,R}\left(\psi_R,\cdots,g_{sR},m_R,\epsilon,\mu_R\right)$$
$$+\mathcal{L}_{CT}\left(\{Z_i(\mu_R)\},\psi_R,\cdots,g_{sR},\epsilon,\mu_R\right)$$

Since LHS does not depends on μ_R

Green's functions computed from RHS will also not depend on μ_R, when it is computed to all orders in coupling constant

This implies $$\mu_R^2\frac{d}{d\mu_R^2} < \Omega|T\{\psi\cdots A_\mu\cdots\overline{\psi}\cdot\cdot\}|\Omega >= 0$$

This leads to renormalisation group equations for

$$\mu_R^2\frac{d}{d\mu_R^2}\log\left(< \Omega|T\{\psi_R\cdots A_{\mu,R}\cdots\overline{\psi}_R\cdot\cdot\}|\Omega >\right)=\Gamma(\mu_R)$$

in the limit $\varepsilon\to 0$

Running coupling

$$g_s = Z_g(\mu_R)\left(\frac{\mu}{\mu_R}\right)^{\frac{\varepsilon}{2}} S_\varepsilon^{-\frac{1}{2}} g_{sR}(\mu_R) \qquad \hat{a}_s = Z_a(\mu_R^2)\left(\frac{\mu^2}{\mu_R^2}\right)^{\frac{\varepsilon}{2}} S_\varepsilon^{-1} a_s(\mu_R^2)$$

where $\hat{a}_s = \frac{g_s^2}{16\pi^2}$, $a_s(\mu_R^2) = \frac{g_{sR}^2(\mu_R)}{16\pi^2}$ $Z_a = Z_g^2$

RG invariance implies

$$\mu_R^2 \frac{d}{d\mu_R^2}\hat{a}_s = 0$$

Compute log on both sides

$$\log(\hat{a}_s) = \log(Z_a(\mu_R^2)) - \frac{\varepsilon}{2}\log\left(\frac{\mu_R^2}{\mu^2}\right) + \log(S_\varepsilon^{-1}) + \log(a_s(\mu_R^2))$$

Differentiate w.r.t μ_R

$$0 = \mu_R^2 \frac{d}{d\mu_R^2}\log(Z_a(\mu_R^2)) - \frac{\varepsilon}{2} + \mu_R^2 \frac{d}{d\mu_R^2}\log(a_s(\mu_R^2))$$

Renormalisation group equation:

$$\mu_R^2 \frac{d}{d\mu_R^2} a_s(\mu_R^2) = -a_s(\mu_R^2)\mu_R^2 \frac{d}{d\mu_R^2}\log(Z_a(\mu_R^2)) - \frac{\varepsilon}{2} a_s(\mu_R^2)$$

$$= \beta(a_s, \varepsilon)$$

Finite and Computable!

Running coupling

$$\mu_R^2 \frac{d}{d\mu_R^2} a_s(\mu_R^2) = \beta\left(a_s(\mu_R^2)\right)$$

$$= -\beta_0\, a_s^2(\mu_R^2) - \beta_1\, a_s^3(\mu_R^2) - \beta_2\, a_s^4(\mu_R^2) - \cdots$$

Solution at three loops

$$a_s(\mu_0^2) = a_s(\mu_R^2)\frac{1}{\omega} + a_s^2(\mu_R^2)\left[\frac{1}{\omega^2}\left(-\eta_1 \ln\omega\right)\right]$$

$$+ a_s^3(\mu_R^2)\left[\frac{1}{\omega^2}\left(\eta_1^2 - \eta_2\right) + \frac{1}{\omega^3}\left(-\eta_1^2 + \eta_2 - \eta_1^2 \ln\omega + \eta_1^2 \ln^2\omega\right)\right]$$

$$\omega = 1 - \beta_0 a_s(\mu_R^2) \log\left(\frac{\mu_R^2}{\mu_0^2}\right) \qquad \eta_i = \frac{\beta_i}{\beta_0}$$

One Loop result

Renormalisation group equation:

$$\mu_R^2 \frac{d}{d\mu_R^2} a_s(\mu_R^2) = -a_s(\mu_R^2)\mu_R^2 \frac{d}{d\mu_R^2} \log(Z_a(\mu_R^2)) = \beta(a_s)$$

Beta function (exact)

$$\beta(a_s) = a_s \frac{\varepsilon}{2} \frac{1}{1 + a_s \frac{d}{da_s} \log Z_a}$$

Perturbative result to one-loop:

$$Z_a = 1 + a_s(\mu_R^2)\frac{1}{\varepsilon}\beta_0 + \mathcal{O}(a_s^2)$$

where,

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}n_f T_f$$

Beta function at one-loop:

$$\beta(a_s) = -a_s^2 \beta_0 + \mathcal{O}(a_s^3)$$

RG equation:

$$\mu_R^2 \frac{da_s(\mu_R^2)}{d\mu_R^2} = -a_s^2 \beta_0 + \mathcal{O}(a_s^3)$$

Beta Function of QCD

Four Loop results for beta function of QCD

$$\mu_R^2 \frac{d}{d\mu_R^2} a_s(\mu_R^2) = \beta\left(a_s(\mu_R^2)\right) = -\beta_0\, a_s^2(\mu_R^2) - \beta_1\, a_s^3(\mu_R^2) - \beta_2\, a_s^4(\mu_R^2) - \cdots$$

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f$$

$$\beta_1 = \frac{34}{3} C_A^2 - 4 C_F T_F n_f - \frac{20}{3} C_A T_F n_f$$

$$\beta_2 = \frac{2857}{54} C_A^3 + 2 C_F^2 T_F n_f - \frac{205}{9} C_F C_A T_F n_f - \frac{1415}{27} C_A^2 T_F n_f + \frac{44}{9} C_F T_F^2 n_f^2 + \frac{158}{27} C_A T_F^2 n_f^2$$

$$\begin{aligned}\beta_3 = {} & C_A^4 \left(\frac{150653}{486} - \frac{44}{9}\zeta_3\right) + C_A^3 T_F n_f \left(-\frac{39143}{81} + \frac{136}{3}\zeta_3\right) \\ & + C_A^2 C_F T_F n_f \left(\frac{7073}{243} - \frac{656}{9}\zeta_3\right) + C_A C_F^2 T_F n_f \left(-\frac{4204}{27} + \frac{352}{9}\zeta_3\right) \\ & + 46 C_F^3 T_F n_f + C_A^2 T_F^2 n_f^2 \left(\frac{7930}{81} + \frac{224}{9}\zeta_3\right) + C_F^2 T_F^2 n_f^2 \left(\frac{1352}{27} - \frac{704}{9}\zeta_3\right) \\ & + C_A C_F T_F^2 n_f^2 \left(\frac{17152}{243} + \frac{448}{9}\zeta_3\right) + \frac{424}{243} C_A T_F^3 n_f^3 + \frac{1232}{243} C_F T_F^3 n_f^3 \\ & + \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left(-\frac{80}{9} + \frac{704}{3}\zeta_3\right) + n_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} \left(\frac{512}{9} - \frac{1664}{3}\zeta_3\right) \\ & + n_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} \left(-\frac{704}{9} + \frac{512}{3}\zeta_3\right)\end{aligned}$$

Asymptotic freedom

RGE at one loop:

$$\mu_R^2 \frac{da_s}{d\mu_R^2} = -a_s^2 \beta_0$$

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} n_f T_f$$

$$C_A = 3, \quad T_f = \frac{1}{2}$$

Integrating from initial to final scales

$$\beta_0 > 0$$

$$\frac{1}{a_s(\mu_f^2)} = \frac{1}{a_s(\mu_i^2)} + \beta_0 \log(\mu_f^2/\mu_i^2)$$

Solution at one loop:

$$a_s(\mu_f^2) = \frac{a_s(\mu_i^2)}{1 + a_s(\mu_i^2)\beta_0 log(\mu_f^2/\mu_i^2)}$$

At large energies as $\mu_f \to \infty$

QCD Coupling constant runs

coupling vanishes $a_s(\mu_f^2) \to 0$

QCD IS ASYMPTOTICALLY FREE

At high energies QCD is weakly interacting

Status of $\alpha_s(\mu_R)$

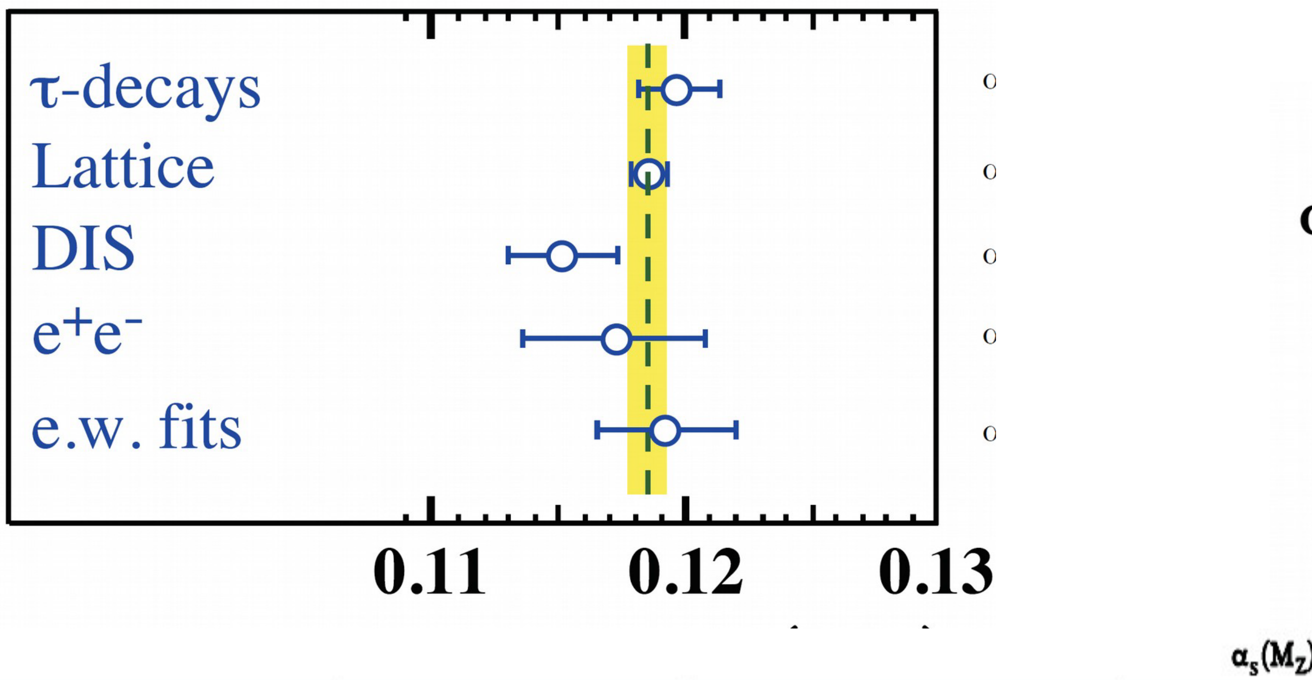
Renormalisation group equation for α_s:

Four Loops

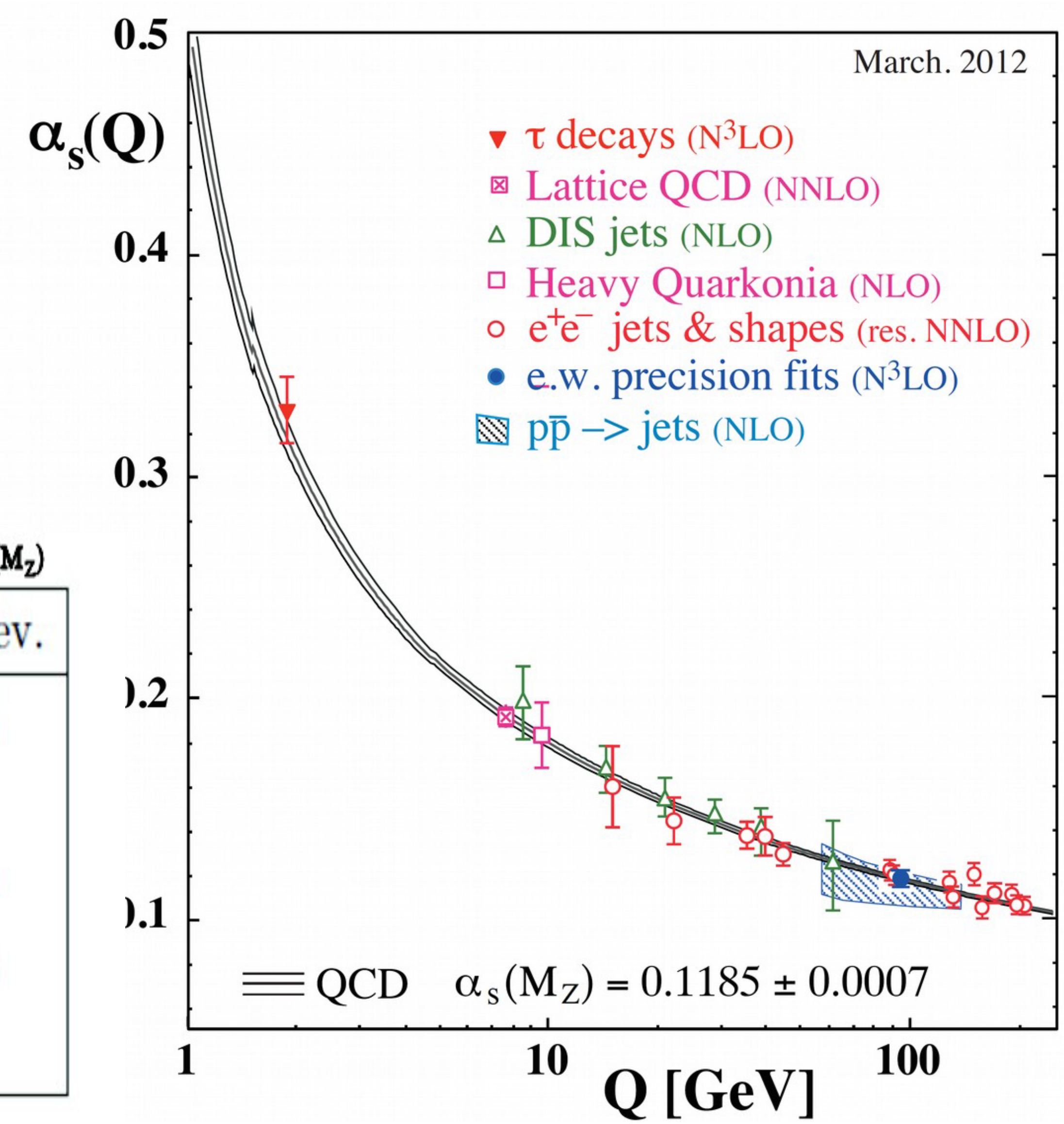
$$\mu_R^2 \frac{d}{d\mu_R^2} a_s(\mu_R^2) = \beta\left(a_s(\mu_R^2)\right)$$
$$= -\beta_0\, a_s^2(\mu_R^2) - \beta_1\, a_s^3(\mu_R^2) - \beta_2\, a_s^4(\mu_R^2) - \cdots$$

$\Lambda_{\overline{MS}}^{(5)} = (214 \pm 9)$ MeV
$\Lambda_{\overline{MS}}^{(4)} = (297 \pm 11)$ MeV

$\alpha_s(M_Z) = 0.1185 \pm 0.0007$



Process	$\alpha_s(M_{Z^0})$	excl. mean $\alpha_s(M_{Z^0})$	std. dev.
τ-decays	0.1197 ± 0.0016	0.1183 ± 0.0007	0.8
Lattice QCD	0.1186 ± 0.0007	0.1182 ± 0.0011	0.3
DIS $[F_2]$	0.1151 ± 0.0022	0.1188 ± 0.0010	1.5
e^+e^- [jets & shps]	0.1172 ± 0.0037	0.1185 ± 0.0006	0.3
ew. prec. data]	0.1192 ± 0.0028	0.1185 ± 0.0006	0.2



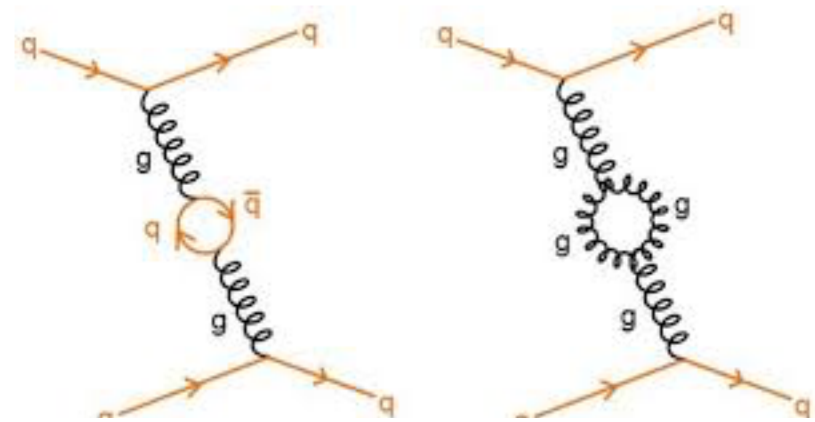
Quantum Chromodynamics

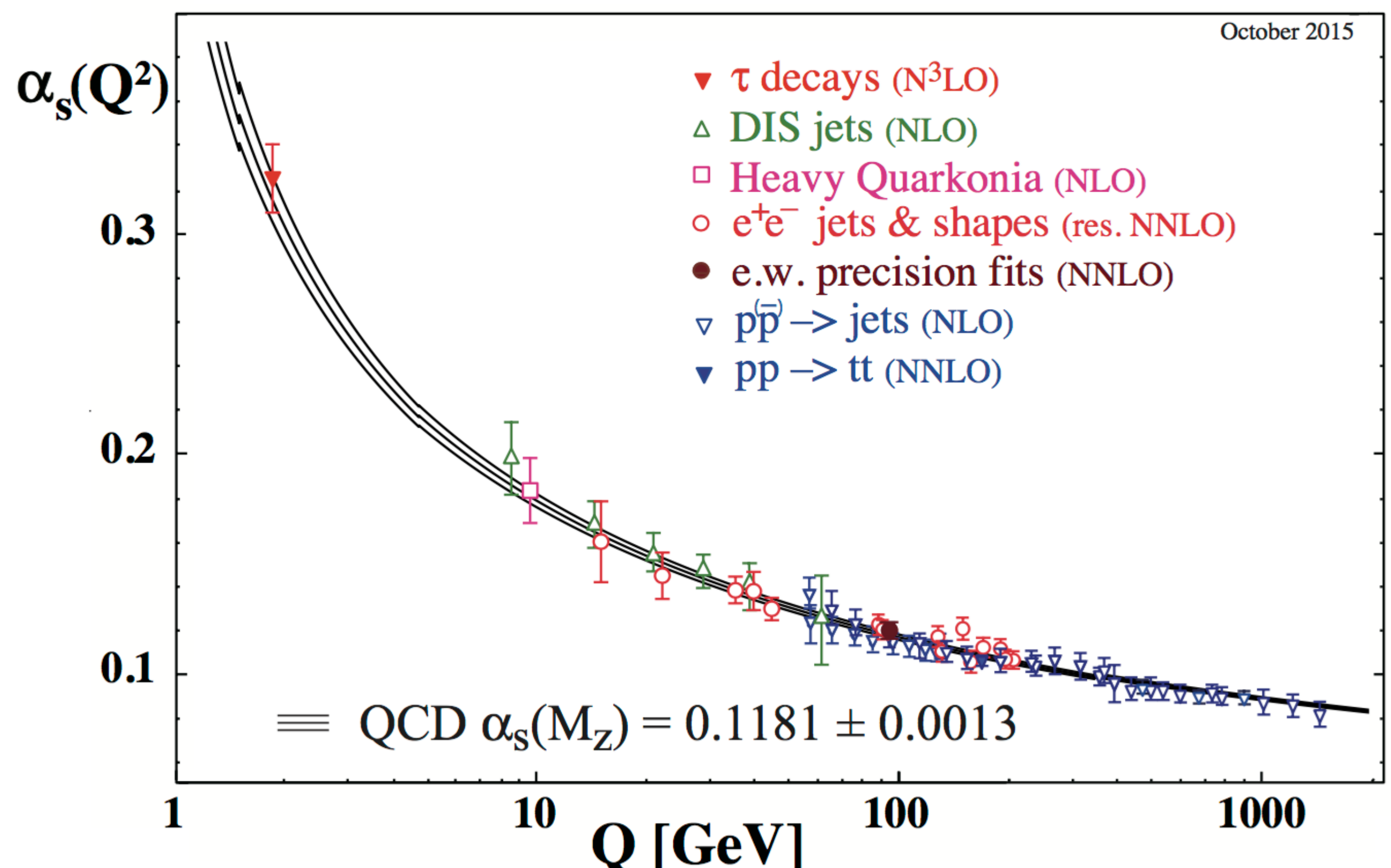
The Nobel Prize in Physics 2004

"for the discovery of asymptotic freedom in the theory of the strong interaction"

David J. Gross | H. David Politzer | Frank Wilczek

Non-Abelian Gauge theory - SU(3)





Asymptotic Freedom

$$Q^2 \to \infty \qquad \alpha_s(Q^2) \to 0$$

Accommodates Bjorken Scaling

Parton Model in QCD

Hadronic Cross section:

$$\sigma^A(\tau, m_A^2) = \sigma^{A,(0)}(\mu_R^2) \sum_{a,b=q,\bar{q},g} \int_\tau^1 dy\, \Phi_{ab}(y, \mu_F^2) \Delta_{ab}^A \left(\frac{\tau}{y}, m_A^2, \mu_R^2, \mu_F^2 \right)$$

Partonic Flux:

$$\Phi_{ab}(y, \mu_F^2) = \int_y^1 \frac{dx}{x} f_a(x, \mu_F^2) f_b \left(\frac{y}{x}, \mu_F^2 \right) ,$$

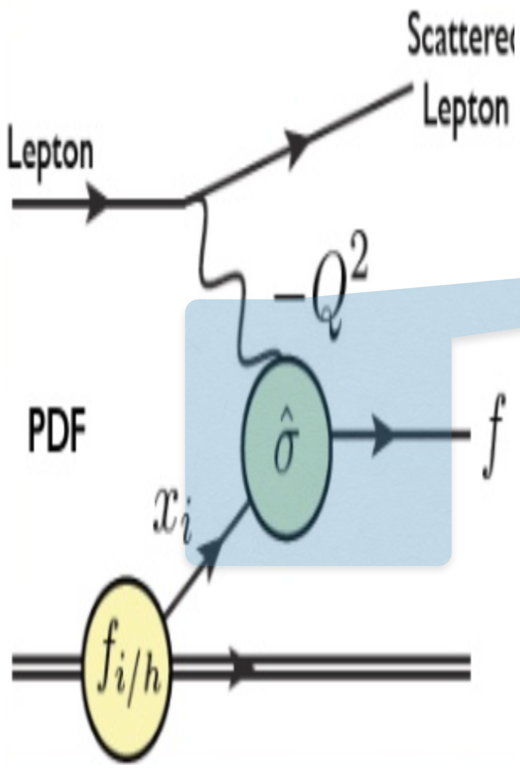
Partonic cross section:

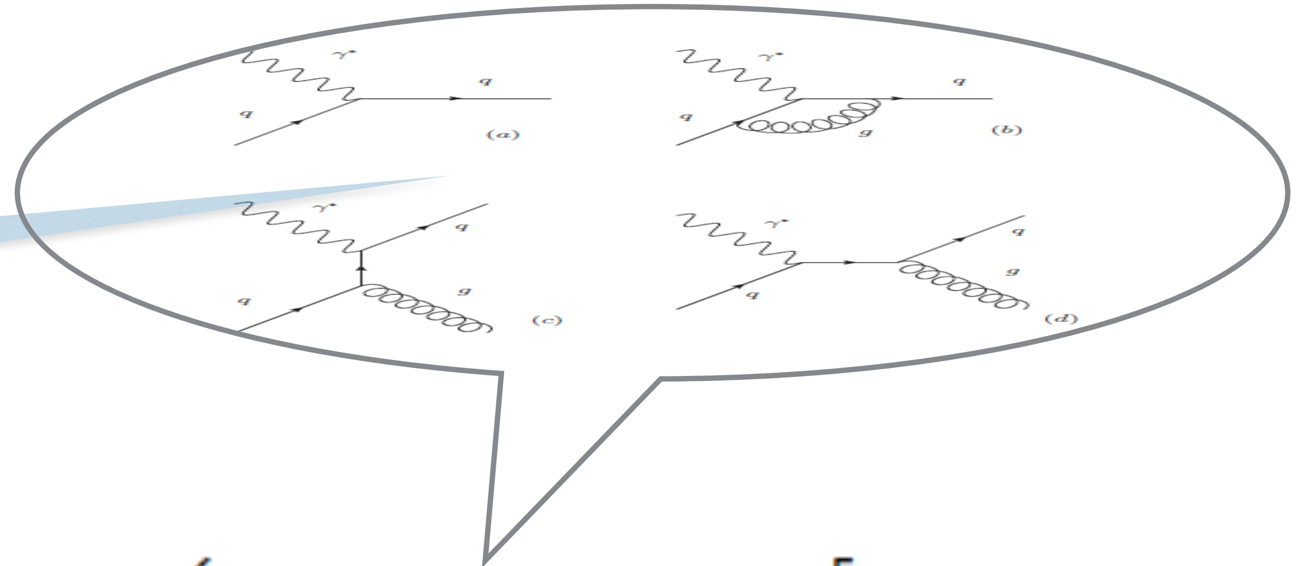
Precision Measurements

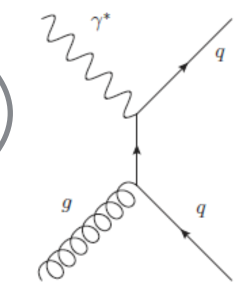
Precise Results

PDFs

QCD improved Parton Model







$$F_2(x,Q^2) = x\sum_{q,\bar{q}} e_q^2 \left\{ q_0(x) + \frac{\alpha_s}{2\pi}\int_0^1 \frac{dz}{z} q_0\left(\frac{x}{z}\right)\left[P(z)\log\left(\frac{Q^2}{m^2}\right) + C(z)\right] + \ldots \right\}$$

Collinear Renormalisation

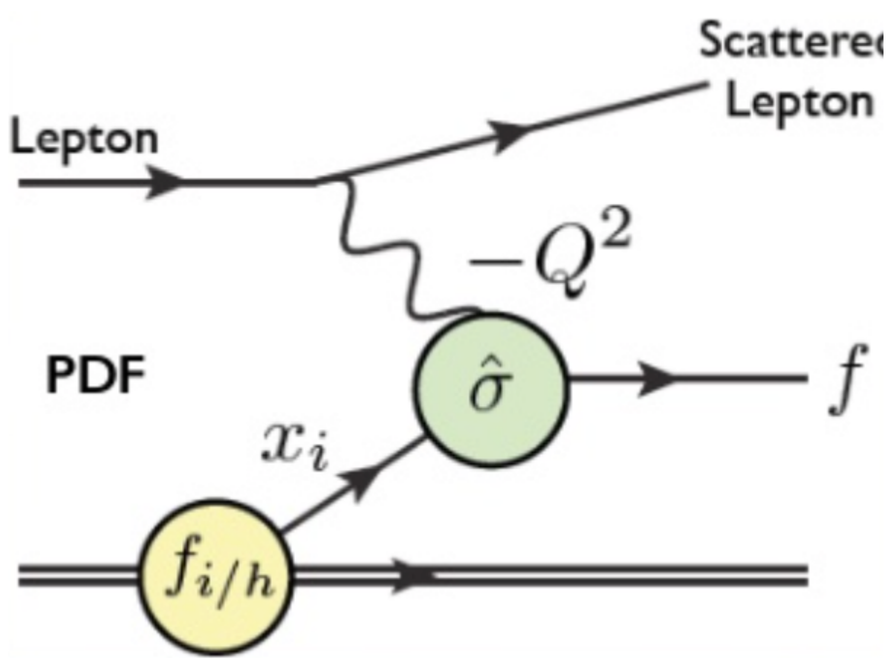
Factorisation Scale

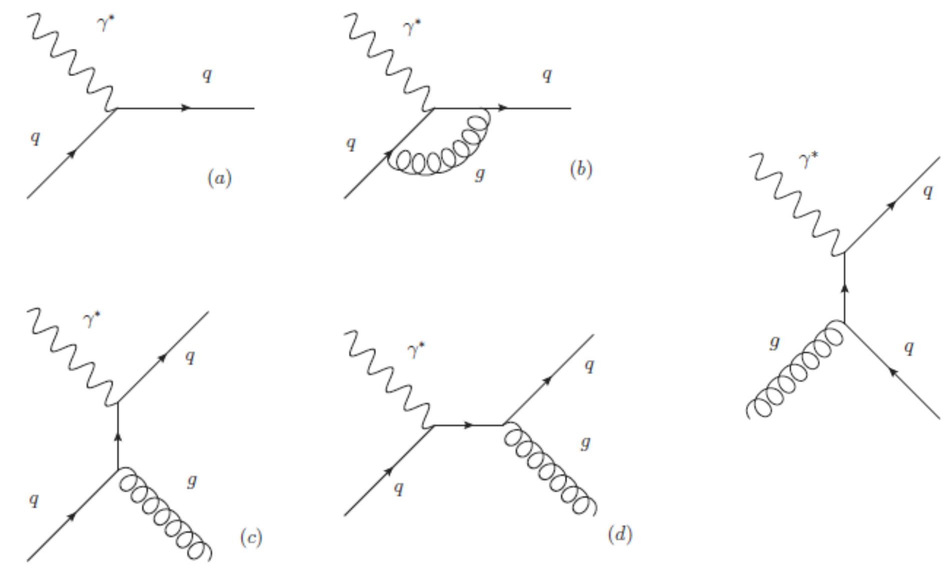
$$q(x,\mu) = q_0(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z} q_0\left(\frac{x}{z}\right)\left[P(z)\log\left(\frac{\mu^2}{m^2}\right) + C(z)\right] + \ldots$$

$$\log\frac{Q^2}{m^2} = \log\frac{Q^2}{\mu^2} + \log\frac{\mu^2}{m^2}.$$

$$\begin{aligned} F_2(x,Q^2) &= x\sum_{q\bar{q}} e_q^2 \int_x^1 \frac{dz}{z} q\left(\frac{x}{z},Q^2\right)\left[\delta(1-z) + \frac{\alpha_s}{2\pi} C_q^{\overline{MS}}(z) + \ldots\right] \\ &\quad + x\sum_{q\bar{q}} e_q^2 \int_x^1 \frac{dz}{z} g\left(\frac{x}{z},Q^2\right)\left[\frac{\alpha_s}{2\pi} C_g^{\overline{MS}}(z) + \ldots\right] \end{aligned}$$

Factorisation Theorem





μ_F - Factorisation Scale

μ_R - Renormalisation Scale

$$\sigma^P(x, Q^2) = \sum_{i=q,\bar{q},g} \int_x^1 \frac{dz}{z} C_i\left(z, Q^2, \mu_R^2, \mu_F^2\right) f_{i/P}\left(\frac{x}{z}, \mu_F^2\right)$$

Process Dependent Coefficient function
Perturbqtively Calculable to all orders

Only Parton and Target dependent
Non-Perturbative

DGLAP Evolution

Collinear Renormalisation

$$q(x,\mu) = q_0(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z} q_0\left(\frac{x}{z}\right)\left[P(z)\log\left(\frac{\mu^2}{m^2}\right) + C(z)\right] + \dots$$

Arbitrariness in the choice of $\mu = \mu_F$

$$\mu^2 \frac{d}{d\mu^2} q_0(z) = 0$$

Collinear Renormalisation Group Equation

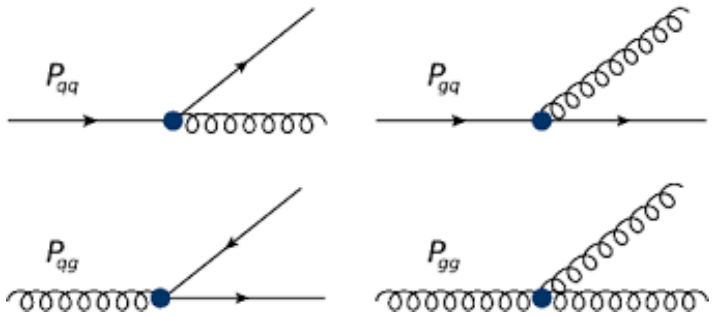
DGLAP Evolution Equation

$$\frac{\partial}{\partial \log \mu^2}\begin{pmatrix} q_i \\ g \end{pmatrix}(x,\mu^2) = \frac{\alpha_s}{2\pi}\int_x^1 \sum_{j=q,\bar{q}} \frac{d\xi}{\xi}\begin{pmatrix} P_{ij}\left(\frac{x}{\xi},\alpha_s\right) & P_{ig}\left(\frac{x}{\xi},\alpha_s\right) \\ P_{gj}\left(\frac{x}{\xi},\alpha_s\right) & P_{gg}\left(\frac{x}{\xi},\alpha_s\right)\end{pmatrix}\begin{pmatrix} q_j \\ g \end{pmatrix}(\xi,\mu^2).$$

In QCD perturbation

$$P_{ij}^{N^mLO}(x,\mu^2) = \sum_{k=0}^{m} a_s^{k+1}(\mu^2) P_{ij}^{(k)}(x).$$

Leading Order

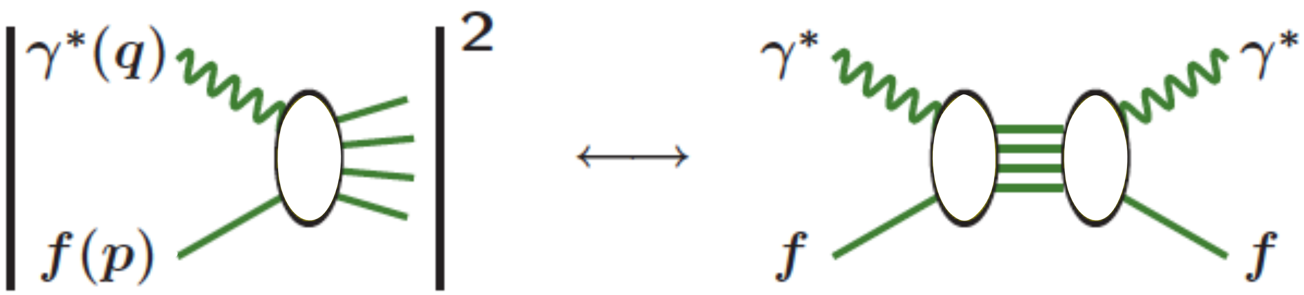


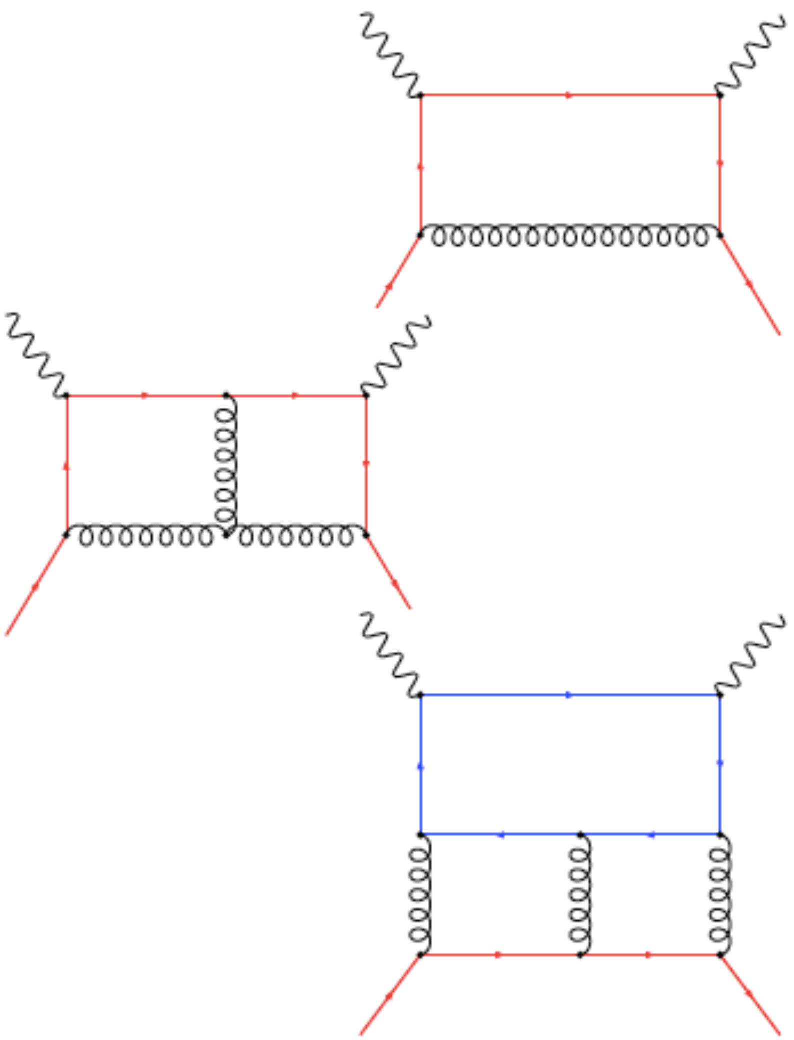
NNLO Results

[Moch,Vogt, Vermaseren]

Optical Theorem

UV + IR Poles in Dim. Regularisation





Poles in Dim. Regularisation

$$P_{ij}(x,\mu^2) = a_s(\mu^2)P^{(0)}_{ij}(x) + a_s^2(\mu^2)P^{(1)}_{ij}(x) + a_s^3(\mu^2)P^{(2)}_{ij}(x)$$

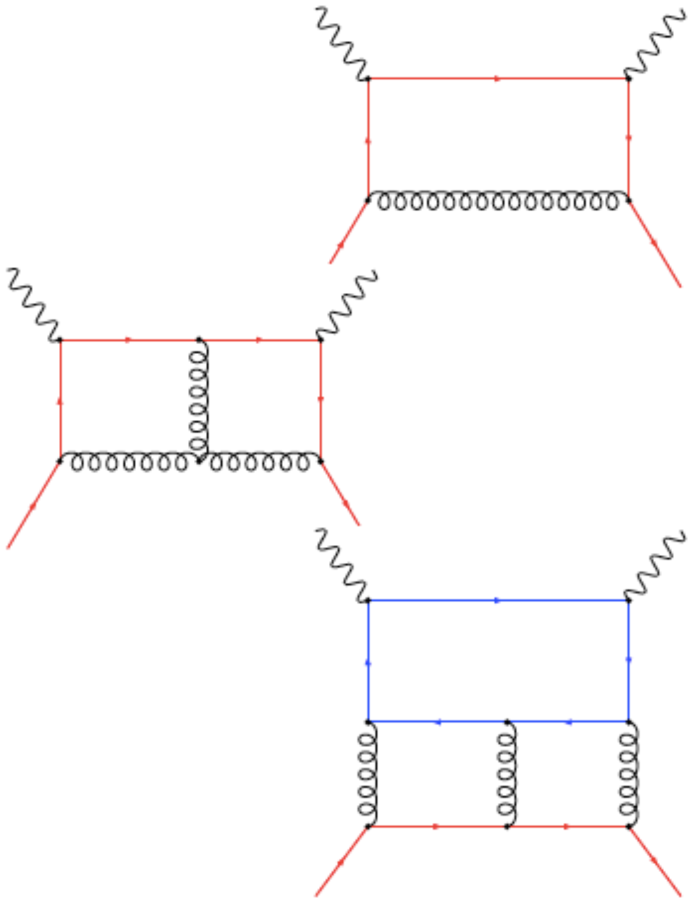
Finite part

$$C_j(x,\mu^2) = C_j^{(0)}(x,\mu^2) + a_s(\mu^2)C_j^{(1)}(x,\mu^2) + a_s^2(\mu^2)C_j^{(2)}(x,\mu^2)$$

NNLO splitting functions

[Moch,Vogt, Vermaseren]

	tree	1-loop	2-loop	3-loop
qγ	1	3	25	359
gγ		2	17	345
hγ			2	56
qW	1	3	32	589
qϕ		1	23	696
gϕ	1	8	218	6378
hϕ		1	33	1184
sum	3	18	350	9607



NNLO splitting functions

[Moch,Vogt, Vermaseren]

$$P_{gg}^{(2)}(x) =$$

$$16C_AC_Fn_f\Big(x^2\Big[\frac{4}{9}H_2+3H_{1,0}-\frac{97}{12}H_1+\frac{8}{3}H_{-2,0}-\frac{2}{3}H_0\zeta_2+\frac{103}{27}H_0-\frac{16}{3}\zeta_2+2H_3$$

$$-6H_{-1,0}+2H_{2,0}+\frac{127}{18}H_{0,0}-\frac{511}{12}\Big]+p_{gg}(x)\Big[2\zeta_3-\frac{55}{24}\Big]+\frac{4}{3}(\frac{1}{x}-x^2)\Big[\frac{17}{24}H_{1,0}-\frac{43}{18}H_0$$

$$-\frac{521}{144}H_1-\frac{6923}{432}-\frac{1}{2}H_{2,1}+2H_1\zeta_2+H_0\zeta_2-2H_{1,0,0}+\frac{1}{12}H_{1,1}-H_{1,1,0}-H_{1,1,1}\Big]-\frac{175}{12}H_2$$

$$+6H_{-1,0}+8H_0\zeta_3-6H_{-2,0}-\frac{53}{6}H_0\zeta_2-\frac{49}{2}H_0+\frac{185}{4}\zeta_2+\frac{511}{12}-\frac{1}{2}H_{2,0}-3H_{1,0}-4H_{0,0,0,0}$$

$$-\frac{67}{12}H_{0,0}+\frac{43}{2}\zeta_3-H_{2,1}+\frac{97}{12}H_1-4\zeta_2^{\,2}-\frac{9}{2}H_3-8H_{-3,0}+\frac{33}{2}H_{0,0,0}+\frac{4}{3}(\frac{1}{x}+x^2)\Big[\frac{1}{2}H_2-H_{2,0}$$

$$+\frac{11}{3}H_{-1,0}+H_{-2,0}+\frac{19}{6}\zeta_2+2\zeta_3-H_{-1}\zeta_2-4H_{-1,-1,0}-\frac{1}{2}H_{-1,0,0}-H_{-1,2}\Big]+(1-x)\Big[9H_1\zeta_2$$

$$+12H_{0,0,0,0}-\frac{293}{108}+\frac{61}{6}H_0\zeta_2-\frac{7}{3}H_{1,0}-\frac{857}{36}H_1-9H_0\zeta_3+16H_{-2,-1,0}-4H_{-2,0,0}+8H_{-2}\zeta_2$$

$$-\frac{13}{2}H_{1,0,0}+\frac{3}{4}H_{1,1}-H_{1,1,0}-H_{1,1,1}\Big]+(1+x)\Big[\frac{1}{6}H_{2,0}-\frac{95}{3}H_{-1,0}-\frac{149}{36}H_2+\frac{3451}{108}H_0$$

$$-7H_{-2,0}+\frac{302}{9}H_{0,0}+\frac{19}{6}H_3-\frac{991}{36}\zeta_2-\frac{163}{6}\zeta_3-\frac{35}{3}H_{0,0,0}+\frac{17}{6}H_{2,1}-\frac{43}{10}\zeta_2^{\,2}+13H_{-1}\zeta_2$$

$$+18H_{-1,-1,0}-H_{3,1}-6H_4-4H_{-1,2}+6H_{0,0}\zeta_2+8H_2\zeta_2-7H_{2,0,0}-2H_{2,1,0}-2H_{2,1,1}-4H_{3,0}$$

$$-9H_{-1,0,0}\Big]-\frac{241}{288}\delta(1-x)\Big)+16C_An_f^{\,2}\Big(\frac{19}{54}H_0-\frac{1}{24}xH_0-\frac{1}{27}p_{gg}(x)+\frac{13}{54}(\frac{1}{x}-x^2)\Big[\frac{5}{3}-H_1\Big]$$

$$+(1-x)\Big[\frac{11}{72}H_1-\frac{71}{216}\Big]+\frac{2}{9}(1+x)\Big[\zeta_2+\frac{13}{12}xH_0-\frac{1}{2}H_{0,0}-H_2\Big]+\frac{29}{288}\delta(1-x)\Big)$$

$$+16C_A^{\,2}n_f\Big(x^2\Big[\zeta_3+\frac{11}{9}\zeta_2+\frac{11}{9}H_{0,0}-\frac{2}{3}H_3+\frac{2}{3}H_0\zeta_2+\frac{1639}{108}H_0-2H_{-2,0}\Big]+\frac{1}{3}p_{gg}(x)\Big[\frac{10}{3}\zeta_2$$

$$-\frac{209}{36}-8\zeta_3-2H_{-2,0}-\frac{1}{2}H_0-\frac{10}{3}H_{0,0}-\frac{20}{3}H_{1,0}-H_{1,0,0}-\frac{20}{3}H_2-H_3\Big]+\frac{10}{9}p_{gg}(-x)\Big[\zeta_2$$

$$+2H_{-1,0}+\frac{3}{10}H_0\zeta_2-H_{0,0}\Big]+\frac{1}{3}(\frac{1}{x}-x^2)\Big[H_3-H_0\zeta_2-\frac{13}{3}H_2+\frac{5443}{108}-3H_1\zeta_2+\frac{205}{36}H_1$$

$$-\frac{13}{3}H_{1,0}+H_{1,0,0}\Big]+(\frac{1}{x}+x^2)\Big[\frac{151}{54}H_0-\frac{8}{3}\zeta_2+\frac{1}{3}H_{-1}\zeta_2-\zeta_3+2H_{-1,-1,0}-\frac{2}{3}H_{-1,0,0}$$

$$-\frac{37}{9}H_{-1,0}+\frac{2}{3}H_{-1,2}\Big]+(1-x)\Big[\frac{5}{6}H_{-2,0}+H_{-3,0}+2H_{0,0,0}-\frac{269}{36}\zeta_2-\frac{4097}{216}-3H_{-2}\zeta_2$$

$$-6H_{-2,-1,0}+3H_{-2,0,0}-\frac{7}{2}H_1\zeta_2+\frac{677}{72}H_1+H_{1,0}+\frac{7}{4}H_{1,0,0}\Big]+(1+x)\Big[\frac{193}{36}H_2-\frac{11}{2}H_{-1}\zeta_2$$

$$+\frac{39}{20}\zeta_2^{\,2}-\frac{7}{12}H_3-\frac{53}{9}H_{0,0}+\frac{7}{12}H_0\zeta_2-\frac{5}{2}H_{0,0}\zeta_2+5\zeta_3-7H_{-1,-1,0}+\frac{77}{6}H_{-1,0}+\frac{9}{2}H_{-1,0,0}$$

$$+2H_{-1,2}-3H_2\zeta_2-\frac{2}{3}H_{2,0}+\frac{3}{2}H_{2,0,0}+\frac{3}{2}H_4\Big]+\frac{1}{9}\zeta_2+7H_{-2,0}+2H_2+\frac{458}{27}H_0+H_{0,0}\zeta_2$$

$$+\frac{3}{2}\zeta_2^{\,2}+4H_{-3,0}-x\Big[\frac{131}{12}H_{0,0}-\frac{8}{3}H_0\zeta_2+\frac{7}{2}H_3-H_{0,0,0,0}+\frac{7}{6}H_{0,0,0}+\frac{1943}{216}H_0+6H_0\zeta_3\Big]$$

$$-\delta(1-x)\Big[\frac{233}{288}+\frac{1}{6}\zeta_2+\frac{1}{12}\zeta_2^{\,2}+\frac{5}{3}\zeta_3\Big]\Big)+16C_A^{\,3}\Big(x^2\Big[33H_{-2,0}+33H_0\zeta_2-\frac{1249}{18}H_{0,0}$$

$$-44H_{0,0,0}-\frac{110}{3}H_3-\frac{44}{3}H_{2,0}+\frac{85}{6}\zeta_2+\frac{6409}{108}H_0\Big]+p_{gg}(x)\Big[\frac{245}{24}-\frac{67}{9}\zeta_2-\frac{3}{10}\zeta_2^{\,2}+\frac{11}{3}\zeta_3$$

$$-4H_{-3,0}+6H_{-2}\zeta_2+4H_{-2,-1,0}+\frac{11}{3}H_{-2,0}-4H_{-2,0,0}-4H_{-2,2}+\frac{1}{6}H_0-7H_0\zeta_3+\frac{67}{9}H_{0,0}$$

$$-8H_{0,0}\zeta_2+4H_{0,0,0,0}-6H_1\zeta_3-4H_{1,-2,0}+10H_{2,0,0}-6H_{1,0}\zeta_2+8H_{1,0,0,0}+8H_{1,1,0,0}+8H_4$$

$$+\frac{134}{9}H_{1,0}+\frac{11}{6}H_{1,0,0}+8H_{1,2,0}+8H_{1,3}+\frac{134}{9}H_2-4H_2\zeta_2+8H_{3,1}+8H_{2,2}+\frac{11}{6}H_3+10H_{3,0}$$

$$+8H_{2,1,0}\Big]+p_{gg}(-x)\Big[\frac{11}{2}\zeta_2^{\,2}-\frac{11}{6}H_0\zeta_2-4H_{-3,0}+16H_{-2}\zeta_2-12H_{-2,2}-\frac{134}{9}H_{-1,0}+2H_2\zeta_2$$

$$+8H_{-2,-1,0}+12H_{-1}\zeta_3-18H_{-2,0,0}+8H_{-1,-2,0}-16H_{-1,-1}\zeta_2+24H_{-1,-1,0,0}+16H_{-1,-1,2}$$

$$+18H_{-1,0}\zeta_2-16H_{-1,0,0,0}-4H_{-1,2,0}-16H_{-1,3}-5H_0\zeta_3-8H_{0,0}\zeta_2+4H_{0,0,0,0}+2H_{3,0}$$

$$-\frac{67}{9}\zeta_2+\frac{67}{9}H_{0,0}+8H_4\Big]+\Big(\frac{1}{x}-x^2\Big)\Big[\frac{16619}{162}+\frac{22}{3}H_{2,0}-\frac{55}{2}\zeta_3-\frac{11}{2}H_0\zeta_2-\frac{67}{9}H_2-\frac{67}{9}H_{1,0}$$

$$-\frac{413}{108}H_1-\frac{11}{2}H_1\zeta_2+\frac{33}{2}H_{1,0,0}\Big]+11(\frac{1}{x}+x^2)\Big[\frac{71}{54}H_0-\frac{1}{6}H_3-\frac{389}{198}\zeta_2-\frac{2}{3}H_{-2,0}-\frac{1}{2}H_{-1}\zeta_2$$

$$+H_{-1,-1,0}-\frac{523}{198}H_{-1,0}+\frac{8}{3}H_{-1,0,0}+H_{-1,2}\Big]+(1-x)\Big[\frac{31}{36}H_1+\frac{27}{2}H_{1,0}-\frac{25}{2}H_{1,0,0}-4H_{-3,0}$$

$$-\frac{263}{12}H_{0,0}-\frac{29}{3}H_{0,0,0}-\frac{19}{3}H_{-2,0}-\frac{11317}{108}-4H_{-2}\zeta_2-8H_{-2,-1,0}-12H_{-2,0,0}-\frac{3}{2}H_1\zeta_2\Big]$$

$$+(1+x)\Big[\frac{27}{2}H_0\zeta_2-\frac{43}{6}H_3+\frac{29}{3}H_{2,0}+\frac{4651}{216}H_0-\frac{329}{18}\zeta_2+\frac{11}{2}(1+x)\zeta_3-\frac{43}{5}\zeta_2^{\,2}-\frac{215}{6}H_{-1,0}$$

$$-22H_{0,0}\zeta_2-8H_0\zeta_3-3H_{-1,-1,0}+38H_{-1,0,0}+25H_{-1,2}+10H_{2,0,0}-4H_2\zeta_2+16H_{3,0}+26H_4$$

$$-\frac{158}{9}H_2-\frac{53}{2}H_{-1}\zeta_2\Big]-29H_{0,0}-\frac{40}{3}H_{0,0,0}+27H_{-2,0}+\frac{41}{3}H_0\zeta_2-20H_3-24H_{2,0}+\frac{53}{6}\zeta_2$$

$$+\frac{601}{12}H_0+24\zeta_3+2\zeta_2^{\,2}+27H_2-4H_{0,0}\zeta_2-16H_0\zeta_3-16H_{-3,0}+28xH_{0,0,0,0}+\delta(1-x)\Big[\frac{79}{32}$$

$$-\zeta_2\zeta_3+\frac{1}{6}\zeta_2+\frac{11}{24}\zeta_2^{\,2}+\frac{67}{6}\zeta_3-5\zeta_5\Big]\Big)+16C_Fn_f^{\,2}\Big(\frac{2}{9}x^2\Big[\frac{11}{6}H_0+H_2-\zeta_2+2H_{0,0}-9\Big]+\frac{1}{3}H_2$$

$$-\frac{1}{3}\zeta_2-\frac{10}{3}H_0-\frac{1}{3}H_{0,0}+2+\frac{2}{9}(\frac{1}{x}-x^2)\Big[\frac{8}{3}H_1-2H_{1,0}-H_{1,1}-\frac{77}{18}\Big]-(1-x)\Big[\frac{1}{3}H_{1,0}+\frac{1}{6}H_{1,1}$$

$$+\frac{4}{9}+\frac{13}{6}H_1+xH_1\Big]+\frac{1}{3}(1+x)\Big[\frac{68}{9}H_0-\frac{4}{3}H_2+\frac{4}{3}\zeta_2+\frac{29}{6}H_{0,0}-\zeta_3+2H_0\zeta_2-H_{0,0,0}-2H_3$$

$$-H_{2,1}-2H_{2,0}\Big]+\frac{11}{144}\delta(1-x)\Big)+16C_F^{\,2}n_f\Big(\frac{4}{3}x^2\Big[\frac{163}{16}+\frac{27}{8}H_0+\frac{7}{2}H_{0,0}-H_{2,0}-\zeta_2+\frac{9}{4}H_{1,0}$$

$$-H_{2,1}+\frac{1}{2}H_{0,0,0}+\frac{85}{16}H_1+H_2-2H_{-2,0}-\frac{3}{2}\zeta_3\Big]+\frac{4}{3}(\frac{1}{x}-x^2)\Big[\frac{31}{16}H_1-\frac{11}{16}-\frac{5}{4}H_{1,0}+\frac{1}{2}H_{1,0,0}$$

$$-H_1\zeta_2-H_{1,1}+H_{1,1,0}+H_{1,1,1}+\zeta_3\Big]+\frac{4}{3}(\frac{1}{x}+x^2)\Big[H_{-1}\zeta_2+2H_{-1,-1,0}-H_{-1,0,0}\Big]+\frac{215}{12}H_{0,0}$$

$$+\frac{20}{3}H_0-\frac{131}{6}+3H_{2,0}+\frac{205}{12}x\zeta_2-3H_{1,0}+H_{2,1}-\frac{85}{12}H_1+\frac{11}{4}H_2+8H_{-2,0}+2\zeta_2^{\,2}-H_0\zeta_2$$

$$+H_3+6H_0\zeta_3+8H_{-3,0}-4xH_{0,0,0}+(1-x)\Big[\frac{107}{12}H_1-\frac{5}{6}H_{1,0}-4\zeta_2+H_0\zeta_3-8H_{-2,-1,0}$$

$$-4H_{-2}\zeta_2+4H_{-2,0,0}-4H_1\zeta_2+\frac{7}{2}H_{1,0,0}-\frac{7}{12}H_{1,1}+H_{1,1,0}+H_{1,1,1}\Big]+(1+x)\Big[\frac{5}{4}H_2+\frac{33}{8}$$

$$-\frac{99}{4}H_{0,0}-8H_{2,0}-\frac{541}{24}H_0-4H_{2,1}-\frac{3}{2}H_{0,0,0}-2x\zeta_3+\frac{9}{2}\zeta_2^{\,2}+5H_0\zeta_2-5H_3-4H_{-1}\zeta_2$$

$$-8H_{-1,-1,0}+\frac{67}{3}H_{-1,0}+4H_{-1,0,0}+2H_{0,0}\zeta_2-2H_{0,0,0,0}-4H_2\zeta_2+3H_{2,0,0}+2H_{2,1,0}$$

$$+2H_{2,1,1}+H_{3,1}-2H_4\Big]+\frac{1}{16}\delta(1-x)\Big)$$

MVV (2004)

Going beyond NNLO

[Moch et al]

Large x double log behaviour of Splitting and Coefficient functions

Trick: Use Physical Evolution Equations (PEE):

PEE: Differential equations w.r.t Q of Structure functions

$$\frac{d}{d \ln Q^2} F = \mathbf{K} F \equiv \sum_{\ell=0}^{\infty} a_s^{\ell+1} \begin{pmatrix} K_{22}^{(\ell)} & K_{2\phi}^{(\ell)} \\ K_{\phi 2}^{(\ell)} & K_{\phi\phi}^{(\ell)} \end{pmatrix} \cdot \begin{pmatrix} F_2 \\ F_\phi \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} F_2 \\ F_\phi \end{pmatrix}$$

Kernels are enhanced by single logs

Going beyond NNLO

[Vogt et al]

Large x Behaviour of 4-loop Splitting and Coefficient functions

From PEE of (F_2, F_ϕ)

4-loop results!

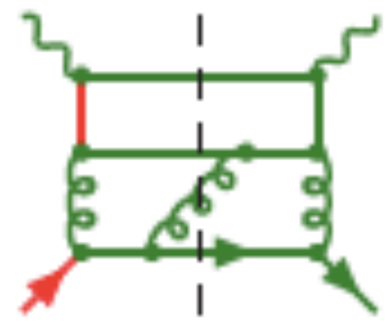
$$
\begin{aligned}
P^{(3)}_{\mathrm{qg}}(x) &= \ln^6(1-x)\cdot 0 \\
&+ \ln^5(1-x)\Big[\frac{22}{27}C_{AF}^3 n_f - \frac{14}{27}C_{AF}^2 C_F n_f - \frac{4}{27}C_{AF}^2 n_f^2\Big] \\
&+ \ln^4(1-x)\Big[\Big(\frac{293}{27} - \frac{80}{9}\zeta_2\Big)C_{AF}^3 n_f + \Big(\frac{4477}{162} - 8\zeta_2\Big)C_{AF}^2 C_F n_f \\
&- \frac{13}{81}C_{AF}C_F^2 n_f - \frac{116}{81}C_{AF}^2 n_f^2 + \frac{17}{81}C_{AF}C_F n_f^2 - \frac{4}{81}C_{AF} n_f^3\Big] \\
&+ \mathcal{O}\left(\ln^3(1-x)\right)
\end{aligned}
$$

From PEE of (F_2, F_ϕ) $\quad and \quad$ (F_2, F_L)

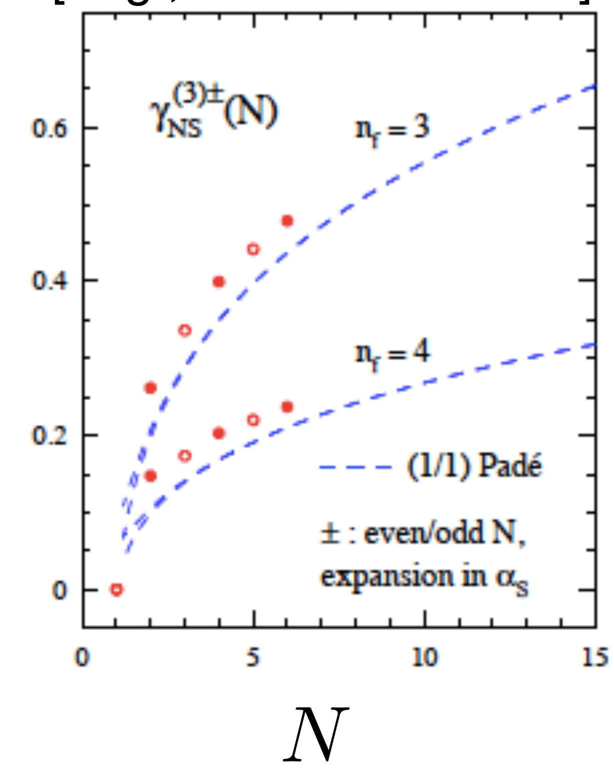
Predictions of $\log^{6,5,4}(1-x)$ $\quad of \quad$ $C_L^{(3)}$

Going beyond NNLO

MINCER to FORCER
for 4 loop results

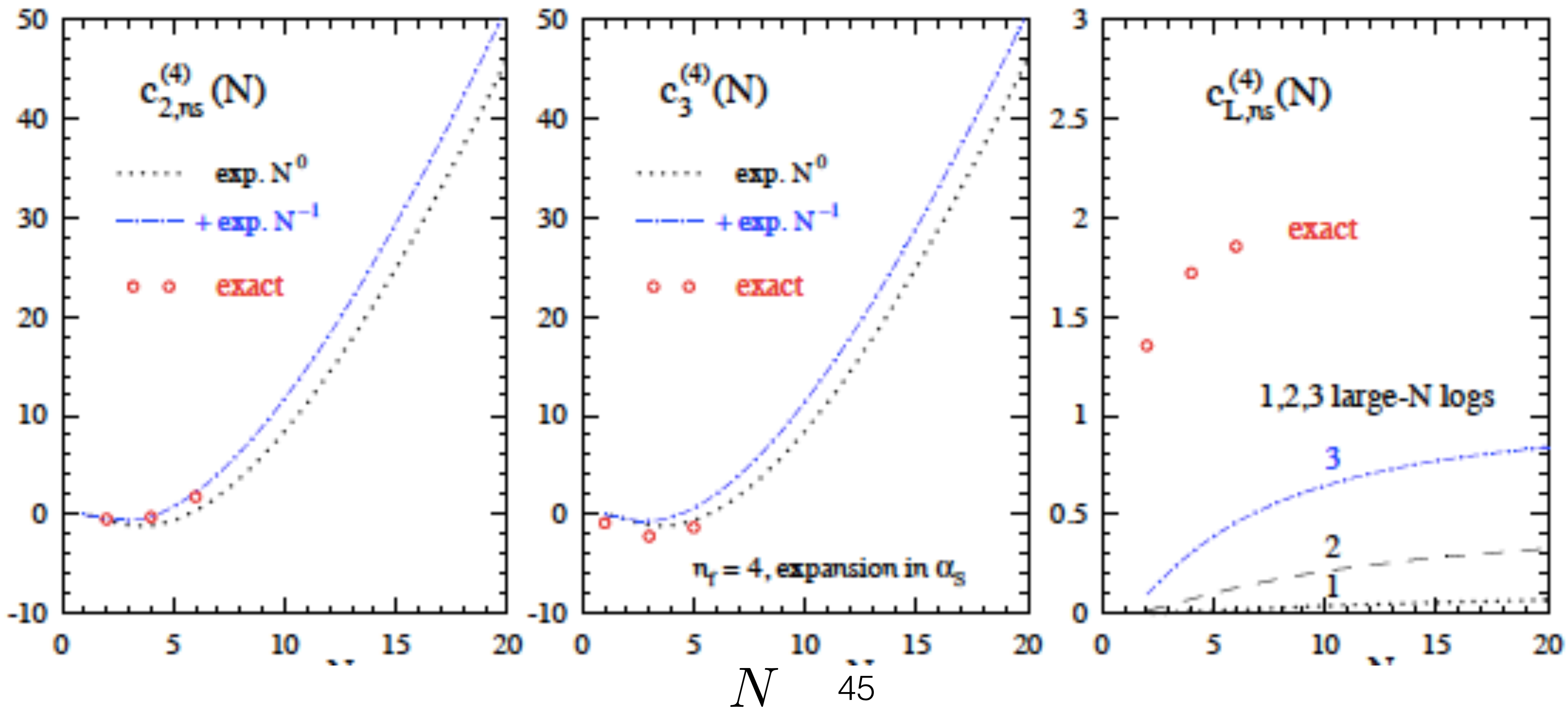


[Vogt, Vermaseren et al]



Third order contributions to Coefficient and splitting functions

$$C^{(3)}_{2,3,L} \qquad P^{(3)}_{ij}(x), \gamma^{(3)}_{ij}(N)$$



Going beyond NNLO

[Baikov et al, Velizhanin]

Non-Singlet Splitting function at 4-loops

$$\gamma_n^{approx:3} = \frac{{\gamma_n^{(2)}}^2}{\gamma_n^{(1)}}$$

$$\gamma_n = -\int dx x^{n-1} P(x)$$

$$\mathcal{O}_{\rm NS}^{a,\mu\nu\rho} = \bar{\psi}\lambda^a\gamma^\mu \mathcal{D}^\nu \mathcal{D}^\rho \psi\,,$$
$$\mathcal{O}_{\rm NS}^{a,\mu\nu\rho\sigma} = \bar{\psi}\lambda^a\gamma^\mu \mathcal{D}^\nu \mathcal{D}^\rho \mathcal{D}^\sigma \psi$$

n=2 moment

$$\gamma_2^{3;NS} = \frac{32}{9}a_s + \frac{9440}{243}a_s^2 + \left[\frac{3936832}{6561} - \frac{10240}{81}\zeta_3\right]a_s^3 + \left[\frac{1680283336}{1777147} - \frac{24873952}{6561}\zeta_3 + \frac{5120}{3}\zeta_4 - \frac{56969}{243}\zeta_5\right]a_s^4$$

n= 3, 4 moment

$$\gamma_{\rm NS}^{4-loop}(3, n_f = 4) = 5.55556\,a_s + 50.39095\,{a_s}^2 + 418.17201\,{a_s}^3 + 4322.89048\,{a_s}^4\,,$$
$$\gamma_{\rm NS}^{4-loop}(4, n_f = 4) = 6.97778\,a_s + 60.07233\,{a_s}^2 + 502.91174\,{a_s}^3 + 5066.33924\,{a_s}^4\,.$$

Pade` 3480

Pade` 4200

Parametrisation of PDFs

Standard form

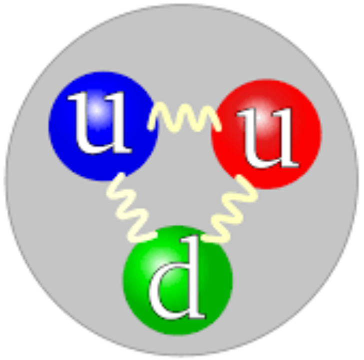
at initial scale μ_0

$$f(x,\mu_o^2) = \alpha_o x^{\alpha_1}(1-x)^{\alpha_2} P(x)$$

where $P(x) = (1+\alpha_3 x + \alpha_4 x^2 + \cdots)e^{\beta_1 x}(1+e^{\beta_4}x)^{\beta_5}$

Simple Constraints

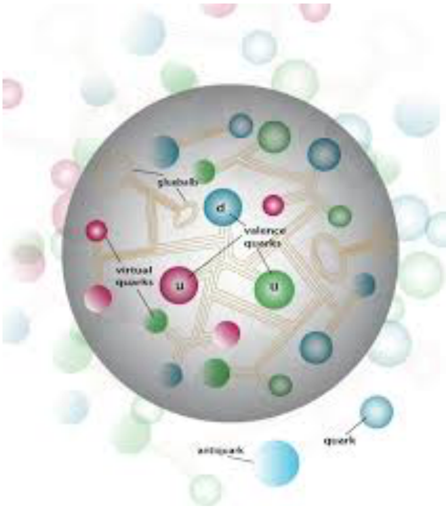
$$\int_0^1 dx\,\left(f_{u/P}(x,\mu^2) - f_{\overline{u}/P}(x,\mu^2)\right) = 2 \qquad \int_0^1 dx\left(f_{d/P}(x,\mu^2) - f_{\overline{d}/P}(x,\mu^2)\right) = 1$$



$$\int_0^1 dx\,\left(f_{s/P}(x,\mu^2) - f_{\overline{s}/P}(x,\mu^2)\right) = 0$$

Momentum sum rule

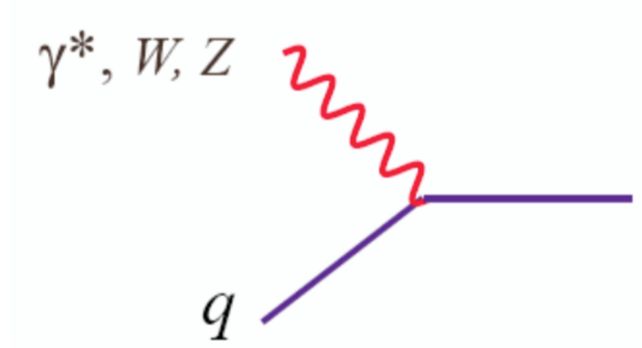
$$\int_0^1 dx\; x\left(\sum_i \left(f_{q_i/P}(x,\mu^2) - f_{\overline{q}_i/P}(x,\mu^2)\right) + f_{g/P}(x,\mu^2)\right) = 1$$



Observables for PDF exactraction

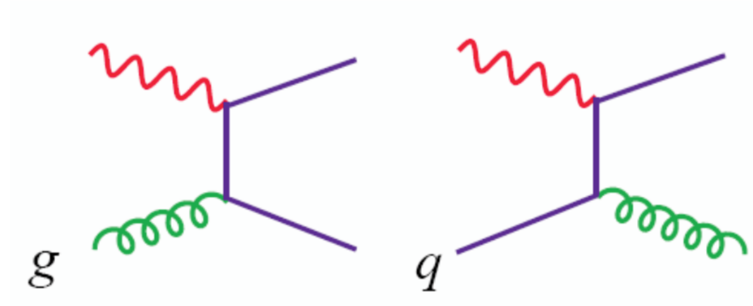
$DIS - eN, \mu N$

(CDHS,CHARM,CCFR,CHORUS,NuTeV)



$DIS - \nu N, \overline{\nu} N$

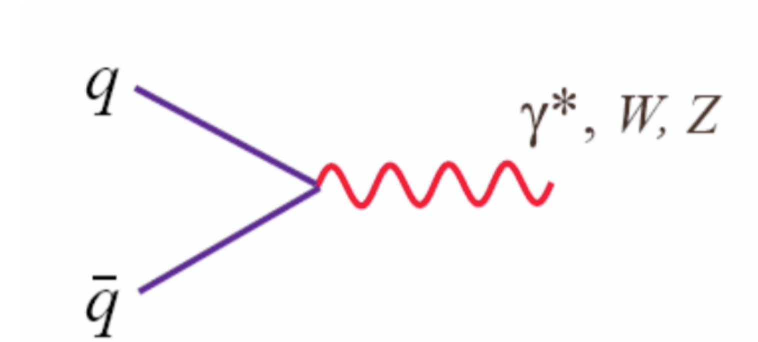
(SLAC,BCDMS,NMC,E665,H1,ZEUS)



$p\overline{p} \rightarrow jets$

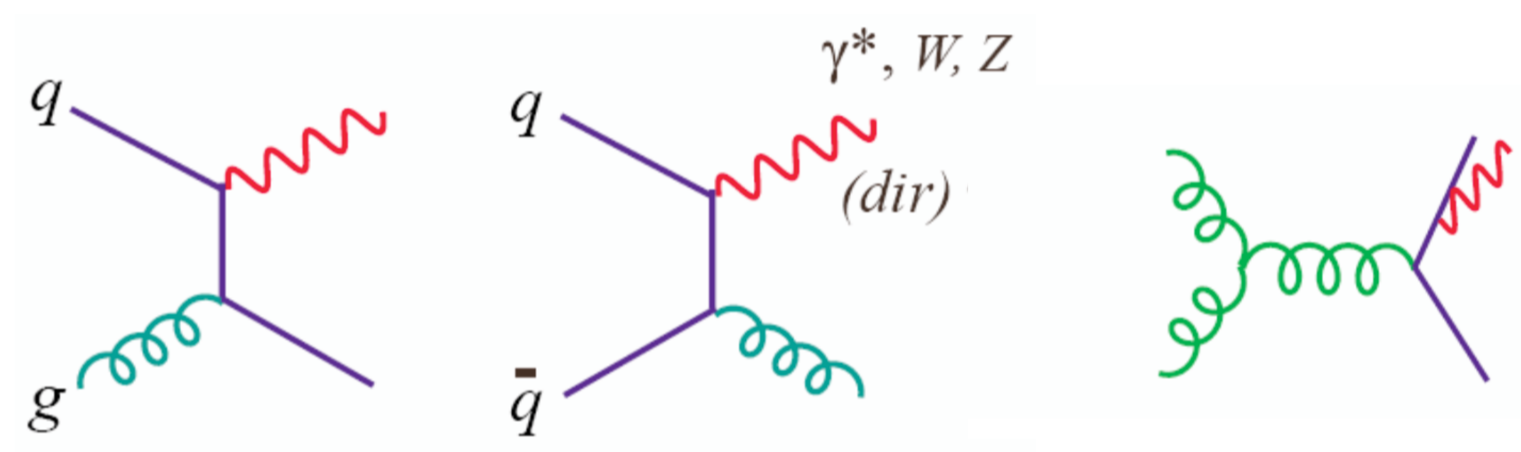
(CDF,D0)

Drell-Yan



Prompt photon

(WA70,UA6,E706)



PDF extraction

GRV, GJR ...

MRST, MSTW ...

CTEQ, CT# ...

NNPDF

ABM, ABKM

Index of /archive/lhapdf/pdfsets/6.1

Name	Last modified	Size	Description
Parent Directory		-	
ATLAS-epWZ12-EIG.tar.gz	23-Apr-2014 21:38	39M	
ATLAS-epWZ12-VAR.tar.gz	23-Apr-2014 21:38	15M	
CJ12max.tar.gz	09-Mar-2016 12:00	3.4M	
CJ12mid.tar.gz	09-Mar-2016 12:00	3.4M	
CJ12min.tar.gz	09-Mar-2016 12:00	3.4M	
CJ15lo.tar.gz	21-Jun-2016 11:34	4.3M	
CJ15nlo.tar.gz	08-Jun-2016 13:36	4.4M	
CT09MC1.tar.gz	13-Apr-2014 08:12	206K	
CT09MC2.tar.gz	13-Apr-2014 08:12	227K	
CT09MCS.tar.gz	13-Apr-2014 08:12	223K	
CT10.tar.gz	13-Apr-2014 08:12	9.8M	
CT10as.tar.gz	29-Oct-2014 12:14	2.0M	
CT10f3.tar.gz	13-Apr-2014 08:12	133K	
CT10f4.tar.gz	13-Apr-2014 08:12	160K	
CT10nlo.tar.gz	13-Apr-2014 08:12	10M	
CT10nlo_as_0112.tar.gz	13-Apr-2014 08:12	190K	
CT10nlo_as_0113.tar.gz	13-Apr-2014 08:12	190K	
CT10nlo_as_0114.tar.gz	13-Apr-2014 08:12	190K	
CT10nlo_as_0115.tar.gz	13-Apr-2014 08:12	190K	
CT10nlo_as_0116.tar.gz	13-Apr-2014 08:12	190K	
CT10nlo_as_0117.tar.gz	13-Apr-2014 08:12	189K	
nCTEQ15npFullNuc_208_82.tar.gz	04-Mar-2016 15:37	3.0M	
nCTEQ15np_1_1.tar.gz	04-Mar-2016 16:37	96K	
nCTEQ15np_3_2.tar.gz	04-Mar-2016 15:37	3.0M	
nCTEQ15np_4_2.tar.gz	04-Mar-2016 15:37	3.0M	
nCTEQ15np_6_3.tar.gz	04-Mar-2016 15:37	3.0M	
nCTEQ15np_7_3.tar.gz	04-Mar-2016 15:37	3.0M	
nCTEQ15np_9_4.tar.gz	04-Mar-2016 15:37	3.0M	
nCTEQ15np_12_6.tar.gz	04-Mar-2016 15:37	3.0M	
nCTEQ15np_14_7.tar.gz	04-Mar-2016 15:37	3.0M	
nCTEQ15np_20_10.tar.gz	04-Mar-2016 15:37	3.0M	
nCTEQ15np_27_13.tar.gz	04-Mar-2016 15:37	3.0M	
nCTEQ15np_40_18.tar.gz	04-Mar-2016 15:37	3.0M	
nCTEQ15np_40_20.tar.gz	04-Mar-2016 15:37	3.0M	
nCTEQ15np_56_26.tar.gz	04-Mar-2016 15:37	3.0M	
nCTEQ15np_64_32.tar.gz	04-Mar-2016 15:37	3.0M	
nCTEQ15np_84_42.tar.gz	04-Mar-2016 15:37	3.0M	
nCTEQ15np_108_54.tar.gz	04-Mar-2016 15:37	3.1M	
nCTEQ15np_119_59.tar.gz	04-Mar-2016 15:37	3.1M	
nCTEQ15np_131_54.tar.gz	04-Mar-2016 15:37	3.1M	
nCTEQ15np_184_74.tar.gz	04-Mar-2016 15:37	3.1M	
nCTEQ15np_197_79.tar.gz	04-Mar-2016 15:37	3.1M	
nCTEQ15np_197_98.tar.gz	04-Mar-2016 15:37	3.1M	
nCTEQ15np_207_103.tar.gz	04-Mar-2016 15:37	3.1M	
nCTEQ15np_208_82.tar.gz	04-Mar-2016 15:37	3.1M	
pdfsets.index	11-Aug-2016 16:08	19K	
unvalidated/	07-Jan-2015 09:52	-	

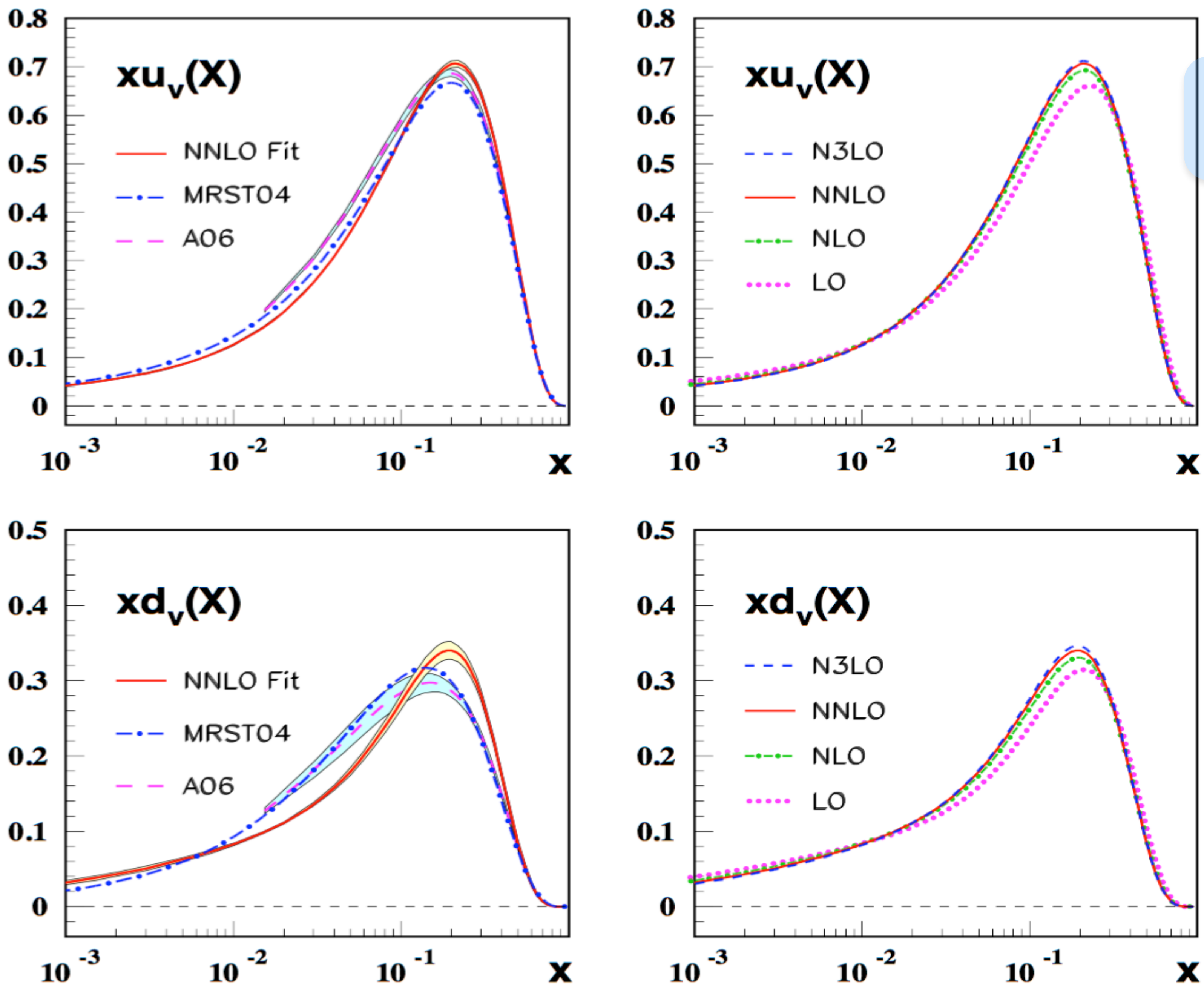
Long List of 19 pages

PDFs at approx. N3LO

[Bluemlein et al]

World Data: NS-analysis

W^2> 12.5 GeV^2 , Q^2 > 4 GeV^2



$$\alpha_s(M_Z) = 0.1145 \pm 0.0009 \,(\text{exp.})$$

Heavy Flavours to DIS

[Bluemlein et al]

Coefficient functions depend on m

$$F_{2,L}(x,Q^2) = \sum_j C_{j,2,L}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes f_j(x,\mu^2)$$

mass of the heavy flavour

Mellin space result

$$C_{j,2,L}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = C_{j,2,L}\left(N, \frac{Q^2}{\mu^2}\right) + H_{j,2,L}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)$$

Light flavour

Heavy flavour

Factorisation

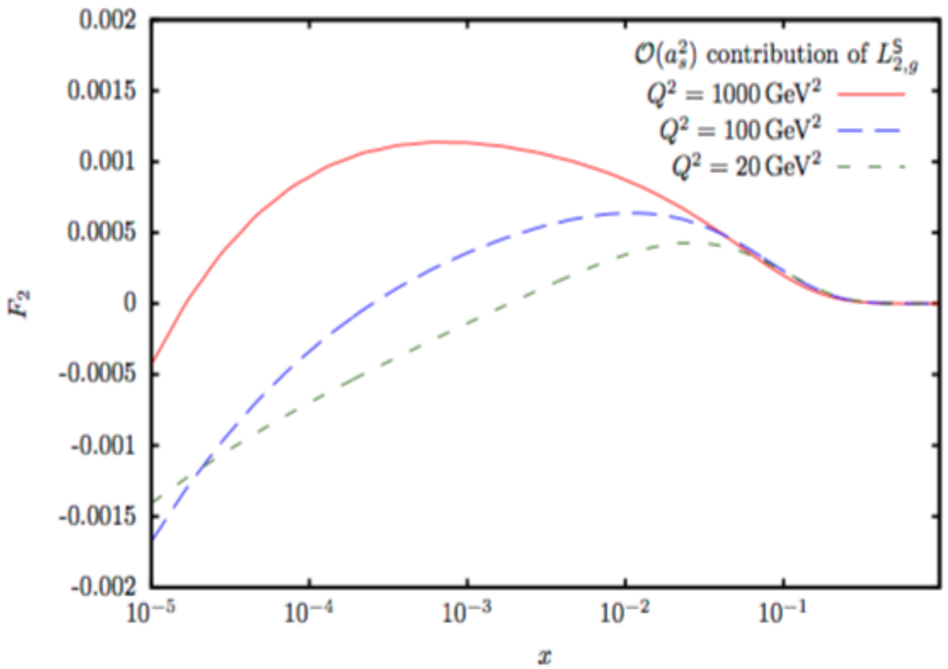
$$H_{j,2,L}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_i C_{i,2,L}\left(N, \frac{Q^2}{\mu^2}\right) A_{ij}\left(\frac{m^2}{\mu^2}, N\right)$$

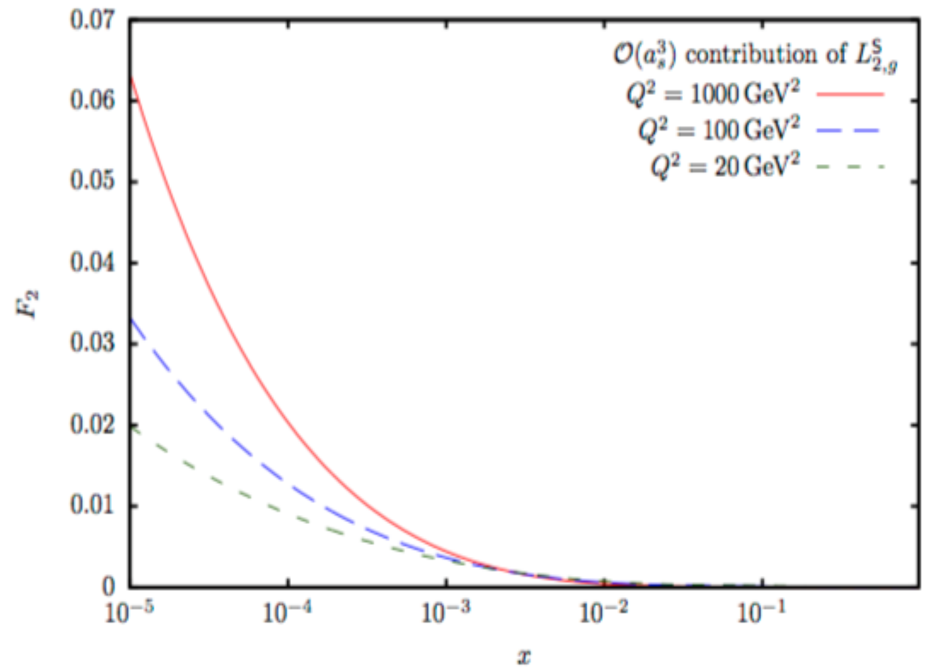
Operator

Perturbatively Calculable: $A_{ij}\left(\frac{m^2}{\mu^2}, N\right) = \langle j \mid O_i \mid j \rangle\ .$

Heavy Flavours to DIS

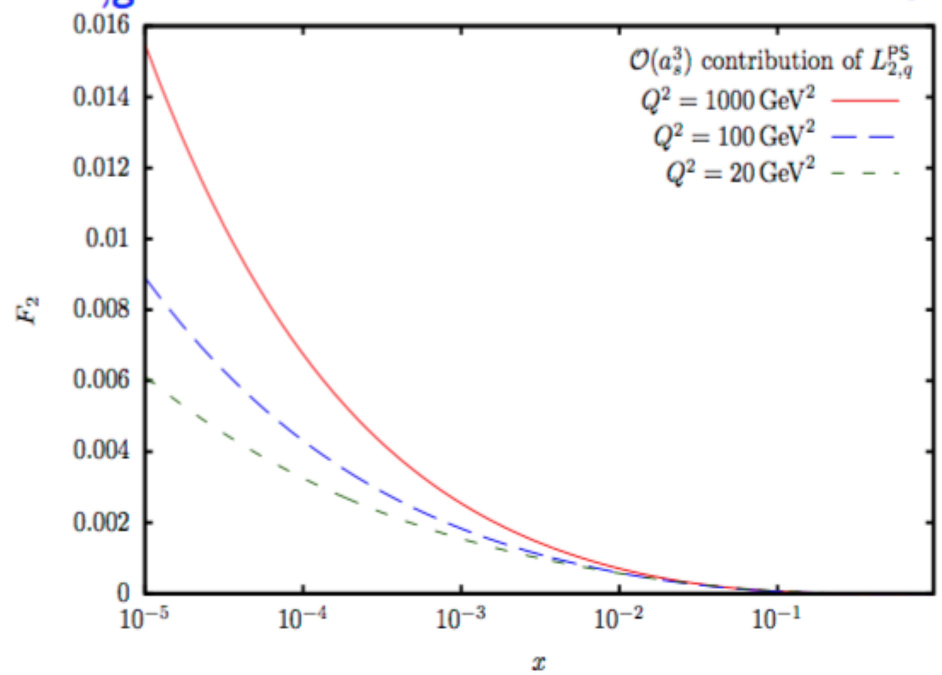
[Bluemlein et al]





$O(a_s^2)$ $L^{\mathrm{S}}_{2,g}$

$O(a_s^3)$ $L^{\mathrm{S}}_{2,g}$

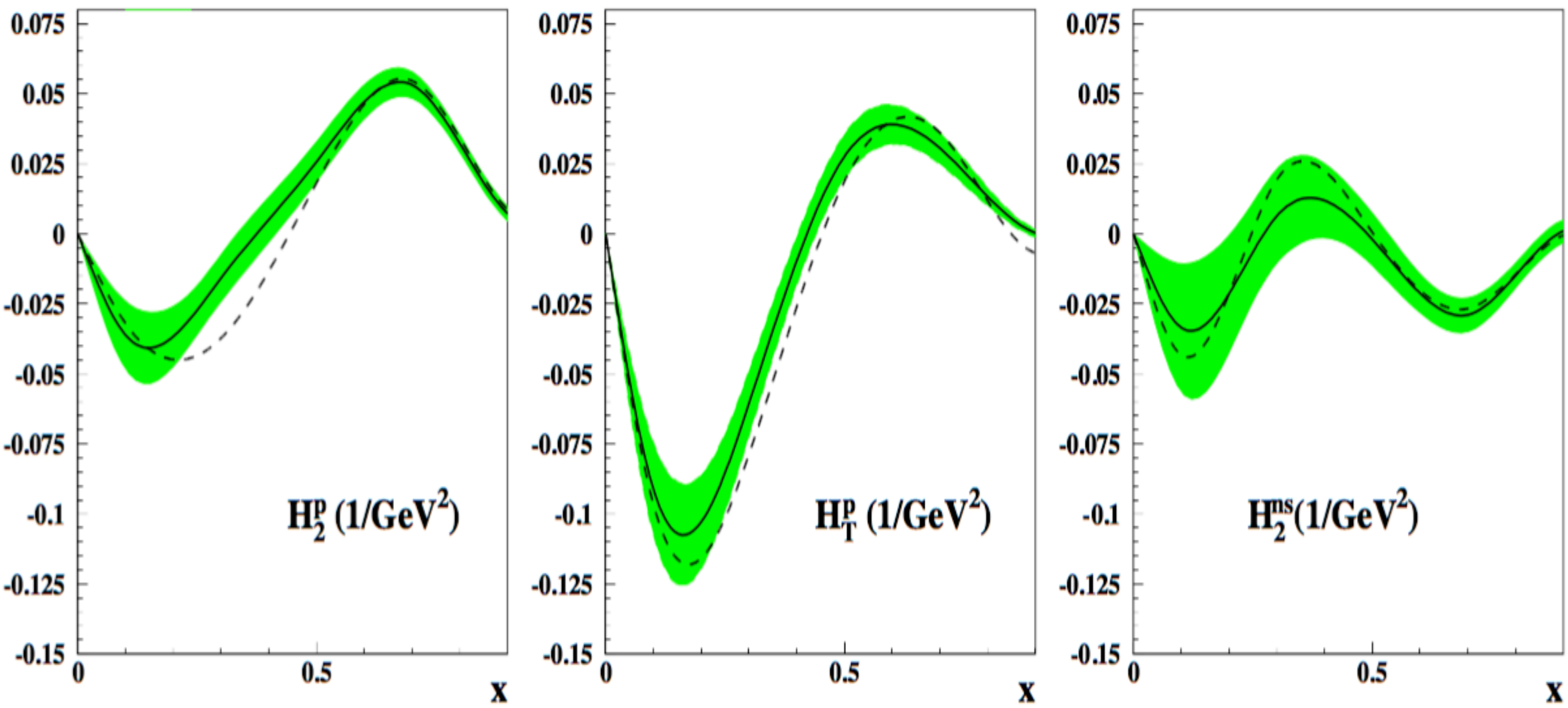


$L^{\mathrm{PS}}_{q,2}$

Higher Twist to DIS

[Bluemlein et al]

$$F_i(x,Q^2) = F_i^{TMC,\tau=2}(x,Q^2) + \frac{H_i^4(x)}{Q^2} + \frac{H_i^6(x)}{Q^4} + \ldots$$



Resummationed PDFs

[Bonvini et al]

Coefficient functions in

$$F_2(x,Q^2) = x\sum_{n=0}^{\infty}\frac{\alpha_s^n(\mu_R^2)}{(2\pi)^n}\sum_{i=q,g}\int_x^1\frac{dz}{z}C_{2,i}^{(n)}(z,Q^2,\mu_R^2,\mu_F^2)\,f_{i/p}\left(\frac{x}{z},\mu_F^2\right)$$

$$\delta(1-z), \quad \left(\frac{\log^j(1-z)}{1-z}\right)$$

$$j = 0, \ldots, \infty$$

$$\frac{1}{z}\log^j z$$

Soft Gluons $z \to 1$

High Energy Gluons $z << 1$

$$\alpha_s^m(\mu_R^2)\; a(x)\log^n b(x) \approx 1$$

when for certain n = g(m)

RESUMMATION to all orders Reliable perturbations predictions

Resummationed PDFs

[Bonvini et al]

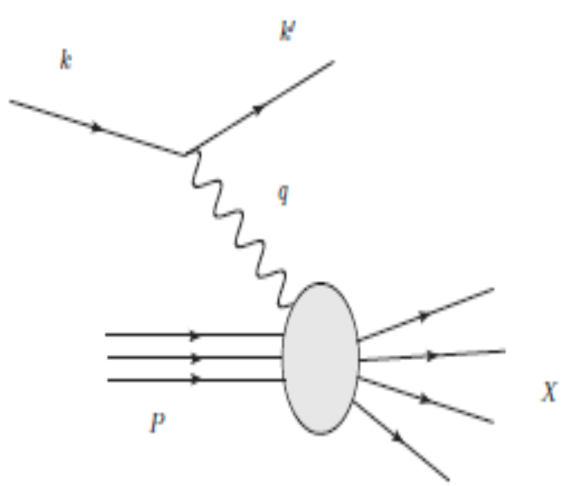
Soft Gluon Resummation: $z \to 1$ Or Mellin $N \to \infty$

$$C_i(z, Q^2, \mu^2) = C_i^{(0)}(z) \quad g_0(\alpha_s) \exp\left(\frac{1}{\alpha_s} g_1(\alpha_s L) + g_2(\alpha_s L) + \alpha_s g_3(\alpha_s L) + \cdots\right)$$

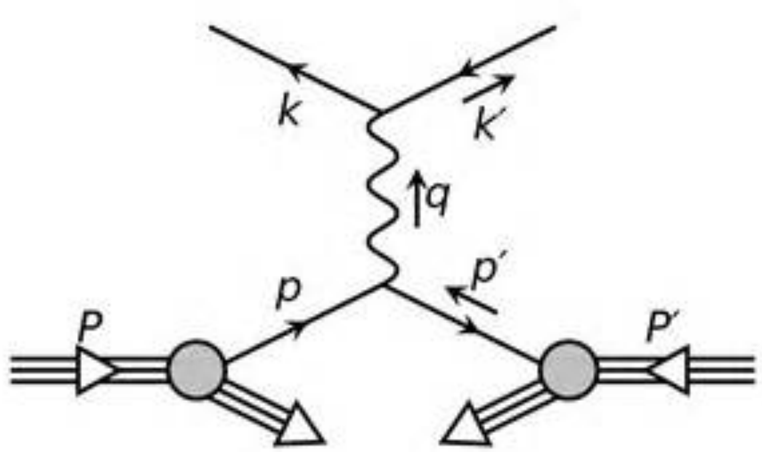
$$L = \frac{\beta_0}{4\pi} \log N$$

N independent

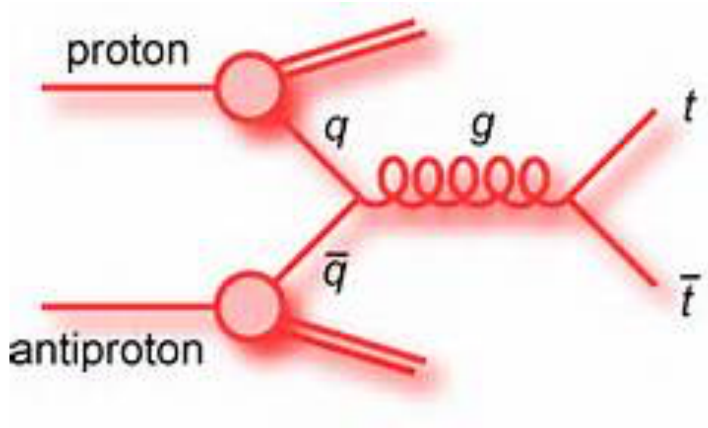
DIS prod.



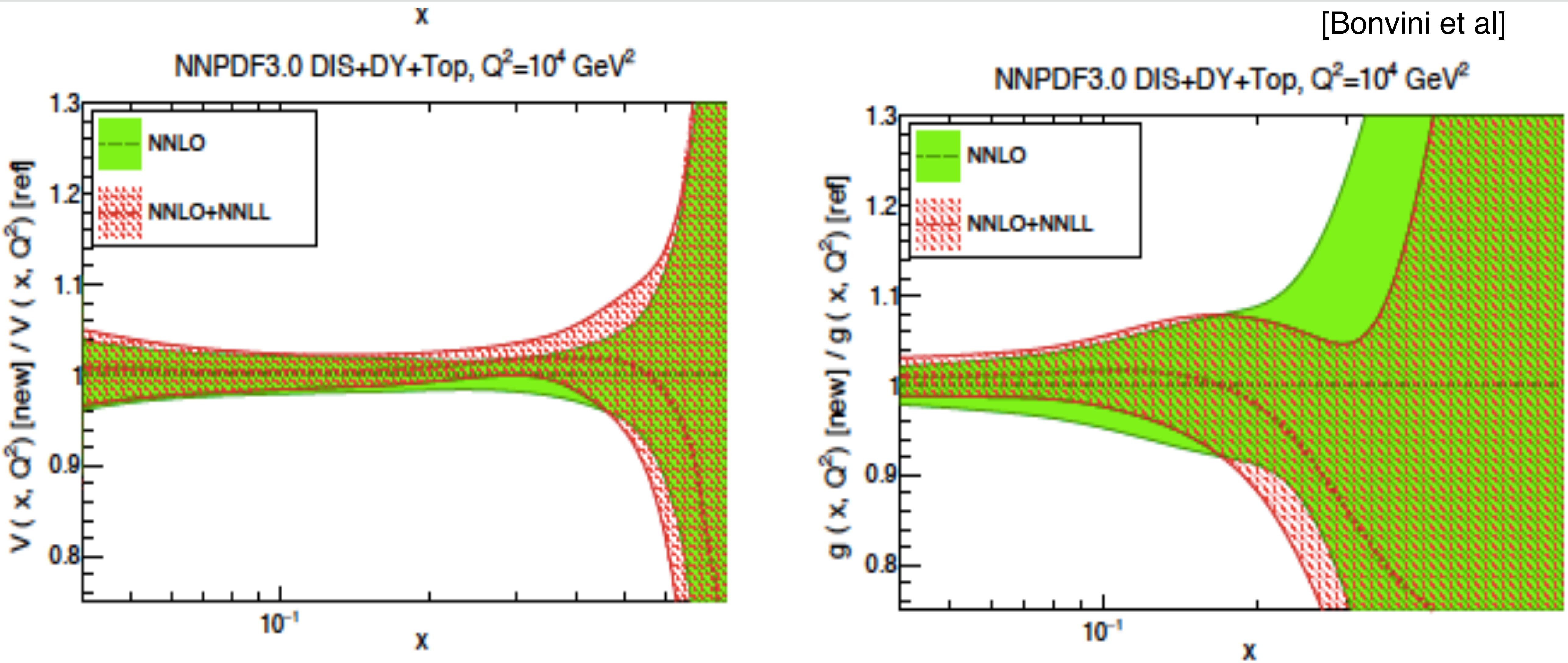
Drell-Yan prod.



Top pair prod.



Resummationed PDFs



- Valence quark are less sensitive to Resumed Coefficient functions
- Large x behaviour of gluons gets modified

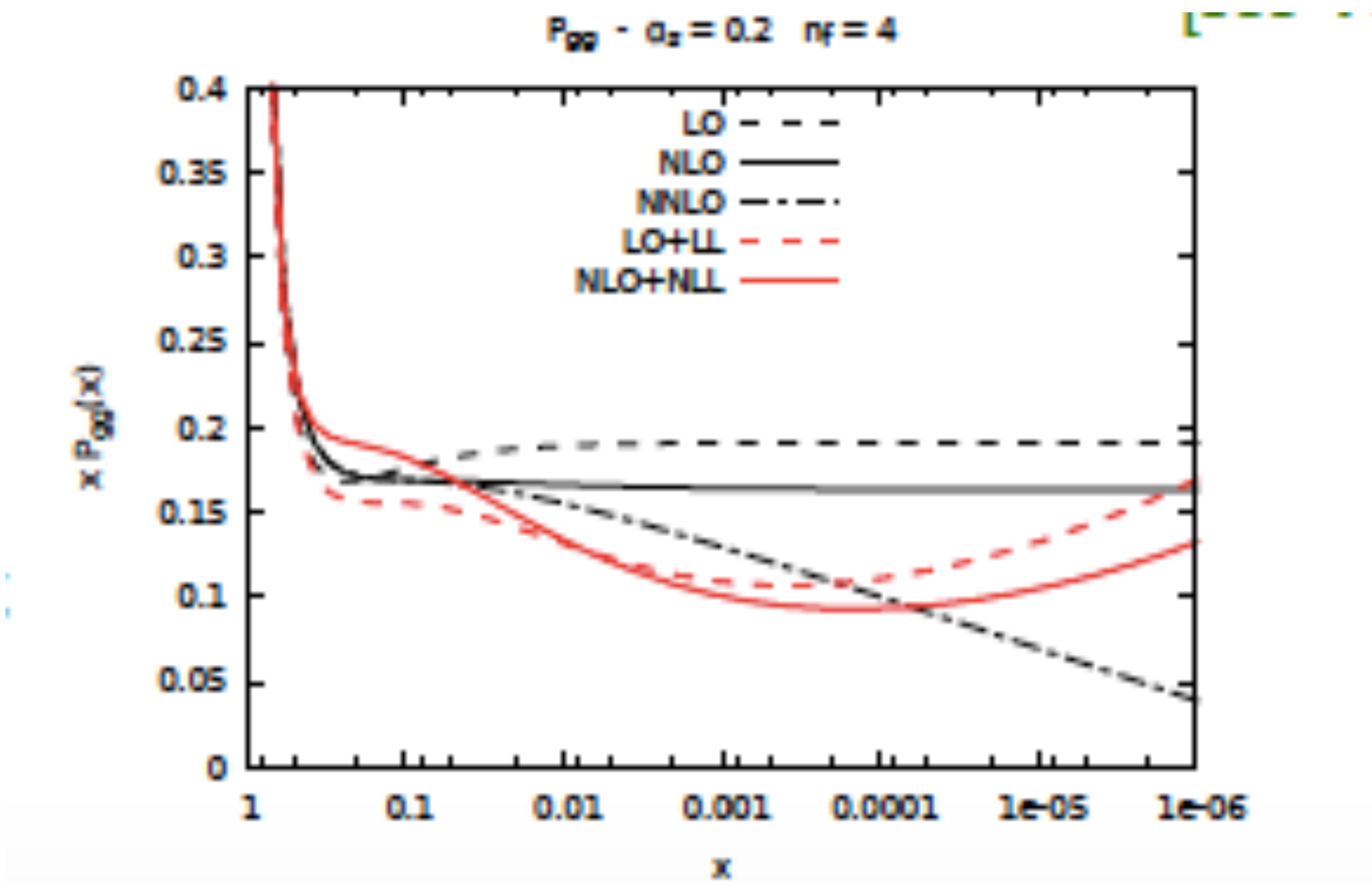
Resummationed PDFs

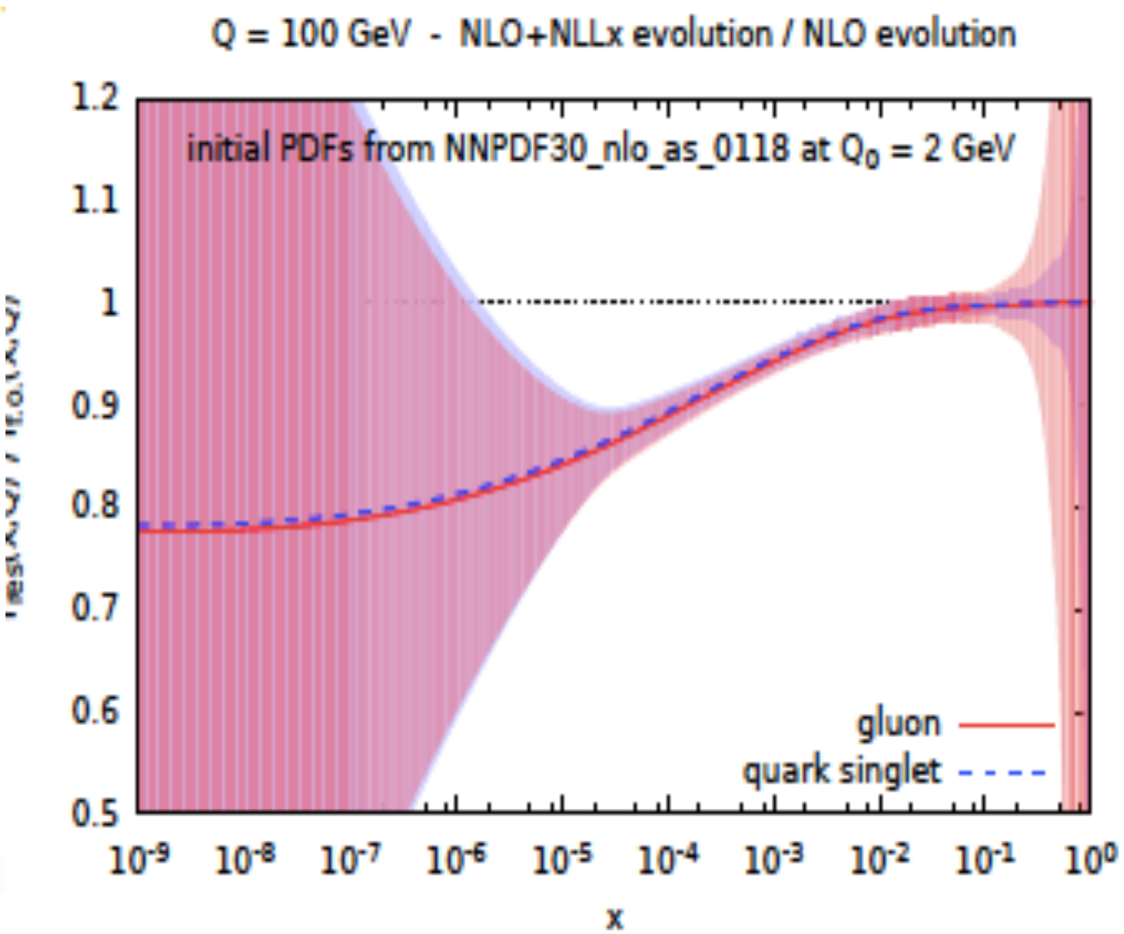
[Bonvini et al]

- Kt or BFKL approach to Small x Resummation
- Altarelli-Ball-Forte procedure to resum small x for both Coefficient and splitting functions

BFKL: $$x\frac{d}{dx}f(x,\mu^2) = \int \frac{d\nu^2}{\nu^2} K\left(x, \frac{\mu^2}{\nu^2}, \alpha_s(\cdot)\right) f(x,\nu^2)$$

HELL (High Energy Large Logarithm) interfaced with APFELL



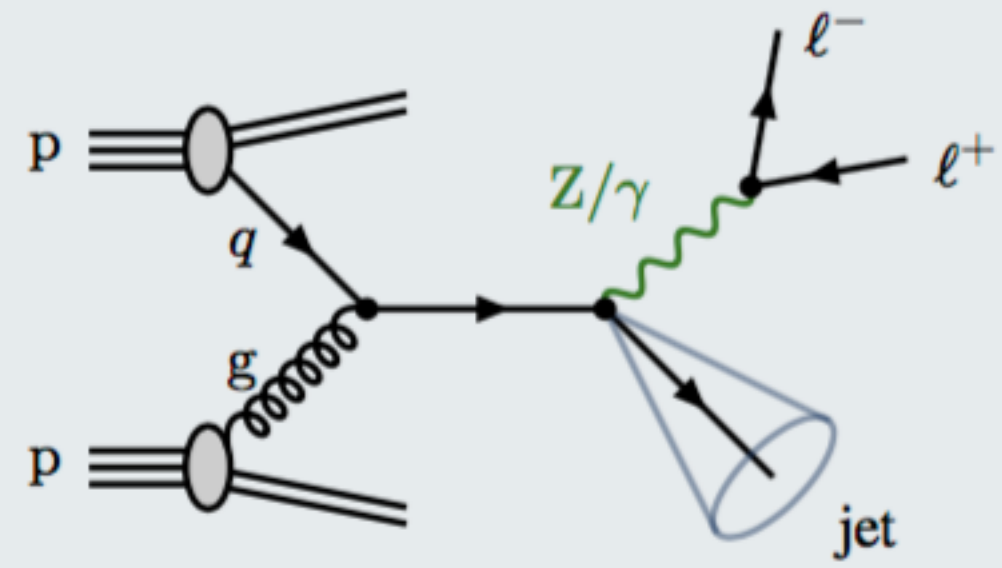


Jets at NNLO for PDF

$$p\,p \to Z/\gamma^* + \text{jet} \to \ell^-\ell^+ + \text{jet} + X$$

- large cross section
- clean leptonic signature

+jet $\rightsquigarrow$ sensitivity to α_s, gluon PDF,...

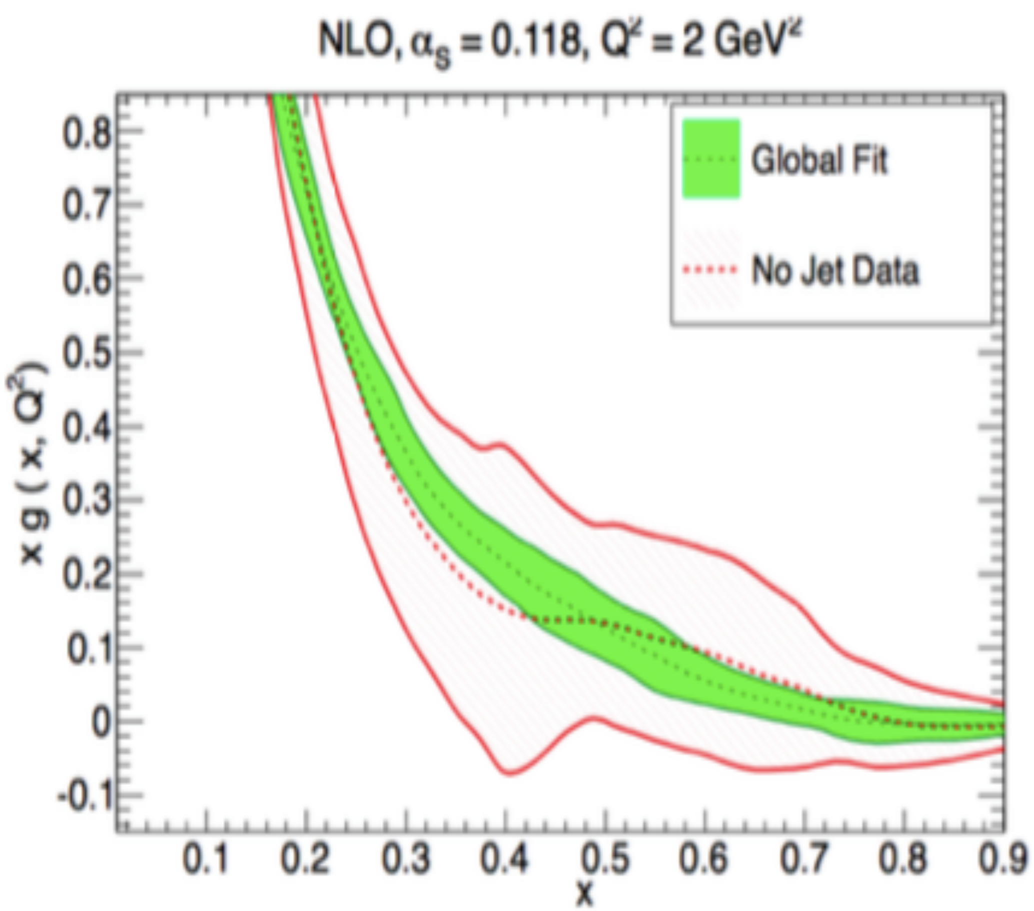


X. Chen, J. Cruz-Martinez, J. Currie, A. Gehrmann-De Ridder, T. Gehrmann, E.W.N. Glover, AH, M. Jaquier, T. Morgan, J. Niehues, J. Pires

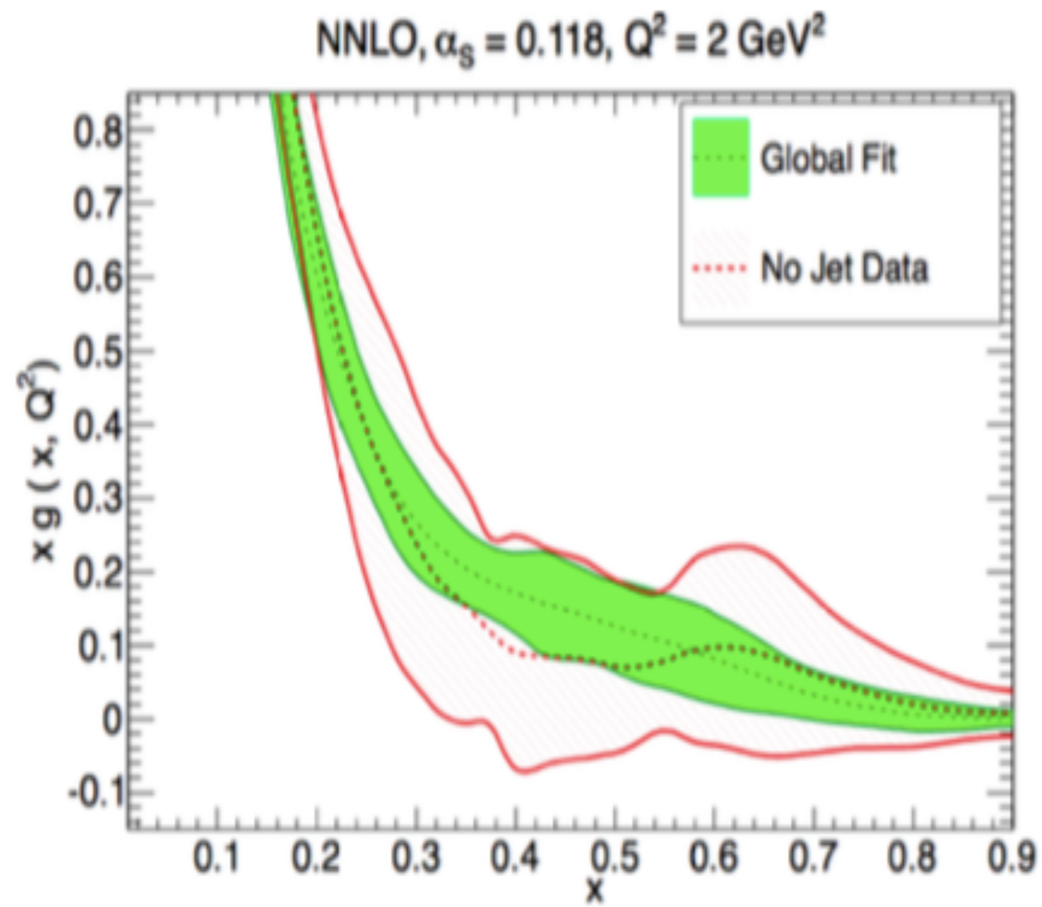
NLO QCD, Giele, Glover, Kosher 93
NLO EW Denner, Dittmaier, Kasprzik, Muck 11

NNLO QCD, Antenna substraction, ..Gehrmann-De Ridder,Gehrmann, Glover, Huss, Morgan
N-jettiness, Boughezal, campbell, Ellis, Focke, Giele, Liu, petriello,15
Boughezal, Liu, Petriello,16

Jet studies for PDFs



NNPDF collaboration



NNPDF collaboration

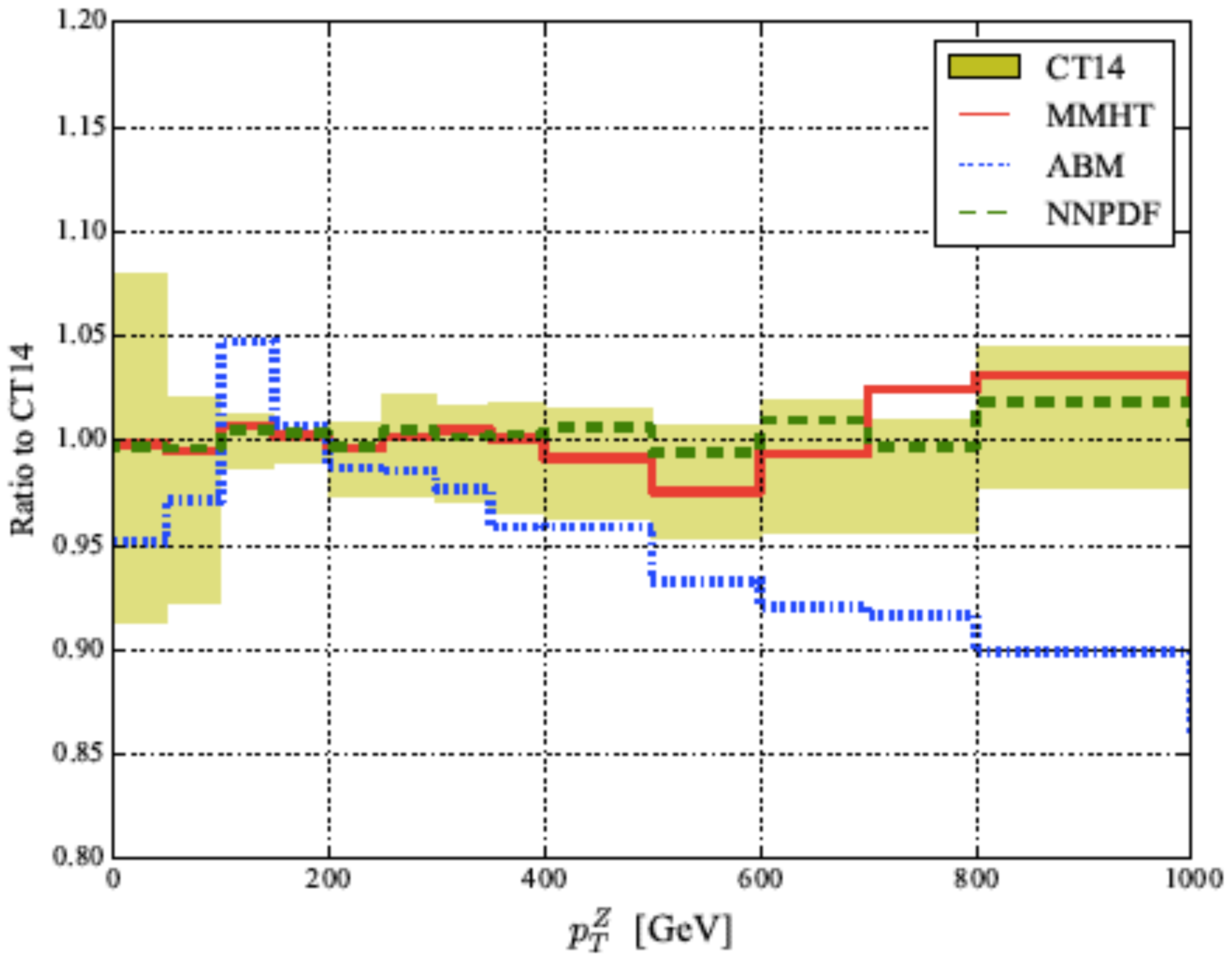
Jet data has a big impact on the medium to large-x gluon PDF

Need exact NNLO all-channel prediction to include full jet dataset

Pt of Z boson in DY for PDFs

Boughezal, Liu, Petriello 2016

- Z boson transverse momentum depends on high x of the gluon
- Fiducial cross section is sensitive to NNLO
- Cross section is dominated at x 10^-2 which is closer to H production region



CT14	ABM12	MMHT2014	NNPDF3.0
15.54 pb	14.98 pb	15.66 pb	15.44 pb

Strong Coupling from PDFs